BIOMETRIKA

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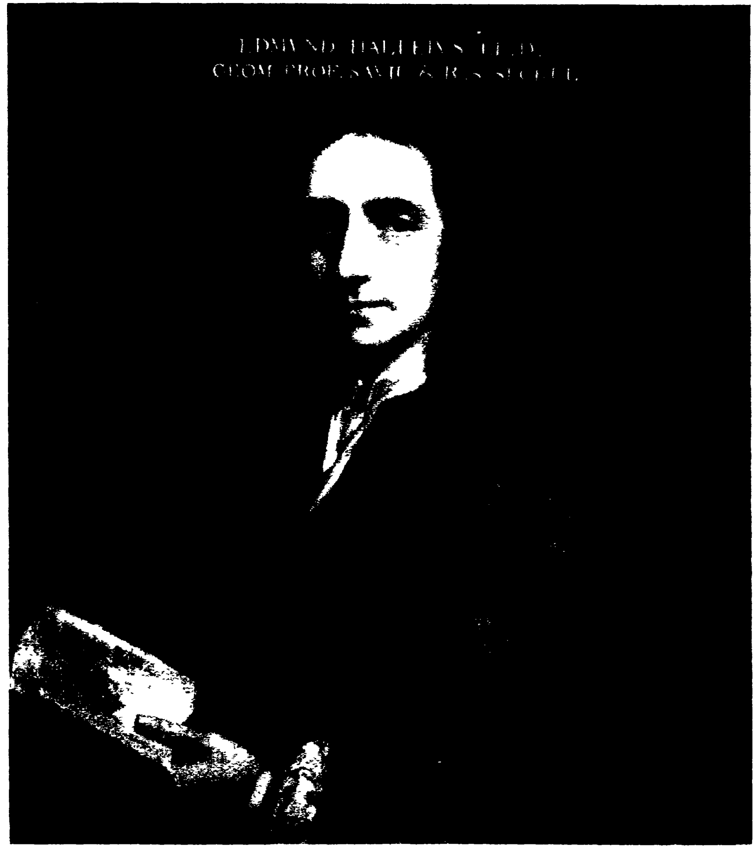

Painted by Michael Dahl. *After Sir G. Kneller.*

Edmund Halley, 1656–1742. From the paintings in the Royal Society's Apartments.
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VOLUME XVIII JULY, 1926 PARTS I AND II

BIOMETRIKA

A THIRD STUDY OF THE ENGLISH SKULL WITH SPECIAL REFERENCE TO THE FARRINGDON STREET CRANIA.

BY BEATRIX G. E. HOOKE, Crewdson-Benington Student in Craniometry, University College, London.

(1) *Account of Material and its Period**. The crania measured and discussed in the present paper were dug up during the removal of certain old houses at the corner of Stonecutter Street and Farringdon Street, and during the resulting excavations for the foundations of the new business premises of Messrs Gordon and Gotch erected on this site. The building operations were started in the spring of 1924 and the present buildings completed in 1925. The site excavated had a frontage of about 63′ on Stonecutter Street, and this part had a breadth of 23′; the total frontage on Farringdon Street had a length of 90′, or excluding the Stonecutter Street rectangle (63′ × 23′), a frontage of 67′, to which there was a breadth of 45′. We shall speak of these as the Stonecutter and Farringdon rectangles. Both rectangles were excavated to a level of 15′ below Farringdon Street. Roughly at 15′ gravel was reached and pits were driven 10′ lower, some 6′ through the gravel and 4′ through the blue clay for the stanchions for the new buildings. The west boundary of the Farringdon rectangle was formed by a nineteenth century wall going down 15′. The west end of the Stonecutter rectangle, extending 18′ to 19′ beyond the western boundary of the Farringdon rectangle, was the floor of a cellar of an 18th century house. The north and south boundaries of the Farringdon rectangle were formed by massive brick walls reaching to a depth of 15′, and a third such wall ran east and west across the Farringdon rectangle dividing it into two unequal rectangles, the north 37′ × 45′ and the south 30′ × 45′. These walls we are informed were certainly built before 1760. In the Stonecutter rectangle no human remains whatever were found. The whole of the Farringdon rectangle was densely packed with human remains to all levels from within 1′ of the surface of Farringdon Street, and they appeared to extend as a mass of osseous debris under Farringdon Street itself. There must have been more than 600 skeletons on the site, and they were clearly not deposited as buried. The graveyard had evidently been violently disturbed, and none of the workmen had come across a skeleton lying in such a position as that in which it might have

* Prepared by the Editor from data collected by G. M. Morant, B. G. E. Hooke and M. Child.

been interred. Fragments of wooden coffins, coffin handles, etc. were found, but in no case fragments of hair or tissues such as occur where coffins have not been broken up and disturbed. In fact the whole was a jumbled mass of human bones, rotten wood, building materials and earth. In many cases mortar or concrete was attached to the bones, showing that in earlier building operations, no respect whatever had been paid to the human remains; they had also largely been piled up against the northern boundary wall of the Farringdon Street rectangle.

A few human skeletons were found in the blue clay though far less plentifully, and at the same level bones of dogs, oxen and pigs, said to be indistinguishable from modern varieties. Except the staining these skeletons from the blue clay were not distinguishable from those found above, and it seems probable that the clay as well as the gravel had been disturbed.

The artifacts found included (*a*) fragments of tiles probably of 17th century, possibly of early 18th century, (*b*) an almost complete unornamented earthenware vessel about 2′ high (now in the London Museum), and the base of a precisely similar one. Mr G. F. Lawrence, Inspector of Excavations to the London Museum, dates these water pots as late 17th or early 18th century, (*c*) clay tobacco pipes, one of the Restoration period and the others 20 or 30 years later, (*d*) coffin handles, most of very simple form, giving no reliable clue to their age, but very possibly of 17th century date. *Three* coffin handles were ornamented, and said to be of Georgian date, "later than 1760." The supposed date of these handles is not in accordance with the bulk of evidence with regard to the interments. (*e*) Two very small and simple head-stones with initials and dates, 1665 and 1666, i.e. the years of the Great Plague and the Great Fire. These head-stones were found under the present pavement of Farringdon Street, under which, as we have noted, the osseous debris extends.

The absence of other domestic articles is noteworthy as the parish records show continual difficulties with the deposit of refuse in the graveyard. The shells of edible shellfish were in abundance at all levels down to the bottom of the cross-walls. Most of the bones were discoloured, according to the depth at which they were found, and some had the bluish green stains of copper salts*.

Such is the information collected on the site by Mr G. M. Morant, who spent several days watching the excavations. We must now turn to the chartographic history of the site. Here we have to bear in mind that originally a part of the present line of Farringdon Street was occupied by the Fleet River†, and that barges used to ascend this river to discharge coal. In 1307 we learn that 10 or 12 ships with merchandise could reach Fleet Bridge, while some of them would go up as far as Holburne Bridge. Even as late as 1502 we hear of vessels with fish and fuel being rowed to the Fleete and to the Holburne bridges. An effort in 1589 to preserve the navigation of the Fleete seems to have failed‡, and it

* Similar staining was also noticed in the Liverpool Street and Whitechapel finds.

† How few Londoners now turn a thought to the Fleet River, the Old Burn, the Wells Beck, the Long Burn, or the Wall Brooke, on which the old water supplies of the London district once depended!

‡ The volume of water was originally adequate to make mills in the Whitefriars area profitable.

became more filthy and decayed than ever. The tanners especially were accused of putting their refuse into it, and from this standpoint, it is of interest to note, that part of the modern Stonecutter Street was originally called "Curriers' Alley."

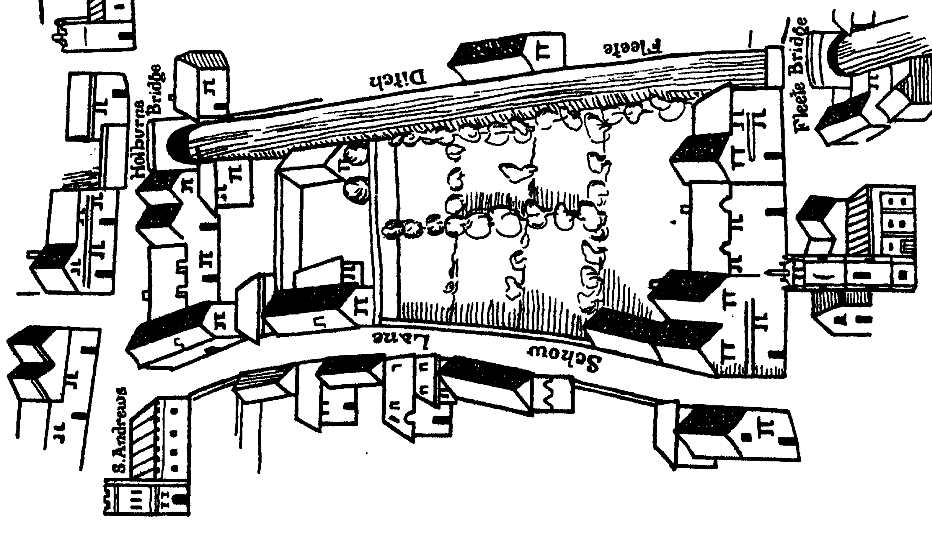
If we examine Agas' map 1560—70 (see Fig. 1) we find a rectanguloid area stretching between St Andrew's Church on the north and St Bride's Church on the south. It is bounded on the south by a row of houses facing Fleete Street, on the east by the Fleete Ditch running from Fleete Bridge to Holburne Bridge, on the north by Holburne and on the west by Schow Lane, now Shoe Lane. While there are houses to the north and south of this area, the middle and southern portion is clearly garden ground with trees; there are no houses on the Fleete Ditch side, but a row of trees and the suggestion of a walk between the trees and the ditch. On the Schow Lane side is a wall with no house until we approach the Fleete Street end*. This space is still open ground in Norden's Map of 1593.

The home of the Sackvilles, Dorset House, in the Whitefriars, had originally been the town residence of the Bishops of Winchester, then it had passed on rental from the Abbot to the Bishops of Salisbury ("Salisbury Court†") and finally into the hands of the Sackvilles from the famous Bishop Jewel who alienated it to Sir Richard Sackville, the founder of the family, and father of the more famous Thomas Sackville, first Earl of Dorset, poet, courtier and statesman. He died in 1608 during a council meeting at Whitehall. His eldest son, Robert, enjoyed the Earldom only for a year, and was in turn succeeded by his elder son Richard (1609), least distinguished of the Sackvilles—except for his debts, and the gift which brings him into our notice here. The burial ground of St Bride's had become overcrowded, and extension was urgent. It was probably to preserve the amenities of Dorset House—there were then orchards between Saint Bride's and the Thames—that Richard Sackville, third Earl of Dorset, who had just had his manor of Salisbury Court confirmed to him by James I, gave a grant of land in the same year 1611 to the parish of St Bride‡ as an additional burial ground. This land was the garden ground between Schow Lane and the Fleete River as shown in Agas' map of 1560—70. The terms of the indenture dated 28th July 8th James I (1611) tell us that Richard, Earl of Dorset, bestows on the parishioners and inhabitants of St Bride's Parish, a piece of ground part of his manor of Holborn to be made, dedicated and employed to and for an addition to the churchyard and burial place of the parish; and also further granted and conveyed

* Agas' map corresponds fairly well with the copy in the Grace Collection of Van der Wyngerde's view of London. In the latter we see the Fleet stretching from the Fleet to the Holborn bridge, with a wall on the west side and barges. The garden ground seems bounded by a wall rather than by trees.

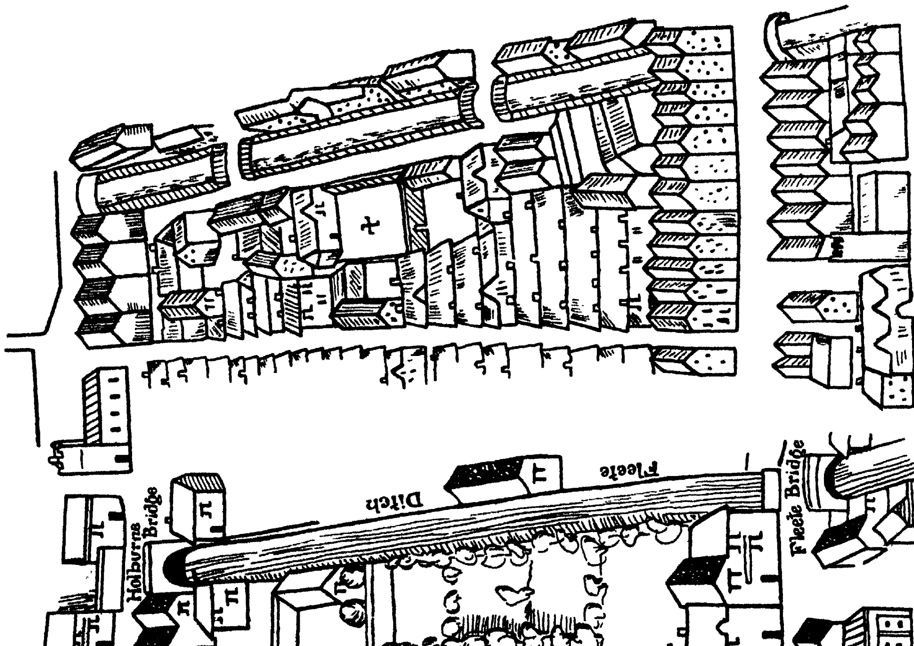
† "The next is Salisburie Court, a place so called for that it belonged to the Bishops of Salisburie, and was their inn, or London house, at such time as they were summoned to come to the parliament, or came for other business; it hath of late time been the dwelling first of Sir Richard Sackville, and now of Sir Thomas Sackville his son, Baron Buckhurst, Lord Treasurer, who hath lately enlarged it with stately buildings." Stowe's *Survey of London*, 1598. Edn. Dent, p. 353.

‡ According to Stowe the parish church of St Bride was of old time a very small thing, until William Venor, Esq., "Warden of the Fleet" about the year 1480, increased it with a large body and south aisles, probably to accommodate the needs of a rapidly growing parish outside the walls.



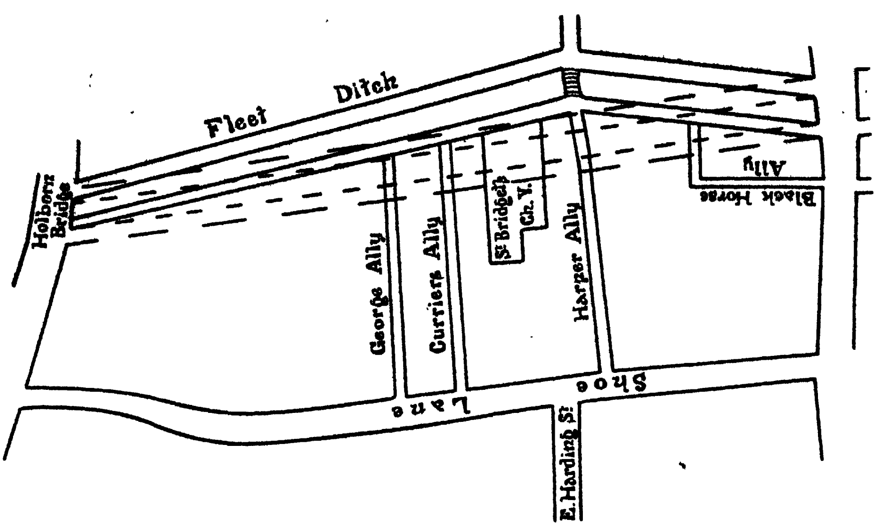
AGAS. 1560–70

Fig. 1.



NEWCOURT. 1658

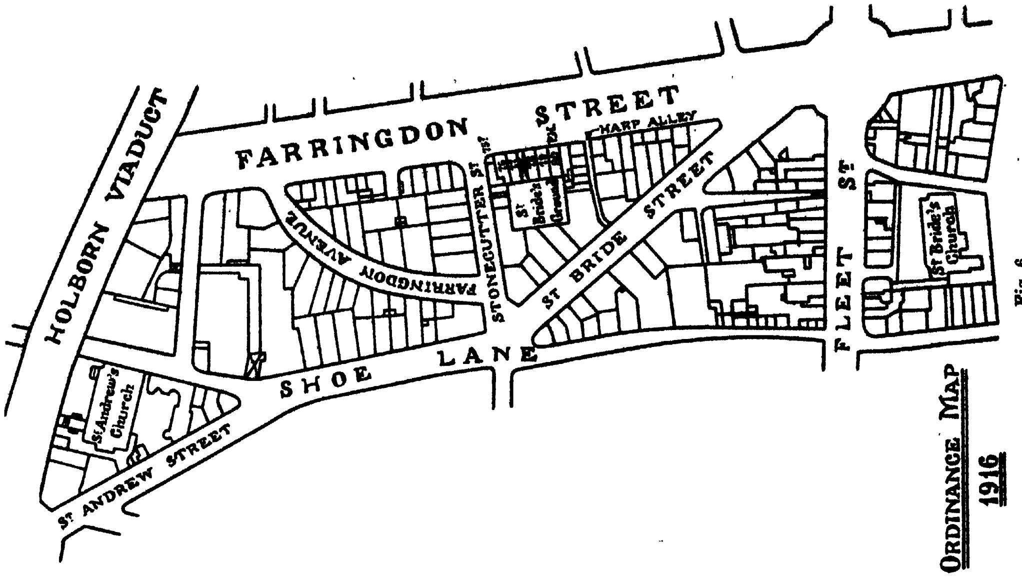
Fig. 2.



– – – Sheweth the enlargement of ye street ways &c

LEAKE. 1666

Fig. 3.

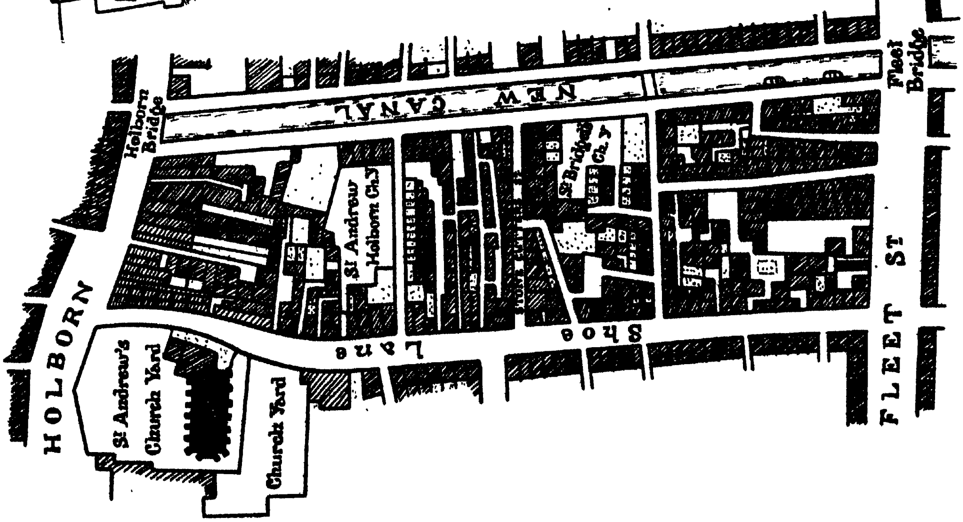


ORDINANCE MAP 1916

Fig. 6.

ROCQUE. 1746

Fig. 5.



OGILBY & MORGAN. 1677

Fig. 4.

to the Vicar and churchwardens for the time being and their successors certain parcels of garden ground to be converted, used, dedicated and employed to and for an enlargement of the burying places of the said parish and *to no other use whatsoever*, unless the said Earl of Dorset, his heirs and assigns or any other well-disposed person or persons should be willing to erect or build any houses or buildings on some part thereof for the relief of the poor of the said parish*.

It might naturally seem from this indenture that the Earl supposed a portion of the land might be devoted to the erection of almshouses, and the remainder to a burial ground; it can hardly be argued that he proposed that what might be used as a burial ground would *afterwards* be used as a building ground even if the rents were employed for charitable purposes.

Owing to the courtesy of the Vicar of St Bride's it was possible to make an examination of the Minute Books of the Vestry. The first Minute Book runs from 1654 to 1664. It is clear from this book that the "churchyard by the Ditch" or the "new churchyard" was already being used for purposes not contemplated by the donor. Thus in 1654 the Vestry had to order that "no souldiers† shalbe admitted to exercise in the new churchyard, nor any doores to be opened into ye churchyard without consent of the Vestrie" (April 4th). These doors into the churchyard were the source of perpetual trouble to the Vestry. They ordered the doors to be nailed up or they granted the door-intruders the privilege of their doors on payment of a yearly rent of 5*s*. if they agree to "doing nothing to prejudice the churchyard" (July 7). Again, June 26, 1657: "Ordered that all the doors, openinge into ye new churchyard, be stopt upp and that none be admitted to any privilege into ye said churchyard, but by the churchwarden's appointment, and that hee take care that nothinge be used there that shall annoy the churchyard or make it unhandsome." Further, August 2, 1660: "Ordered that the doors of ye new ch. yard by the ditchside be nailed or made upp speedilie it being a great annoyance to the ch. yard and not to have any liberty therein for private use for rent and the ch. wardens and constables are desired to see this order performed."

It is clear from these and other passages that houses were growing up on the open space round the churchyard, or even on the churchyard itself, and that the inhabitants of these houses were encroaching on the proper uses of the churchyard to carry on their trades or deposit their rubbish‡. It is not very easy

* *Endowed Charities (County of London). Parish of St Bride, Fleet Street*, 243, I. 1901. Eyre and Spottiswoode, p. 15.

† In 1896 Mrs Basil Holmes writes in *The London Burial Grounds* (p. 319), Additional ground for St Bride's, Fleet Street:—"This is off Farringdon Street, is about 750 square yds. in extent and used as a volunteer drill ground. There are no tombstones and the ground is untidy." The absence of tombstones (as evidenced also in the excavations) was probably due to its being a burial ground for the poor.

‡ April 27, 1663, we meet again with complaints of "the dirt and rubbish constantly thrown into the said church yard, making it a great annoyance." Even a wall of the burial ground had fallen down and many of the bricks were stolen: the Vestry ordered the churchwardens "to viewe the said ch. yard wall and other annoyances and consider what should be done to remedy it."

to settle when the Vestry itself began to build *on* the churchyard. But it is evident that by 1659 the Parish had built houses *by* the new churchyard, for on June 22, 1659, the churchwardens are directed to inspect the houses in Harpe Alley, the *tenants* "complaining of the chimnies that are lately fallen downe and to enquire whose right (!) it is to repaire." It is from Harpe Alley that several of the doors were made into the churchyard. But it is clear that by April 3, 1656, the actual building *on* the churchyard has become a discussable proposition: "It was thought fitt that the churchwardens doe returne Mr Palmer, late minister of this place, thanks for his charitable thoughts towards the poore of this parish, and that it was the will of the Vestry, that if he would build houses for poore people of this parishe to live in that he should have such proportion of ground out of the churchyard by the ditchside to build upon as he should think fitt." This matter might seem to have been dropped for three years; for on July 13, 1659, we read: "It is thought fitt that the goodwill of Mr Palmer be hearkened to and be accepted of and some buildings be made as hee shall desire to raise a * towards ye relief of the Poore in consideration of it, alsoe some buildings to be made to joyne to it for the moneys given by Mrs Looman." Again on April 25, 1660, there is more consideration of Mrs Looman's gift, and "some parishioner of the parishe" is willing to advance "the same at present for ye building of some houses on ye ch. yard by the Ditchside for to raise £6 yearly benefitt to be paid for the use of ye poore for ever." Presumably Mrs Looman's gift was out at interest and could not be immediately recalled.

Now there are, I think, two schemes here, (*a*) that of Mr Palmer of 1656 to build almshouses for the poor of the parish, and (*b*) the scheme of 1659 to build houses at the cost of Mr Palmer and Mrs Looman to be rented and provide an annual sum for the poor of the parish. My reason for this is that we read on August 28, 1660: "Att this Vestrie it was taken into consideration that the six houses by the ditchside adjoyning to the churchyard were filled with poore people who neither pay rent nor keepe them in Repaire...ordered that they be all turned out and the overseers of the poore do take order for the putting them in Repaire and afterwards lettinge them."

These six houses are a mystery. Where did they stand? It would certainly appear from this that already in 1660 houses had been built "by the Ditchside," i.e. on the border of the churchyard facing the ditch. It is noteworthy that six messuages on the west side of Farringdon Street still belong to St Bride's Institute or Foundation, i.e. Nos. 75—80 inclusive. If these 1660 houses were first put up by Mr Palmer in 1656, it seems unlikely that they would be in bad repair by 1660. It appears more likely that they had been built on a portion of the churchyard before the date 1654 at which the Vestry Minute Book starts. Such would be in accordance with the suggestion of the Earl of Dorset's grant. Yet in Newcourt's map of 1658 (see Fig. 2) although the churchyard† is built up with houses on

* Word illegible, but the sense is clear that the rental of the buildings was to provide an annual sum for poor relief.

† The space marked with a cross.

the north (one with a door into it), and on the west, there is still only a wall towards the Fleete. On the south towards Harpe Alley, however, there do not appear those houses with doors entering the churchyard, which are so constantly referred to in the minutes. There are houses facing the Ditch south of the churchyard, and possibly these are the houses occupied by the non-rent-paying poor which had fallen out of repair. In the Minute Book for October and November, 1660, the building on the churchyard is several times mentioned. Thus:

October 10th. Ordered to pay the bricklayers' bill "if possible."

October 23rd and 25th. "The former order concerning the bricks laid in the new churchyard for the building there was considered, which building the vestry supposed could not be performed in regard to the many corps that are found to be buryed there, and therefore they considered of another place for the building, of [a] place nominated in the south side of the old churchyard, and certain persons were ordered to review the ground and report their opinion touching it to the vestry."

"Mr Sampson made report concerning the building that it was the opinion of some that it could not be performed in that place neither [New churchyard?] without an order from the Bishop and Deane, and some were against building in the said churchyard in regard it was appointed for another use...many more desired the Vestry to consider again of the old churchyard and thereupon this vestry ordered to meet the next day at 3 precisely to view the ground in the chyard in order to build if it was convenient."

October 25th. "Again it was put to the vote when the building should be performed in the old c. yard, according as was mentioned in the former vestry, and by most hands it was agreed to. Whereupon Mr Read the carpenter was called in to be discoursed with about the doing thereof. But in regard to the Season of the year was not thought by some of the Vestry to be so convenient for it as the spring, it was by most hands agreed that it should be deferred till then, and it was debated how the Bricklayers and Carpenters bills should be paid, but nothing concluded concerning them."

November 2nd. "The charge incurred about the intended building to be defrayed and the workers paid...but where [? whether] the building should be forthwith or at spring was now again indespute." [Matter referred to a Committee.]

January 28th, 1660—61. Promised again to pay the carpenter and bricklayer.

April 24th, 1661. "Vestrie doth agree that the churchwardens shall lett a lease to John Fuller, Cutler of the house where Widdow Hillman dwelt by the New Churchyarde, Ditchside."

During the following years we find no reference to the building, but a house is purchased in Whitefriars with Mrs Looman's gift (1662—3). So that gift was not spent in building by the Ditchside.

June 15th, 1663. "Ordered that the old houses of the parish by the new churchyard much out of repair be speedily putt in repair to preserve them from being utterly ruinated."

July 8th, 1663. "That the two churchwardens doe take care for the speedy puttinge in repaire of the Parish Houses by the Lower churchyard* formerly ordered, but nothing done in it."

It cannot be argued from these minutes that actual building on the occupied churchyard took place between 1654 and 1664, although it was clearly proposed and bricks taken to the churchyard. At the same time it is clear that Parish Houses for the poor had been built on the land granted by the Earl of Dorset, and that these were already old and dilapidated.

But a new state of affairs was coming. In 1665 the plague broke out and on July 7, 1665, we read: "Part of ye churchyard by the Ditchside to be inclosed for the bearers to be in."

At the first onset of the plague St Bride's had relieved the necessity of the neighbouring parish of St Dunstan's in Fleet Street by allowing the use of the new churchyard for its dead and also lending its bearers. But as by August both churchyards were becoming *choked*, the Vestry appointed two persons, who should give orders for digging plague pits and pay the labourers. Before the plague ceased St Bride's had suffered as severely as any other parish†. We thus have definite evidence that before 1666 the new churchyard was choked. There is no need on account of the mass of skeletons recently exhumed to appeal to centuries later than the seventeenth!

Let us turn to what happened after the Great Fire in 1666. This is well illustrated in Leake's map of 1666, which "sheweth the enlargement of ye street ways, etc." (see Fig. 3). According to this scheme not only was the Fleet Ditch straightened, but its actual channel and the passages on either side of it were to be carried right across "St Bridget's Ch. Y." reducing the size of it by about a half. This would fully account for the osseous debris to be found under the pavement of Farringdon Street. That this change was carried out within ten years is evidenced by Ogilby and Morgan's Map of 1677 (see Fig. 4). Fleete Ditch now appears as the "New Canal." The distance along Shoe Lane between Curriers' Alley (Stonecutter Street) and Harper Alley is the same on the two maps, but St Bridget's Churchyard is reduced to about half its original extent; its decrease in dimensions east to west is precisely that indicated in Leake's plan‡ of reconstruction for 1666. Again while Newcourt and Leake show a straight northern boundary to the churchyard, Ogilby and Morgan indicate that after the reconstruction the yards of two houses in Stonecutter Lane encroached on the church-

* "Lower churchyard" here certainly means the New churchyard by the Ditch, but as farther from the Thames, one might easily suppose it the higher.

† W. G. Bell, *The Great Plague in London in* 1665, pp. 275—6.

‡ Leake's plan may be taken as a very careful survey; William Leybourn, a mathematician and trigonometrician, was among those who took part in its production; it was prepared for the Commissioners for the reconstruction.

yard, and that a house had been built in Farringdon Street facing on to the Ditch. This house must have been upon a portion of the site, whence our remains were excavated (see Fig. 4). With regard to the alteration of the Fleet Ditch, it would appear that £27,000 was voted for the canal to Holborn (i.e. "Cutting a navigable river to Holborn-bridge"), and we read: "The attempt to make Fleet-brook or ditch navigable to Holborn-bridge was a mighty chargeable and beautiful work: and though it did not fully answer the designed purpose, it was remarkable for the curious stone bridges over it, and the many huge vaults on each side thereof, to treasure up Newcastle coals for the use of the poor*." If any of these "huge vaults on each side thereof" was made on the Fleet Market frontage of St Bridget's churchyard, we have a further explanation of the origins of our osseous debris.

We will now return to our maps. Overton, 1676, does not mark the churchyard, but he claims to show the "Citty" as now rebuilt. There are houses all down the west side of Fleet Market. This is repeated in Overton's map of 1706 and seems to be indicated in Jeffreys, of 1735. But in A. Lea and R. Glynne (? 1680) the churchyard adjoins the market. None of these maps is of much value. On the other hand Rocque of 1746 shows houses all along, except for a narrow neck or passage to the churchyard (see Fig. 5) between Fleet Market and St Bride's churchyard. Thus the maps seem to indicate that probably a good deal before 1746 the reduction of the churchyard which commenced with the construction of the New Canal had been pushed still farther. The actual area between Fleet Market, Shoe Lane, Stonecutter Street and Harp Alley is very nearly the same in Ogilby and Morgan's map of 1677 and Rocque's map of 1746, but the boundaries of the burial ground have been thrown inwards, not only on the north and east but also on the west (see our Figs. 4 and 5). The modern ordnance map of 1916 reduced to about the size of Rocque's shows a quadrangle, of which the maximum east to west dimension is the same as Rocque's, and likewise the north to south maximum dimension, but the north-east and south-west cuttings off of Rocque's and earlier maps do not appear (cf. Figs. 5 and 6). In other words the yard appears larger in 1916 than in 1746! That the yard approaches more closely Stonecutter Street is probably due to a considerable broadening of the latter street. Such is the history of the Earl of Dorset's gift.

Apart from the excavations for building Gordon House, hundreds of skeletons must still lie under the surface of the 1916 quadrangle. This we understand it is desired to convert into a garage—presumably without excavating the quadrangle—could sanction be obtained. It is at present a waste yard with rubbish heaps.

Now let us return to the documents at St Bride's. Unfortunately the Minute Books of the Vestry from 1665 to 1680 appear at the present time to be missing. This is very regrettable as we should have doubtless been able to trace the events which led to the first halving of the burial ground, and the disappearance under

* *The City Remembrancer*, Vol. II. pp. 31—32. London, 1769.

the pavements of Farringdon Street of the eastern end interments. The following are extracts from the minutes:

June 18*th*, 1692. "It is left to Messrs .... to inspect the Lower Churchyard [see note, p. 9] about making of shoppes, warehouses to ye front and to report it to ye next Vestry."

March 13*th*, 1692—3. "Ordered that ye 2 Churchwardens and ...... be a Committee to take a view of a jetty that is to be made on the Inside of ye Lower Churchyard Wall by Thomas Bye and to report." [The "Inside" is significant.]

October 27*th*, 1696. "Ordered to inspect into ye books about ye ground and soyl enclosed on ye West side of Fleet Ditch*."

June 28*th*, 1699. The next entry is very important for it seems to indicate that some building on the churchyard had after all been done for Mr Palmer's benefaction. See our p. 7.

"A Lease granted to Benjamin Tesdale and Thomas Kittrich, Cityzins and Skinners of London of all ye Toft, piece or parcel of grounds...in front from N. to S., 32 foot, from E. to W., 24 foot, the East part of which doth front or ly next unto and against the Street next Fleet River, the North part thereof doth adjoin or ly next to and against a Messuage or Tenement now in the possession of W. Noades, Broker, and the E. [? W.] and S. part doth adjoyn ye Churchyard belonging to the said Parish of St Bridgett als Bride called the lower Church yard† which said Toft......is scituate lying and being in the said Parish of St Bridgett als Bride, London, and on which now standeth an Almshous belonging to the said Parish of St Bridgett als Bride London with liberty for them the said B. T. and T. K. in case they shall think fitting and convenient in ye building of the said Toft......to project, erect, make lay or carry the said Building......in ye ground floor thereof 4 foot over and above ye said twenty four foot into ye said churchyard on the West part of the said Toft......, and also to project etc. the said building by them intended as aforesaid in the 2nd floor thereof seven foot over and above the said 24 ft. into the said churchyard. And the said lease is to commence from Michas next and to hold and continue thenceforth for the term of 99 years, at ye yearly rent of £5 payable quarterly. And in consideration thereof the said B. T. and T. K. are to build one or more messuages or Tenements upon the said premises. It is hereby further ordered that the Alms Hous now standing upon the said premises be granted, bargained and sold to them, the said B. T. and T. K., for the consideration of the said fine by them agreed to be paid and of the charge they are to be at in building as aforesaid, and the said B. T. and T. K. are to hold the same for ever without rendering any account for the same, and to pull down or use the same if they shall think fitting when they shall begin to build upon the same premises."

* This seems to suggest that on the construction of the New Canal a portion of the former churchyard on the Ditch side had been separately enclosed, query with a view to building.

† This passage adequately identifies our burial ground with the "lower churchyard."

Now in interpreting this deed we have to remember that it dates from 1699, thirty-three years after the Great Fire and after the construction of the New Canal, and the cutting off under Fleet Market of half the burial ground. But the Alms House referred to clearly faces Fleet Market and is adjacent to a house in Stonecutter Lane. It seems therefore that this Alms House and the building which was to supplement or replace it were in the north-east corner of the burial ground, just where a house appears in Ogilby and Morgan's map of 1677 (see our Fig. 4). The Alms House must therefore have been built after the Great Fire, and as the recent excavations show right on top of a mass of osseous debris, probably resulting from the reconstructions involved in the New Canal and the Fleet Market on the west side of it. Now the lease just cited should have run out in 1799, but whether that lease was really granted to Benjamin Tesdale and Thomas Kittrich Cityzins and Skinners of London, we cannot say. It does, however, identify the site of the Alms House. Among the documents, however, in the chest in St Bride's Vestry is a lease of land for building purposes, *then occupied by Almshouses*, at the corner of Stonecutter Lane and Fleet Market. The date of that lease is 1788. The Charity Commissioners' Report on St Bride's Foundation shows No. 75 Farringdon Street granted on a lease for 99 years from Lady Day, 1787, i.e. 1788 of the new reckoning. We have therefore four disturbances of the north-east corner of the Dorset burial ground, i.e. (i) the cutting away of half of it for the reconstruction after the Great Fire, (ii) the building of Alms Houses between 1666 and 1677, (iii) the probable building by the firm of Skinners, and (iv) the building again in 1788. Except in so far as these successive buildings may have encroached farther into the burial ground, the mass of broken-up skeletons forming the debris found in the excavation of 1924 must be due to interments taking place before the building of the Almshouses between 1666 and 1677. Thus much for the corner in which the great mass of material was found.

We can, however, throw light on the date at which the remainder of the Fleet Market frontage was built upon. A faculty of the date February 20, 1718, was discovered by the Charity Commissioners and published by them in their Report. In this, after reciting the gift of the 3rd Earl of Dorset (see our p. 3), Lionel Duke of Dorset for himself and his heirs "as much as in him lay" gives leave to the Parishioners of St Bride to erect such houses on the burial ground and grant such leases as they desire. What the parishioners desired to do we are told in the deed itself, namely "to build on part of the premises, containing in length from N. to S. towards Fleet Ditch 69 feet, and from E. to W. 35 feet or thereabouts, the rents and profits of the buildings to be applied towards the support and relief of the poor inhabitants of the said parish." It will be noted that the 24 feet with 7 feet projection of the second story over the graveyard has now increased to 35 feet. Measurements on the modern ordnance map show, omitting the Almshouse site, for the *four* houses between the graveyard and Farringdon Street N. to S. 80 feet* and E. to W. about 43 feet. If we omit the public-

* We may allow for the neck to the burial ground, which would still give the purchaser over measure! Was the measurer the purchasing churchwarden?

house the reach would be 60 feet and not 69, but the purchaser would hardly accept an underestimate. There were, however, other steps necessary to the safe conclusion of the project. The Dean and Chapter of St Peter's, Westminster, were proprietors and owners of the rectory and patrons of the vicarage of the said church of St Bride. They consented to the proposal, but their consent seems to have been linked with the condition that in lieu of the space taken from the already diminished burial ground by the Ditch the Parishioners should set apart a piece of ground belonging to the Parish, adjoining the ancient church, this piece to be converted into, dedicated and employed as a burial place for the use of the parish. We have no evidence whether this condition was carried out; it does not, however, concern our present enquiry. The parishioners next presented a petition to the Bishop for a licence or faculty for these alterations, and he granted the vicar, churchwardens, parishioners and inhabitants of the parish his licence to erect and build upon the said piece of ground such houses as might be convenient or necessary for the support and maintenance of the poor inhabitants of the parish. The recital does not state whether or not the faculty was conditional on the re-interment of the bodies buried in this portion of the graveyard. If it was, the condition appears to have been very imperfectly, if at all, carried out*. The Charity Commissioners' Report shows that the 99 years' building leases of houses on this property must have run out from 1822 to 1829, for new short term leases were granted during that period. This indicates that the building leases were granted before 1723 or skeletons taken up from below these houses must have been before that date, probably much before, as we know the ground was "choked" at the time of the Great Plague. Having got their faculty, how did the parishioners proceed? This is duly described in the Vestry Minutes, to which we once more turn.

August 12, 1720. Ordered that Lord Dorsett's grant of part of the Lower Church Yard to build on be produced and read at the next Vestry.

December 19*th*, 1722. Ordered that that part of the churchyard which is intended to be built upon be cleared from corps at the parish charge, and it is further ordered that the said ground be built upon.

Ordered that the churchwardens give direcions for having the s[d] ground cleared with expedition. Ordered that the unanimous thanks of this Vestry be given to the Revd. W. Evans the Vicar for his kind offer (wh. this parish will thankfully accept off) of buying a piece of ground to inlarge their lower churchyard †.

* Most of the above facts are taken from the recital of the Faculty, provided in the Charity Commissioners' Report. According to the Report the deeds referred to in the Faculty were not to be found, and that instrument itself was not known till the enquiry. The income from the estate had been paid into the churchwardens' account and had been applied to the usual purposes of a church rate. The parish officers undertook to appropriate the rents to the purposes declared in the Faculty, i.e. solely to the maintenance of the poor inhabitants of the parish.

† If this purchase was made it may account for the difference in shape and size of the yard between 1746 and 1916. See our p. 10.

[The excavations under the former sites of Nos. 75—79 Farringdon Street show how very imperfectly the order that the land "To be built upon be cleared from corps" was carried out !]

February 5th, 1722—23. Ordered that a building Lease or leases be lett of a piece of ground on the west side of Fleet Ditch which belongs to this parish, whereon three houses shall be built the terms to be for 61 yeares, and the ground rent to be reserved on each house 3 pounds per annum.

Ordered that an advertisement thereof be insirted in ten publick Newspapers that persons may have due notice to come and bidd. Ordered that the present Churchwardens...be joynd with the Trustees named in a deed from the L^{d} Dorsett as a Com^{ee} to receive proposalls and treat ab^{t} letting these grounds, the s^{d} Comt^{ee} to report to a Gen^{le} Vestry.

February 22nd, 1722. Ordered that a Lease of a piece of ground by ffleet Ditch part of the Lower Church yard containing in front 69 foot and in depth 35 foot or thereabouts be lett to the best bidder for erecting thereon 3 substantiall brickhouses at the annuall rent of nine pounds.

Ordered that a gateway into the s^{d} Churchyard six foot in front and the heighth of the first story be reserved to the Parish.

Ordered that each bidder for the s^{d} ground advance 5 pounds every bidding.

Ordered that the s^{d} ground be disposed of this day* to the best bidder for a 61 yeares' lease. Then the question being put whether the s^{d} ground be lett entire or in parcelles, Twas ordered that the said ground be let intire.

Ordered That a lease be made of the s^{d} ground from Lady-day next att a "peppercorne" rent for the first yeare & at the rent of nine pounds p. ann. for the remainder of the terme. Then† Mr Churchwarden W^{m} Parsons putt up the s^{d} ground and the wall thereon standing at £300 & severall persons bidding, at last the said W^{m} Parsons bidd £455 for the same, no person advancing higher. It was ordered *Nem. Con.* that M^{r} Parsons have a lease of the s^{d} ground for £455 fine for 61 yrs att nine pounds p. ann. for the last 60 yeare & at pepper corne rent for the first yeare.

Such is the history of the buildings on the Fleet Market side of the Ditch burial ground. We do not know when the Almshouse was built, but the site and building were presumably sold to the firm of Skinners in 1699. The remainder of the site recently excavated was built on in 1722. It is clear therefore that the mass of corpses forming the osseous debris were buried before 1699; the smaller but far from insignificant number taken from the more southern portion of the "Farringdon Street rectangle" (see our p. 2) may to some extent have been buried between 1699 and 1722. But the very perfunctory clearance of corpses, if

* The previous order of *public* announcement could hardly have been carried out, for the terms of the lease and bidding are now settled, and the bidding is to take place on *this day*.

† It would appear that as soon as the terms of the lease had been settled, Senior Churchwarden William Parsons (in the Vestry ?) put up the ground for sale, and after a few bids knocked it down to himself!

such took place at all, would most probably have removed those more recently interred and nearer the surface. It is safe therefore to say that the limits of our series are 1610 to 1722, but that the great bulk of the material dates 1610 to 1666, and probably belongs chiefly to interments made when the graveyard became choked at the time of the Great Plague. Material in the eastern side of the burial ground was probably thrown, when the New Canal was carried across the burial ground, helter-skelter into the remainder of the burial ground, possibly into dug-out pits against the north-wall of the Farringdon Street rectangle, and upon this osseous debris the Almshouse must have been built at some date after the Great Fire, and again the whole mass would be disturbed, when Tesdale and Kittrich built their premises on the site of the Almshouse. Their lease should have run out about 1798, but it was probably surrendered in 1788, when a new 99 years' lease was granted for building purposes. It seems therefore impossible that the three ornamented coffin handles found, and said to date from "later than 1760," could be due to interments made under houses built over the burial ground in 1699 and 1722. If they really date later than 1760, then they must be part of refuse from the burial ground thrown into the backyards of the Farringdon Street houses, or into their excavations on rebuilding. The burial ground could hardly encroach again on the sites covered by William Parsons' houses.

The whole story has been developed here, not only because it fixes within fairly narrow limits the period of the crania measured, but because in itself it is extremely instructive. The savage has respect, or at any rate fear, before the bones of his dead. The Egyptian buried his dead on the desert border of his productive delta, where there would be no chance of the growth of population disturbing their bones. The Londoner of late mediaeval times thought nothing of a mass of human corpses as a foundation for his house, and laid down his concrete and mortar so as to form a conglomerate with the bones of his ancestors. To-day where a garden might have been made we desire to lay down the floor of a garage. And thus all surface traces of the Earl of Dorset's burial ground would disappear. But there under the pavement of Farringdon Street its occupants will lie until changes in the roadway or the rebuilding of houses are again undertaken; then the bones may perchance provide material for a scientist who will treat them with greater reverence than the Restoration Vestry or the builder, whose object was to put them as quickly as possible out of sight.

We have to thank very heartily Dr W. G. Howarth, Medical Officer of Health for the City, for his great aid in enabling us to measure this valuable cranial material.

(2) *Measurements and Methods of Measurement.* The direct measurements were similar to those taken by previous workers in the Biometric Laboratory, and detailed accounts of them are given in *Biometrika,* Vol. I. pp. 412—419, and Vol. XIV. pp. 196—200. The definitions of cranial "points" given in the latter paper were followed in the present investigation.

F = Flower's Ophryo-occipital length. L' = Glabellar projective length. L = Glabellar-occipital length. B = Maximum parietal breadth. B' = Least forehead

breadth. H' = Basio-bregmatic height. H = Basion to point vertically above it with skull adjusted on craniophor to Frankfurt horizontal. OH = Craniophor auricular height. LB = Basion to nasion. Q = Craniophor transverse arc through "apex" and terminating at ear rods. Q' = Transverse arc terminating at auricular points. S = Arc from nasion to opisthion. S_1 = Arc nasion to bregma. S_2 = Arc bregma to lambda. S_3 = Arc lambda to opisthion. S_1' = Chord nasion to bregma. S_2' = Chord bregma to lambda. S_3' = Chord lambda to opisthion. U = Horizontal circumference. PH = Alveolar point to tip of anterior nasal spine. $G'H$ = Nasion to alveolar point. GB = Distance between points where zygomatic-maxillary sutures cross lower front ridges of zygomatic arches. J = Zygomatic breadth. NH' = Nasion to base of anterior nasal spine. NH, R and L = Nasion to lowest edge, right and left, of pyriform aperture. NB = Greatest breadth of pyriform aperture. DS = Shortest subtense from bridge of nose to dacryal chord with Mérejkowsky's simometer. DC = Chord dacryon to dacryon. DA = Arc dacryon to dacryon. SS = Shortest subtense from nasal bridge to simotic chord with simometer. SC = Minimum chord between naso-maxillary sutures. O_1 = Breadth of orbit, R and L, using curvature method. O_1' = Breadth of right orbit from dacryon. O_2 = Height of orbit, R and L. EOW = distance between points where the borders of the orbital ridges, right and left, meet the fronto-malar sutures. It is the same as Martin's *innere orbitale Gesichtsbreite*. G_1 = distance from point of spina nasalis posterior to imaginary line tangential to inner rims of alveoli of middle incisors. G_1' = Similar to G_1, but from base of spine. G_2 = From inner alveolar walls at second molars. EH = Palate height; EB = Palate breadth both taken with Pearson's uraniscometer. GL = Basion to alveolar point. fml = Basion to opisthion. fmb = Greatest breadth of foramen magnum. $P\angle$ = profile angle, found by means of Ranke's goniometer when the skull is in the Frankfurt horizontal position on the craniophor. The angles ($N\angle$, $A\angle$, $B\angle$) of the fundamental triangle were calculated with the aid of Pearson's trigonometer. θ_1, the basio-nasal horizontal angle, is obtained by subtracting $N\angle$ from the supplement of $P\angle$; θ_2, the basio-alveolar horizontal angle, by subtracting $A\angle$ from $P\angle$. The Occipital Index

$$100\,\frac{S_3}{S_3'}\sqrt{\frac{S_3}{24\,(S_3-S_3')}},$$

was obtained from Tildesley's table of this function*.

The capacities were determined by tight packing with mustard seed and weighing in the manner described by Macdonell†.

The weight of seed held by the "crâne étalon" γ was determined at the beginning of, and at intervals during, the weeks devoted to this measurement, and the following results were obtained:

(1) 1376·0 gms. (2) 1378·5 gms. (3) 1379·5 gms. (4) 1380·0 gms. (5) 1379·0 gms.

* *Biometrika*, Vol. XIII. p. 261.

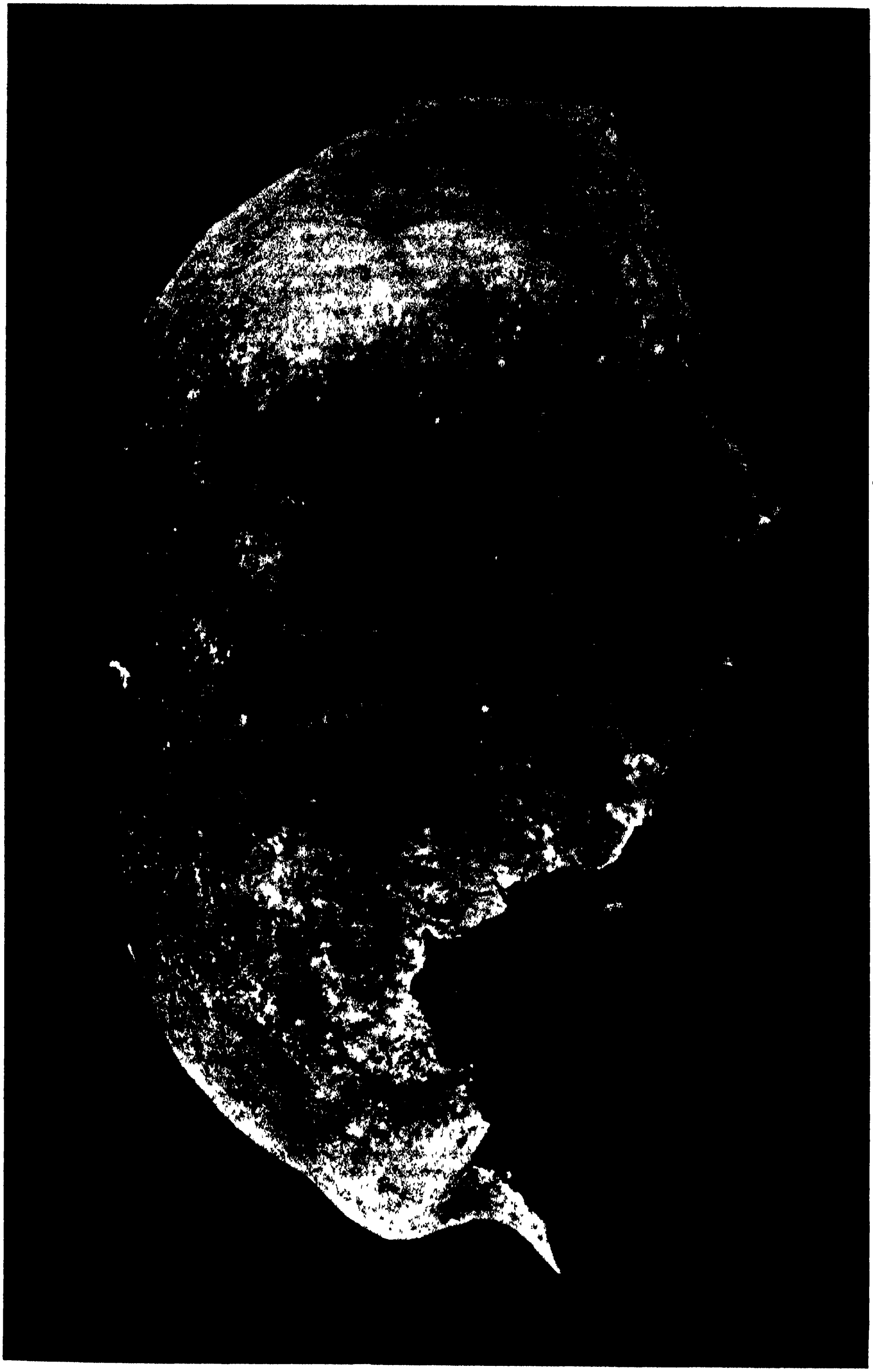
† *Biometrika*, Vol. III. pp. 203—206.

B. Hooke, *Study of English Skull*

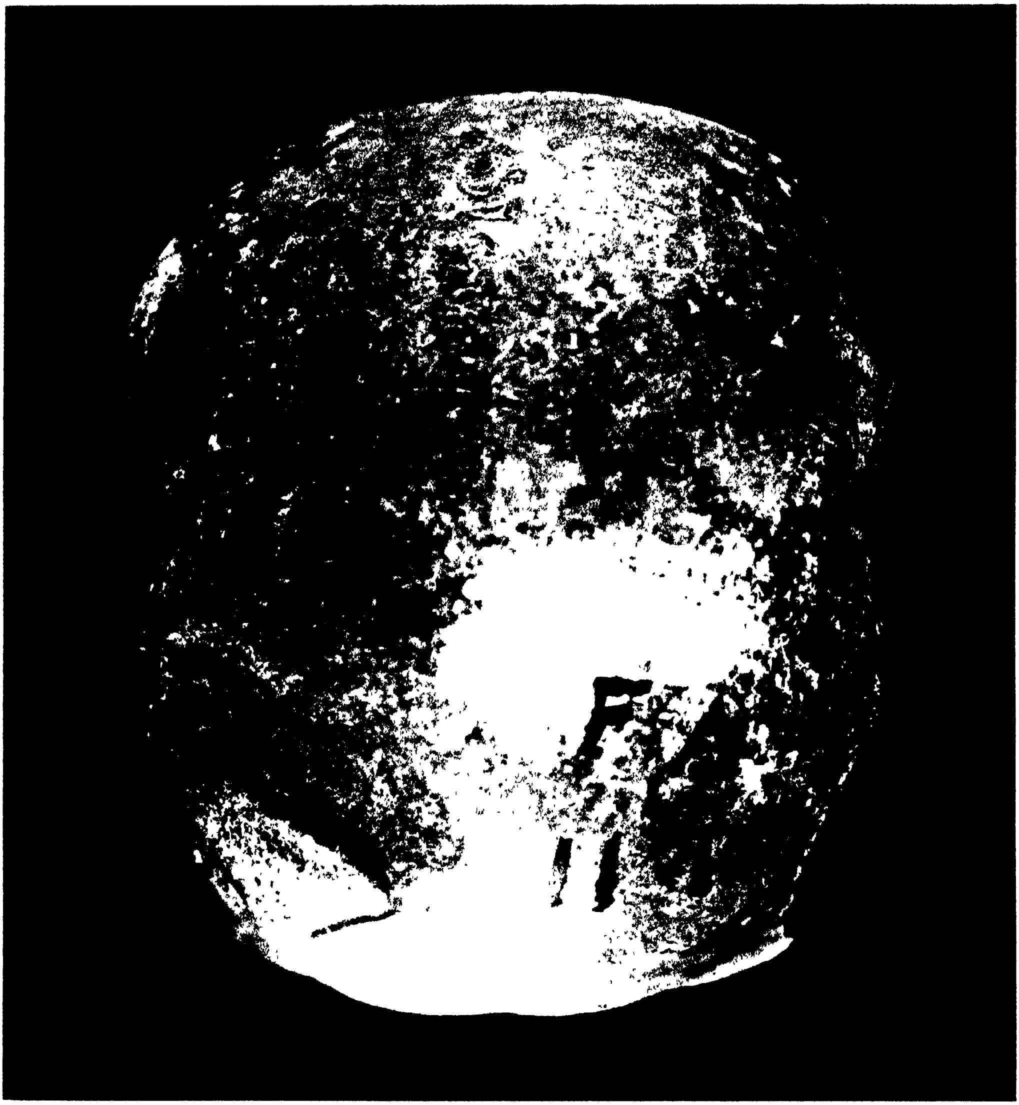
Special Skull FA. 44.

Wormian Bone (? Ossicle of Asterion).

Special Skull FA. 89.



Special Skull FA. 136.
Lower occipital Flattening.



Special Skull FA. 114.

Partially healed Injury on R. frontal Bone.

Spec al Skull FA. 264.

Trans erse occipital Groove

Special Skull FA. 88.

Po. crotaphitico-buccinatorius divided into two externally.

Special Skull FA. 1.

(a) Anomalous process on Pterygoid Plate.

γ was also filled several times with water, the volume of which was determined by pouring it into a large measuring glass, and was found to be 1755 cm.^{3} Taking 1380 gms. as the weight of seed, the following result is obtained:

$$\text{Vol. of 1000 gms. seed} = \frac{1000}{786 \cdot 3} \text{ cm.}^3 = 1272 \text{ cm.}^3,$$

and this was used to calculate the capacities of the skulls.

Previous determinations on the standard skull γ by Macdonell and Morant gave 765·7 gms. and 772·2 gms. respectively as the weight of 1000 cm.^{3} of seed. The increase in the present instance is partly due to shrinkage of the seed and partly owing to repairs effected on the standard skull which may have altered its capacity to some extent.

The weight of the skulls was not taken as they were too much damaged to give results of any value.

Sex. The skulls were examined first by Mr G. M. Morant with a view to sexing them. Those of doubtful sex were afterwards re-examined by Professor Pearson and Sir George Thane.

Mandibles. Several hundred mandibles were unearthed, but in no case were they attached to the skulls. Many were incomplete, the teeth in the mandibles were not in as good condition as in the upper jaws, and in many instances there was absorption of the alveolus due to loss of teeth and old age. Since the number on which a complete set of measurements could be taken was small, no attempt to determine the sex has been made at present, and in this paper records are given only of those mandibles which were in a good state of preservation. The measurements taken were in accordance with those given in *Biometrika*, Vol. XIV. p. 253.

(3) "*Remarks*" *on Individual Crania.* The series was examined for anomalies by Sir George Thane and I am much indebted to him for his help in this respect as well as in the question of sexing. A description of these anomalies, together with other remarks concerning the condition of the skulls, will be found accompanying the tables of individual measurements in the Appendices.

The following abbreviations have been used: cal. = calvarium = skull without mandible, f. = face, dome = roof of the skull only, frag. = fragment, br. = broken.

The skulls were examined for the following characters and anomalies, and if no mention be made of any particular feature, it may be assumed that it did not exist in the skull in question.

Age. If no mention of age is made, the skull may be regarded as adult. Obliteration of sutures, considerable loss of teeth and absorption of the alveolus due to old age, have been noted. A record was kept of teeth in process of coming through and basilar synchondrosis was looked for.

Teeth. Many of the teeth were missing, but those left were in good condition and many were well worn. Carious teeth were noted. Also accessory cusps on

the 2nd and 3rd molars. In six cases, one or both of the canines were delayed, and in four of these, the tooth was coming down behind the lateral incisor.

Palate. Each was examined for the presence of a bony ridge over the palatine grooves leading from the pterygo-palatine canal*. It is described as an outer or inner palate bridge according as it is across the outer or inner groove. On account of the damaged condition of the skulls it is very probable that our observations give an entirely inadequate value for the frequency of its occurrence. The occurrence of a torus palatinus was also noted.

Base of Pyriform Aperture. This is represented by the letters P.B. It refers first to the base of the floor of the nasal aperture immediately behind the edge, and is described as "flat," sloping "upwards and outwards" or "downwards and outwards."

Secondly, the edge itself is described as sharp, blunt or rounded. Occasionally the floor and the edge form together an unbroken curve and they are then described as "P.B. rounded." Double edges were also noted. The frequencies found are given in the tables below.

Edge of Pyriform Aperture.

Slope of Floor.

Males	Sharp	Blunt	Rounded	Totals
Upwards ...	34	12	9	55
Flat	13	9	5	27
Downwards...	1	3	0	4
Totals	48	24	14	86

There were 3 cases of *PB* rounded and 5 had double edges.

Edge of Pyriform Aperture

Slope of Floor.

Females	Sharp	Blunt	Rounded	Totals
Upwards ...	40	11	6	57
Flat	6	9	5	20
Downwards...	2	1	2	5
Totals	48	21	13	82

There were 2 cases of "P.B. rounded" and 3 had double edges.

In both sexes there was a marked preponderance in favour of P.B. upwards with a sharp edge.

Asymmetry. The Sylvian depressions on each skull were examined and their relative sizes were observed and denoted SR, SL according as the groove was

* See Le Double, *Variations des os de la Face*, p. 266.

greater on the *R* or *L* side. Similar observations were made on the jugular foramen.

	Sylvian Depression		Jugular Foramen	
	♂	♀	♂	♀
Greater on *R* side	42	64	54	66
Equal	36	42	14	21
Greater on *L* side	23	13	22	39

It has been noticed in several other series that the balance is in favour of the right side, and this is borne out by the above observations, but the proportion is distinctly smaller in the present instance.

Asymmetry of other parts of the skull was also noted.

Metopism. The metopic suture was persistent in 14 males, 23 females and 1 child. In 1 of the males, and 5 of the females, part of the suture was completely closed. Contact between the frontal and parietal bones is denoted by LF + RP or RF + LP and the length of the suture between them is given. In 1 male and 5 females the suture was in line with the sagittal suture.

Contact	♂	♀
LF+RP	11	12
RF+LP	1	3

The frequent occurrence of contact between the right parietal and left frontal was very noticeable.

In the Whitechapel crania there were 24 skulls in which the metopic suture was persistent, while in the Moorfields there were 8. The total number of instances in the Farringdon Street skulls is 27, the proportion of metopic skulls is thus much the same in all three series.

Macdonell observed that a persistent frontal suture was accompanied by a slight increase in forehead width although the maximum head breadth was affected to little or no extent*. This conclusion is confirmed by a comparison of these two mean measurements on the metopic skulls of the Farringdon Street series with those for the whole series.

Maximum Head Breadth.

	Farringdon Street		Whitechapel		Moorfields	
	♂	♀	♂	♀	♂	♀
Whole Series...	142·4 (141)	135·7 (180)	141 (135)	135 (140)	143·0 (46)	137·6 (62)
Metopic Skulls	144·7 (14)	134·5 (20)	142 (9)	135 (15)	144·2 (5)	138·6 (3)

* *Biometrika*, Vol. v. p. 102.

Least Forehead Breadth.

	Farringdon Street		Whitechapel		Moorfields	
	♂	♀	♂	♀	♂	♀
Whole Series...	96·8 (152)	93·3 (199)	98 (132)	93 (147)	98·5 (47)	95·2 (64)
Metopic Skulls	100·6 (14)	95·0 (22)	100 (9)	96 (15)	103·7 (5)	101·0 (3)

Interparietals. Unlike the Whitechapel crania, interparietals do not occur frequently in the Farringdon Street series. One skull had a small simple interparietal, one showed traces of separation into *os pentagonale* and *ossa triangularia* and a third had an *os triangulare* on the right side.

Ossicles. All cases of ossicles of considerable size have been noted. In skull *FA*. 44 there was a large Wormian bone 50 × 50 mm. between the parietal bone and the mastoid on the left side. (See Plate I.)

Conformation of the Pterion. Each skull was examined for the presence of ossicles of the pterion; the thrusting forward of a process of the squama temporalis to meet the frontal bone has also been mentioned. Two instances of pterions in *K* were observed.

Post-Coronal Depression. This feature was present in 10 males and in 18 females, but in 6 of the males and 7 of the females it was noted as faint. Post-coronal constriction was noted in 1 male and in 1 female skull. Combining the two we have about 12 per cent. showing this feature as compared with 16 per cent. in the Whitechapel and 29 per cent. in the Moorfields series.

Tympanic Perforation. A record was kept of all cases of perforation of the tympanic plate.

Bathrocephaly. As in the Whitechapel and Moorfields crania, a number of bathrocephalic skulls were observed, 7 males and 7 females exhibiting this feature to a marked extent, and 6 of each sex showing it in a slight degree. This proportion is higher than in the Whitechapel series, where 15 out of 292 were bathrocephalic, but not so great as in the Moorfields crania, where 13 cases were observed out of 120.

Other Features. In addition to the above, the presence of the sutura notha, precondylar growths, and flattening of the obelion have been noted, together with any other singularity of interest.

There were no cases of a horizontal suture across the malar bone.

In 2 skulls the porus crotaphitico-buccinatorius was observed, and in one there was a further bony growth dividing the foramen into two externally, thereby giving separate egress to the two branches of the nervus crotaphitico-buccinatorius. (Plate VI *b*.)

All cases of the pterygo-spinous bridge and foramen were recorded; also transverse perforation of the sphenoidal spine. In a number of skulls signs of beginnings

of these bony ridges were observed, but a record was kept only when the process was very nearly complete.

In 7 skulls there was a projection on the anterior outer surface of the pterygoid plate, probably offering support to the internal maxillary artery. (Plate VI *a*.)

Five skulls were diseased (syphilitic) and one had suffered a severe injury during life of the frontal bone, from which a partial recovery had been made. (Plate IV.)

In skulls *FA*. 264 and 329, the occipital bone was crossed by a deep transverse groove, giving rise to a marked cerebellar overlap. (Plate V.)

Two skulls, *FA*. 136 and *FA*. 165, showed pronounced occipital flattening. (Plate III.) On the right temporal squama of skull *FA*. 89, there was a styliform process, 18 mm. long. (Plate II.)

Except for the lack of interparietals, which were a very noticeable feature of the Whitechapel crania, the Farringdon Street skulls are as remarkable for their anomalies as the other London series.

(4) *Comparative Material.* A detailed study of two series of London crania has already been made, and the chief interest in the present investigation lies in a consideration of the relationship of the Farringdon Street crania to the abovementioned series, namely the Whitechapel and Moorfields crania, measured by Dr Macdonell, detailed accounts of which will be found in *Biometrika*, Vol. III. p. 191, and Vol. V. p. 86.

These series have been regarded as representative of the typical Londoner of the 17th century, and the aim of the present enquiry is to determine how far Macdonell's conclusions are confirmed and elaborated by the study of a third series of London skulls, in all probability of the same date, although perhaps representative of a lower class of the population.

Macdonell drew attention to the similarity between the London crania and the Long Barrow skulls. He also noted the prevalence of features of a primitive or debased type and confessed "to a certain feeling of unrest, so long as the two largest series of English skulls, of which we have complete measurements, namely the Whitechapel and Moorfields series, give the English these not very flattering cranial characters*."

The Moorfields series is small, consisting of only 120 skulls, but the Whitechapel with 292 and the Farringdon Street with 389 are sufficiently large series to allow deductions to be made with a reasonable degree of reliability.

Although the majority of measurements on the three series are taken in the same way, there are a few for which the methods differ and the characters are not truly comparable. Attention is drawn to these differences in due course. It

* *Biometrika*, Vol. v. p. 99. [More recent investigations of individual crania such as those of Jeremy Bentham, Sir Thomas Browne, and Robert the Bruce (as also in the case of Charles Darwin), suggest that a retreating forehead, if compensated in the parietal region, is by no means incompatible with great intellectual force and strength of character. ED.]

should, however, be noted here that the basio-bregmatic height denoted by Macdonell as H is the same as our H', and in this paper it is tabled as such, while H refers to the height from the basion to the point vertically above it, taken while the skull is on the craniophor and orientated to the Frankfurt plane. This measurement was not taken by Macdonell. A few additional measurements have been taken on the Whitechapel crania. Attention is drawn to these in Table V.

Other English skulls to which we may turn for comparative purposes are the Hythe and Rothwell crania measured by Professor Parsons*.

The Hythe crania are, in all probability, those of Kentish men, dating back to the 14th and 15th centuries, some 200 years earlier than the London series. Measurements were made on 590 crania selected from at least double that number.

The Rothwell crania are believed to be contemporaneous with the Hythe series, and measurements were taken on 127 individuals selected from a population of five or six thousand. 100 of these were classed as male and the remaining 27 as female.

Direct measurements on the Rothwell and Hythe series were made to the nearest mm., whereas in the London series they are given to ·1 mm. in many cases and to ·5 mm. in the remainder.

The characters measured by Parsons are far fewer in number than those taken in this Laboratory, and his indices are calculated from the mean direct measurements and not from the individual indices. Hence a detailed comparison between these and the London series is not possible, but the differences are sufficiently striking to suggest some broad general conclusions.

In his paper on the Whitechapel crania†, Macdonell refers to Sir William Turner's paper on Scottish skulls published in the *Transactions of the Royal Society of Edinburgh* (Vol. XL. Part III. pp. 547—613). The skulls were obtained from various parts of Scotland and were of different dates, and consequently the series could not be considered homogeneous. Mr G. M. Morant has suggested that the skulls should be classified first into eleven groups, and he has calculated the Coefficients of Racial Likeness between them. The eleven groups are termed: (i) Mid Lothian Rural District, (ii) Mid Lothian Sea Coast Village, (iii) Renfrew, (iv) Practical Rooms, (v) Lowlands, (vi) Highlands and Hebrides, (vii) Fifeshire, (viii) East Lothian, (ix) North East Counties, (x) Edinburgh, and (xi) Shetlands.

The coefficients of racial likeness between the first six groups are so small that it was considered they were representative of one type and they were accordingly pooled and termed Lowland Scottish. Similarly the next three have equally low coefficients with one another, and were grouped as Eastern Scottish. The coefficients between the Edinburgh and Shetlands groups or any of the others were high and

* *Journal of the Royal Anthropological Institute*, Vol. XXXVIII. p. 419, and Vol. XL. p. 483.

† *Biometrika*, Vol. III. p. 192.

they must be supposed to represent types different from all the others. Also, since they are based on small numbers, I have ignored them for the present purpose.

I have to thank Mr G. M. Morant for permission to use the above data and for the coefficients of racial likeness (C.R.L.) given in Table III.

(5) *Coefficient of Racial Likeness.* The coefficient of racial likeness

$$\frac{1}{m} S \left\{ \frac{(M_s - M_s')^2}{\frac{\sigma_s^2}{n_s} + \frac{\sigma_s'^2}{n_s'}} \right\} - 1$$

provides a measure of the degree of relationship between two races, based on the consideration of a number of characters between which the correlation is known to be small or non-existent. It was suggested by Professor Pearson and first used by Tildesley*, and a further discussion of its use will be found in *Biometrika*, Vol. XIV. p. 205, and Vol. XVI. p. 11, wherein are also found the standard deviations which are used to replace those of the series under consideration, since the number of crania is, as a rule, too small to enable a determination of the variability of their characters to be made with any degree of accuracy.

In order that coefficients may be comparable it is necessary that the same list of characters should be employed in each determination, and as far as possible I have adhered to the characters used by Morant, a list not identical with that used by Tildesley.

In evaluating the coefficients between the Farringdon Street crania and the series measured by Macdonell it has been necessary to replace H by H', and B/H' has been calculated from the mean values of B and H' for the Whitechapel and Moorfields series. NH', except for the males of the Whitechapel group, O_1 and Q are not strictly comparable owing to the different methods employed in taking the measurements. NH, according to Macdonell, was the nasal height measured to the lowest edge of the pyriform aperture, and this was sometimes on the right side and sometimes on the left. On the Farringdon Street skulls measurements were taken from the nasion to the lowest edge on the right side and on the left, and also to the base of the anterior nasal spine. This would mean that his mean measurement would tend to be longer than any of ours.

For the orbital breadth, O_1, Macdonell used the geodesic method, whereas I have used the curvature method, but the dacryal breadths, O_1', were taken in the same way.

Lastly, both Q and Q' in our case were taken through the "apex," but Macdonell took his measurement through the bregma.

Tables I and II give the coefficients of racial likeness between the three London series.

In deducing the relationship between these series from the coefficients of racial likeness, we must bear in mind that the mean measurements of skulls in

* *Biometrika*, Vol. XIII. p. 247.

the Farringdon Street and Whitechapel series for the characters used in computing the coefficients are approximately the same, while in the Moorfields the number is much less.

TABLE I.

Coefficients of Racial Likeness.

Males	All Characters	Indices and Angles
Farringdon Street (91·9)* and Whitechapel (90·3)	4·15 ± ·18 (27)†	4·85 ± ·29 (10)
Farringdon Street (94·0) and Moorfields (27·5) ...	1·86 ± ·18 (26)	1·41 ± ·29 (10)
Whitechapel (87·2) and Moorfields (24·2) ...	2·05‡ ± ·18 (26)	1·73 ± ·30 (9)

TABLE II.

Coefficients of Racial Likeness.

Females	All Characters	Indices and Angles
Farringdon Street (110·7) and Whitechapel (88·5)	3·92 ± ·20 (22)	5·77 ± ·32 (8)
Farringdon Street (113·2) and Moorfields (39·6)...	2·89 ± ·18 (26)	0·96 ± ·29 (10)
Whitechapel (86·0) and Moorfields (34·8) ...	5·20 ± ·18 (27)	5·36 ± ·29 (10)

We may then compare directly the coefficients between the Moorfields and the other two series, and, whether we judge by all characters or by angles and indices alone, the males of the Moorfields series appear to be as closely related to the Farringdon Street as to the Whitechapel crania. Between the two latter the coefficient is higher. This does not necessarily suggest that the Farringdon Street crania are further removed from the Whitechapel than they are from the Moorfields, since in the latter series the material is hardly sufficient to enable us to make a reliable determination of the degree of difference between it and any other series.

For the females, the coefficients are on the whole higher, markedly so between the Whitechapel and Moorfields.

Judging by the indices, the closest resemblance in shape is found between the Farringdon Street and Moorfields crania, and the widest divergence between the Farringdon Street and Whitechapel. An examination of the individual characters and indices demonstrates, as will be shown later, that the chief points of difference between these latter series lie in their height and parietal breadth, and it is these factors and the index B/H' ($\alpha = 33{\cdot}07$) which are chiefly responsible for the higher coefficient between these series.

The C.R.L.'s, then, suggest that in our new series we are dealing with a type very closely allied to the 17th century crania previously known to us, but, at the

* Mean number of skulls for characters used in computing C.R.L.

† Number of characters on which the coefficient is based.

‡ *Biometrika*, Vol. xiv. p. 209.

same time, it appears that there are significant differences between them. The characters in which these differences are most apparent will be discussed in a later section of this paper.

The coefficients of racial likeness between the Farringdon Street and the Hythe and Rothwell series have been calculated but the numbers of characters available for this purpose in the latter two series are very few. A comparison with these crania is reserved to the next section of this paper.

We will now turn to the Scottish skulls and endeavour to ascertain the degree of their relationship to the Londoners.

In Tables III and IV are given the coefficients of racial likeness between them.

TABLE III.*

Coefficients of Racial Likeness. Males.

	Lowland Scottish (48·2)		Eastern Scottish (21·1)	
	All Characters	Indices and Angles	All Characters	Indices and Angles
Farringdon Street (100·3)	5·61 ± ·19 (23)	9·38 ± ·33 (7)	9·78 ± ·19 (23)	15·82 ± ·33 (7)
Whitechapel (98·4) ...	3·37 ± ·19 (23)	7·06 ± ·33 (7)	13·94 ± ·19 (23)	26·46 ± ·33 (7)
Moorfields (30·4) ...	3·87 ± ·21 (19)	6·32 ± ·38 (5)	7·31 ± ·21 (19)	13·85 ± ·38 (5)

TABLE IV.

Coefficients of Racial Likeness. Females.

	Lowland Scottish (24·0)		Eastern Scottish (6·0)	
	All Characters	Indices and Angles	All Characters	Indices and Angles
Farringdon Street (120·2)	8·34 ± ·19 (23)	12·52 ± ·33 (7)	0·61 ± ·19 (23)	1·10 ± ·33 (7)
Whitechapel (99·0) ...	4·65 ± ·21 (20)	7·78 ± ·36 (6)	-0·07 ± ·21 (20)	0·37 ± ·36 (6)
Moorfields (41·1) ...	6·41 ± ·20 (21)	11·23 ± ·36 (6)	1·05 ± ·20 (21)	1·32 ± ·36 (6)

The coefficients of racial likeness for all characters suggest that the Lowland Scottish males are moderately like the Londoners, the least resemblance being with the Farringdon Street crania. The Eastern Scottish appear to be quite distinct from the English and indeed from the Lowland Scottish also, the coefficients between the Scottish types for males being 7·89 ± ·19 for 23 characters and 15·14 ± ·33 for 7 indices and angles.

The females of the Lowland Scottish type exhibit less affinity for the English. In the case of the Moorfields crania the coefficient is considerably higher than is

* The coefficients between the Scottish, Whitechapel and Moorfields series were calculated by Mr G. M. Morant.

the case for the males, as was observed in the comparison of the Moorfields with the Whitechapel series.

With regard to the Eastern Scottish female type we are unable on the basis of our present data to assert a real difference between it and the English.

A noticeable feature in both tables is the high value of the coefficient determined from angles and indices alone, in most cases it being nearly twice as great as that determined from all characters, a difference much more marked than occurred in the coefficients between the English series themselves. It appears, then, that the Scottish skulls are chiefly differentiated from the English by the shape of their heads, but it should be noted that this is equally true of the two types of Scottish crania themselves.

(6) *Comparison of Mean Direct Measurements.* The C.R.L.'s have indicated a close resemblance between the various London series, suggesting, at the same time, that there are certain significant differences, and these will appear in a detailed examination of the mean characters themselves.

Table V gives the mean direct measurements of the three series with their probable errors.

The values of α, where

$$\alpha = \frac{(M_s - M_s')^2}{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}},$$ [*]

between various characters have been determined and are given in Table VI. The function has been calculated on the usual supposition that the variabilities of the characters are the same for different races and equal to those of the long series of Egyptian skulls, Series E. The values of

$$\alpha = \frac{(M_s - M_s')^2}{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s}{n_s}}$$

using the standard deviations of the characters of the English series themselves, have also been determined, and are given in the columns denoted by α'.

As usual, if the value of α exceeds 2·7 the characters are regarded as exhibiting differences, which are possibly significant provided α is less than 6·1, but above that value the probability of their representing samples from the same population is exceedingly small.

The Farringdon Street and Whitechapel series are both of considerable length, and since the mean values of most of the characters are based on approximately the same number of skulls in each series, it should be possible to make a reliable comparison.

Turning first to the males, we see there are 9 characters for which α is greater than 6·1, and these will indicate the chief points of difference between the two series. The most striking values are those of the indices H'/L and B/H', which

* *Biometrika*, Vol. XIV. p. 212.

are evidently due to the superior height of the Whitechapel crania, together with their narrower breadth across the parietal bones. It is this discrepancy in height which is the most noticeable feature of difference between the two series. The larger circumference of the Farringdon Street skulls is due to their excess in breadth, although actually in forehead width the Whitechapel are significantly greater.

The length from basion to nasion is markedly different in the two series, the Whitechapel being the longer.

The foraminal length is distinctly greater in the Farringdon Street crania, but the foraminal breadths and indices do not differ significantly.

As was mentioned in the previous section, the orbital breadth was taken from different points in the two series. This, however, does not affect the measurement of the height, which is greater in the Farringdon Street. The dacryal breadth is greater in the Whitechapel skulls by 0·7 mm., which is actually the difference between the O_1 measurements, so we may safely conclude that their orbits are significantly wider than the orbits in our series.

The Farringdon Street crania, then, are distinguished from the Whitechapel in that they are lower, wider, although with narrower foreheads, and have somewhat rounder orbits.

Of the characters exhibiting differences of doubtful significance, the values of α lying between 2·7 and 6·1, the most distinctive feature is afforded by G_1, the longer palate occurring in the Farringdon Street series. The palate widths of the Farringdon Street and Moorfields crania agree closely. Macdonell gave 36·8 as the mean value of the palate width of the Whitechapel series, based on 66 skulls.

The measurement has been retaken independently by Mr G. M. Morant, and I have to acknowledge my thanks to him for permission to use this and other measurements that he has taken on the same series. Reference to these has been made in Table V. He found only 32 skulls on which he considered the measurement, G_2, could be made with a reasonable degree of accuracy, and their mean value was 39·6, a result in very close agreement with the mean palate width of the Farringdon Street and Moorfields crania. This is a measurement for which the personal equation is large, and caution must be exercised in comparing results obtained by different workers.

The ophryo-occipital length is greater in the Whitechapel skulls, the difference is less marked in the glabellar-occipital length, and the glabellar-projective length taken when the skull is on the craniophor is actually greater in the Farringdon Street.

The nasal angle is a little smaller in the Farringdon Street crania, but the difference is barely significant.

The remaining characters exhibit no difference in type. They include most of the facial indices and measurements, $G'H/GB$, NB/NH, $P\angle$, $A\angle$, $G'H$, J, NH, R, NB and also B/L', fmb/fml, Q, S, fml, and C.

TABLE V. *Comparative Table of Means.*

Character	Males			Females		
	Farringdon Street	Whitechapel	Moorfields	Farringdon Street	Whitechapel	Moorfields
C	1481·5 ± 9·46 (86)	1476·9 ± 9·73 (72)	1473·8 ± 19·01 (22)	1296·5 ± 6·12 (132)	1299·9 ± 8·51 (80)	1365·3 ± 13·68 (31)
F	186·1 ± ·36 (140)	187·4 ± ·35 (138)	186·9 ± ·59 (45)	180·0 ± ·31 (188)	180·1 ± ·36 (143)	182·5 ± ·52 (65)
L'	188·6 ± ·50 (73)	187·8 ± ·45 (72)	188·0 ± 1·02 (19)	179·2 ± ·59 (68)	180·1 ± ·57 (57)	182·5 ± ·86 (23)
L	188·8 ± ·37 (139)	189·1 ± ·36 (137)	189·2 ± ·57 (44)	181·6 ± ·34 (182)	180·4 ± ·35 (140)	183·4 ± ·51 (63)
B	142·4 ± ·34 (141)	140·7 ± ·31 (135)	143·0 ± ·53 (46)	135·7 ± ·25 (180)	134·7 ± ·27 (140)	137·6 ± ·45 (62)
B'	96·8 ± ·25 (152)	98·0 ± ·25 (132)	98·5 ± ·41 (47)	93·3 ± ·19 (199)	93·1 ± ·23 (147)	95·2 ± ·34 (64)
H	130·4 ± ·42 (73)	—	—	122·8 ± ·40 (66)	—	—
H'	129·7 ± ·31 (118)	132·0 ± ·34 (122)	129·8 ± ·75 (34)	122·5 ± ·29 (163)	124·6 ± ·30 (124)	123·6 ± ·46 (47)
OH	110·0 ± ·40 (76)	112·1 ± ·25 (135)	111·3 ± ·47 (46)	105·2 ± ·44 (69)	106·7 ± ·25 (143)	106·9 ± ·38 (59)
LB	100·1 ± ·28 (118)	101·6 ± ·25 (119)	98·5 ± ·52 (35)	95·8 ± ·24 (154)	95·3 ± ·24 (122)	95·9 ± ·43 (46)
Q	306·3 ± 1·00 (75)	307·9 ± ·72 (115)	305·4 ± 1·49 (32)	290·4 ± 1·00 (67)	294·0 ± ·71 (122)	293·1 ± 1·16 (42)
Q'	309·0 ± 1·04 (75)	—	—	293·1 ± ·92 (67)	—	—
S	378·8 ± ·85 (128)	377·1 ± ·81 (131)	378·5 ± 1·28 (40)	362·3 ± ·68 (167)	362·8 ± ·84 (130)	365·6 ± 1·26 (53)
S_1	129·3 ± ·35 (153)	—	129·3 ± ·56 (44)	124·1 ± ·33 (185)	—	125·8 ± ·57 (53)
S_2	128·1 ± ·44 (147)	—	128·7 ± ·78 (43)	123·8 ± ·37 (201)	—	123·6 ± ·68 (52)
S_3	120·6 ± ·46 (135)	—	120·5 ± ·90 (40)	114·8 ± ·36 (182)	—	117·0 ± ·59 (49)
S_1'	113·3 ± ·26 (153)	—	—	108·4 ± ·25 (187)	—	—
S_2'	114·5 ± ·34 (148)	—	—	110·3 ± ·29 (201)	—	—
S_3'	97·3 ± ·30 (132)	—	98·1 ± ·63 (40)	94·0 ± ·24 (184)	—	95·9 ± ·59 (49)
U	530·0 ± ·96 (126)	524·3 ± ·88 (126)	527·1 ± 1·60 (37)	507·8 ± ·82 (166)	503·8 ± ·85 (136)	512·7 ± 1·53 (56)
PH	19·2 ± ·21 (80)	—	—	18·2 ± ·22 (71)	—	—
$G'H$	70·5 ± ·33 (82)	70·2 ± ·30 (75)	68·1 ± ·62 (20)	65·7 ± ·35 (73)	65·9 ± ·40 (62)	64·2 ± ·47 (27)
GB	91·4 ± ·48 (74)	90·9 ± ·45 (55)	93·9 ± ·77 (15)	87·0 ± ·35 (64)	84·9 ± ·41 (58)	86·9 ± ·79 (18)
J	131·0 ± ·50 (43)	130·1 ± ·57 (43)	129·0 ± 1·19 (7)	120·4 ± ·45 (52)	120·3 ± ·58 (33)	122·0 ± ·69 (18)
NH'	51·0 ± ·25 (84)	* 50·3 (76)	—	47·2 ± ·23 (79)	—	—
NH, R	51·8 ± ·21 (86)	† 51·2 ± ·20 (79)	† 50·4 ± ·39 (20)	48·1 ± ·20 (83)	† 48·7 ± ·22 (67)	† 48·0 ± ·36 (27)
NH, L	51·7 ± ·22 (89)	—	—	48·0 ± ·20 (79)	—	—
NB	24·6 ± ·15 (83)	24·3 ± ·17 (70)	24·0 ± ·30 (18)	23·5 ± ·14 (80)	23·2 ± ·14 (64)	23·4 ± ·25 (26)
DS	12·8 ± ·13 (79)	‡ 12·1 ± ·11 (50)	—	11·6 ± ·16 (74)	‡ 10·8 ± ·12 (50)	—
DC	22·2 ± ·16 (84)	‡ 21·5 ± ·21 (50)	—	21·3 ± ·18 (81)	‡ 20·6 ± ·19 (50)	—
DA	36·1 ± ·27 (78)	‡ 34·8 ± ·26 (50)	—	33·0 ± ·30 (75)	‡ 31·5 ± ·28 (50)	—
SS	4·6 ± ·82 (82)	‡ 4·7 ± ·10 (50)	—	4·0 ± ·08 (76)	‡ 4·2 ± ·07 (50)	—
SC	9·2 ± ·14 (81)	‡ 9·4 ± ·17 (50)	—	9·0 ± ·13 (80)	‡ 9·1 ± ·17 (50)	—
O_1, R	42·3 ± ·12 (81)	43·0 ± ·16 (68)	42·3 ± ·22 (18)	40·3 ± ·13 (75)	41·0 ± ·14 (62)	40·9 ± ·23 (25)
O_1, L	42·4 ± ·11 (85)	43·1 ± ·15 (63)	41·8 ± ·23 (20)	40·7 ± ·13 (74)	41·2 ± ·13 (57)	40·9 ± ·24 (22)
O_2, R	34·3 ± ·17 (83)	33·4 ± ·18 (69)	32·8 ± ·34 (18)	33·7 ± ·15 (78)	33·7 ± ·13 (64)	32·6 ± ·27 (25)
O_2, L	34·3 ± ·16 (89)	33·5 ± ·15 (67)	32·8 ± ·32 (20)	33·6 ± ·15 (72)	33·6 ± ·12 (64)	32·8 ± ·34 (22)
O_1'	39·8 ± ·13 (81)	*40·5 (67)	—	38·5 ± ·13 (80)	—	—
EOW	98·1 ± ·30 (77)	—	—	94·1 ± ·29 (79)	—	—

* Measured by G. M. Morant. † Macdonell's NH, to lowest point of pyriform aperture R or L.
‡ Ryley and Bell, *Biometrika*, Vol. IX. p. 403.

TABLE V.—(*continued*). *Comparative Table of Means.*

Character	Males: Farringdon Street	Males: Whitechapel	Males: Moorfields	Females: Farringdon Street	Females: Whitechapel	Females: Moorfields
G_1	49·7 ± ·23 (67)	48·3 ± ·22 (69)	48·1 ± ·69 (17)	46·8 ± ·26 (65)	45·1 ± ·26 (57)	45·9 ± ·43 (20)
G_1'	46·0 ± ·22 (69)	—	—	43·8 ± ·22 (69)	—	—
G_2	39·3 ± ·28 (53)	§ 36·8 ± ·24 (66)	39·7 ± ·41 (15)	37·5 ± ·28 (48)	35·2 ± ·24 (58)	37·0 ± ·40 (22)
GL	94·4 ± ·45 (64)	—	—	90·8 ± ·45 (61)	90·4 ± ·40 (58)	92·1 ± ·76 (25)
EH	11·1 ± ·24 (53)	—	—	10·3 ± ·29 (48)	—	—
EB	39·3 ± ·28 (53)	—	—	37·5 ± ·28 (48)	—	—
fml	36·8 ± ·19 (115)	* 35·9 (117)	35·4 ± ·32 (36)	34·8 ± ·14 (157)	—	34·3 ± ·24 (50)
fmb	30·6 ± ·14 (106)	* 30·3 (112)	29·7 ± ·21 (34)	29·1 ± ·12 (152)	—	29·0 ± ·23 (50)
$100\ B/L'$	75·5 ± ·27 (73)	75·2 ± ·24 (69)	75·1 ± ·52 (18)	75·0 ± ·23 (66)	74·6 ± ·27 (55)	75·4 ± ·30 (21)
$100\ H/L'$	69·1 ± ·25 (72)	—	—	68·8 ± ·29 (65)	—	—
$100\ B/L$	75·4 ± ·20 (132)	74·3 ± ·19 (131)	75·5 ± ·31 (42)	74·8 ± ·15 (167)	74·7 ± ·18 (130)	75·1 ± ·21 (57)
$100\ H/L$	69·3 ± ·26 (71)	—	—	68·9 ± ·29 (65)	—	—
$100\ B/H$	109·1 ± ·37 (73)	—	—	109·1 ± ·32 (64)	—	—
$100\ H'/L$	68·6 ± ·20 (115)	70·0 ± ·20 (120)	68·4 ± ·42 (31)	67·9 ± ·19 (155)	69·1 ± ·18 (117)	67·2 ± ·28 (44)
$100\ B/H'$	109·8 ± ·33 (117)	‖ {106·6} (122)	{110·2} (34)	110·6 ± ·30 (158)	{108·1} (124)	{111·3} (44)
$100\ (B-H')/L$	6·7 ± ·23 (113)	{4·3} (120)	{7·1} (42)	7·0 ± ·20 (149)	{5·6} (117)	{7·9} (44)
$100\ G'H/GB$	77·1 ± ·52 (66)	76·5 ± ·50 (53)	72·8 ± ·86 (14)	75·8 ± ·53 (56)	77·9 ± ·57 (54)	73·6 ± ·64 (18)
$100\ NB/NH,\ R$	47·5 ± ·32 (81)	¶ 47·6 ± ·37 (70)	47·6 ± ·66 (18)	49·0 ± ·33 (77)	47·8 ± ·33 (64)	48·7 ± ·52 (26)
$100\ NB/NH,\ L$	47·5 ± ·33 (81)	—	—	49·4 ± ·35 (76)	—	—
$100\ O_2/O_1,\ R$	81·0 ± ·46 (79)	77·7 ± ·38 (68)	77·4 ± ·69 (17)	83·8 ± ·34 (75)	82·5 ± ·37 (62)	79·8 ± ·75 (25)
$100\ O_2/O_1,\ L$	80·9 ± ·38 (85)	77·9 ± ·32 (63)	78·5 ± ·64 (20)	82·8 ± ·45 (72)	81·7 ± ·38 (57)	80·3 ± ·84 (22)
$100\ O_2/O_1'$	85·9 ± ·49 (79)	{82·5} (67)	—	87·8 ± ·39 (76)	—	—
$100\ fmb/fml$	83·4 ± ·39 (106)	* 84·5 (112)	84·2 ± ·77 (34)	84·1 ± ·31 (149)	—	84·5 ± ·60 (47)
$100\ DS/DC$	58·1 ± ·71 (78)	‡ 56·8 ± ·71 (50)	—	54·5 ± ·64 (74)	‡ 52·9 ± ·63 (50)	—
$100\ SS/SC$	50·7 ± ·96 (81)	‡ 50·8 ± 1·20 (50)	—	44·6 ± ·79 (76)	‡ 46·6 ± ·8 (50)	—
$100\ G_2/G_1$	78·8 ± ·55 (46)	§ 76·3 ± ·55 (61)	82·7 ± 1·05 (13)	79·8 ± ·65 (42)	77·7 ± ·62 (51)	81·2 ± ·71 (19)
$100\ EH/EB$	28·5 ± ·64 (51)	—	—	27·6 ± ·71 (48)	—	—
$P\angle$	85°·9 ± ·26 (65)	86°·1 ± ·33 (63)	84°·5 ± ·69 (15)	85°·6 ± ·24 (57)	87°·1 ± ·27 (52)	84°·8 ± ·35 (19)
$N\angle$	64°·2 ± ·31 (64)	65°·2 ± ·29 (69)	66°·6 ± ·58 (17)	64°·6 ± ·35 (60)	64°·7 ± ·23 (57)	66°·7 ± ·37 (26)
$A\angle$	73°·3 ± ·31 (64)	73°·4 ± ·28 (69)	72°·4 ± ·61 (17)	74°·1 ± ·34 (60)	73°·9 ± ·29 (57)	73°·3 ± ·30 (26)
$B\angle$	42°·5 ± ·31 (64)	41°·4 ± ·20 (69)	41°·1 ± ·44 (17)	41°·3 ± ·29 (60)	41°·4 ± ·27 (57)	40°·0 ± ·35 (26)
θ_1	29°·8 ± ·29 (60)	28°·7 ± ·22 (59)	29°·4 ± ·53 (14)	29°·8 ± ·32 (53)	28°·1 ± ·24 (50)	28°·5 ± ·33 (19)
θ_2	12°·5 ± ·28 (60)	12°·9 ± ·29 (59)	11°·6 ± ·43 (14)	11°·7 ± ·29 (53)	13°·1 ± ·34 (50)	11°·5 ± ·37 (19)
$Oc.\ I.$	58·0 ± ·16 (131)	—	58·5 ± ·23 (40)	59·1 ± ·12 (181)	—	59·2 ± ·28 (49)

* Measured by G. M. Morant.

‡ Ryley and Bell, *Biometrika*, Vol. IX. p. 403.

§ This measurement was retaken recently by Mr G. M. Morant. He found only 32 skulls which he considered sufficiently well preserved to give accurate measurements. Their mean is 39·6 and the corresponding value of G_2/G_1 is {82·0}.

‖ Indices in curled brackets are calculated from the mean direct measurements.

¶ Nasal Index from G. M. Morant's NH' $\{NB/NH'\} = 48{\cdot}3$ (70).

TABLE VI.

Values of $\alpha = \dfrac{n_s n_s'}{n_s + n_s'}\left(\dfrac{M_s - M_s'}{\sigma_s}\right)^2$ *between the London Crania.*

Character	Farringdon Street and Whitechapel				Farringdon Street and Moorfields			
	Males		Females		Males		Females	
	a	a'	a	a'	a	a'	a	a'
B/L'	0·45	0·31	0·73	0·57	0·32	0·21	0·39	0·51
H'/L	13·31	11·03	11·74	9·91	0·11	0·08	2·05	1·94
B/H'	33·07	22·02	26·21	14·20	0·23	0·14	1·07	0·55
Oc. I.	—	—	—	—	0·70	1·46	0·04	0·02
$G'H/GB$	0·43	0·32	5·75	3·30	8·68	8·35	3·44	3·37
$NB/NH, R$	1·92	1·28	3·54	3·05	0·01	0·01	0·12	0·11
fmb/fml	1·97	1·89	—	—	0·49	0·40	0·14	0·12
G_2/G_1	4·19	6·27	2·34	2·58	3·34	4·95	0·68	1·09
$P\angle$	0·12	0·10	7·03	7·87	2·28	1·61	2·13	1·61
$N\angle$	3·03	2·56	0·25	0·23	7·06	5·10	8·64	8·73
$A\angle$	0·03	0·03	0·10	0·09	0·91	0·80	1·00	1·41
F	3·57	2·98	0·04	0·02	0·66	0·60	13·27	7·68
B	8·80	6·37	3·85	3·28	0·55	0·42	8·14	6·09
B'	6·20	5·30	0·24	0·20	6·33	5·79	12·19	10·75
OH	12·63	9·05	7·86	3·99	2·85	2·04	6·90	3·86
LB	8·46	7·19	1·37	1·01	4·38	3·34	0·03	0·02
S	1·20	0·96	0·17	0·10	0·02	0·02	3·97	2·43
U	11·20	8·86	8·68	5·21	1·27	1·10	7·27	3·61
$G'H$	0·20	0·20	0·09	0·06	5·38	5·36	3·14	2·95
J	0·93	0·71	0·01	0·01	1·15	1·10	1·78	1·72
NH, R	1·74	2·68	1·98	1·79	3·75	4·50	0·03	0·03
NB	1·09	0·78	1·19	1·04	1·70	1·44	0·07	0·05
O_2, R	9·30	5·85	0·00	0·00	9·18	7·10	6·77	5·63
G_1	6·00	8·74	10·02	9·77	3·13	2·17	1·41	1·49
G_2	0·26	0·19	20·60	17·54	0·27	0·29	0·59	0·47
fml	7·70	5·22	—	—	8·81	6·42	1·98	1·52
fmb	1·06	1·01	—	—	4·51	5·66	0·09	0·07
C	0·06	0·05	0·07	0·05	0·07	0·05	14·00	9·59

α' = value of $\dfrac{n_s n_s' (M_s - M_s')^2}{n_s \sigma_s'^2 + n_s' \sigma_s^2}$, where σ_s and σ_s' are found from the series themselves, here adequate in number.

Consideration of the females of the two series leads us to much the same conclusions. We find the differences in height and breadth somewhat less marked, but the highest values of α are still found for the indices H'/L, B/H'. In forehead width the two groups are in close agreement.

The horizontal circumference is again significantly greater in the Farringdon Street, but the sagittal arcs are almost identical. The difference between the transverse arcs is more marked than was the case in the males.

The orbital measurements are in close agreement, there being no significant difference in the heights; the Whitechapel appear to be wider.

The angles of the fundamental triangle are identical in the two series, but the profile angle in the Farringdon Street crania is smaller.

The ophryo-occipital lengths are the same, and the similarity in length is confirmed by L and L'. No differences are found in B/L', LB, $G'H$, J, NB, and C.

The difference in nasal heights is small, and since Macdonell's result should be reduced to be comparable with ours, it is extremely likely that the two are very similar.

Between the Moorfields and Farringdon Street crania the greatest differences are shown by the characters $G'H/GB$, $N\angle$, B', O_2, R, and *fml*. The high value of $G'H/GB$ is due to the excess in facial height on the part of the Farringdon Street crania, coupled with their defect in facial width. In forehead width the Moorfields are significantly greater, but there is no difference in the breadth across the parietal bones.

The two series are in close agreement as regards breadth, in height the Moorfields have a slight advantage, but it is barely significant, and the indices B/L', H'/L, B/H' are very nearly the same.

The orbits of the Farringdon Street crania are again significantly higher, and probably there is very little difference in their widths.

The palatine widths are in good accordance; the length is greater in the Farringdon Street.

The nasal height is significantly greater in the Farringdon Street series, the difference in breadth being insignificant.

The remaining characters in which the two series resemble one another closely are *Oc. I*, *fmb/fml*, $P\angle$, $A\angle$, S, U, J, and C.

The agreement between the females is not so close and it should be noted that the numbers in each series are larger.

The superiority in length of the Moorfields is borne out by F, L, and L'; they are significantly wider, and consequently they have a considerably greater capacity. The indices B/L', H'/L, B/H' agree very well, and the facial measurements are more alike than was the case for the males. It thus appears that the distinctive difference between them is that of size.

A study, then, of the direct measurements of the three series indicates a very close relationship between them. The Farringdon Street are lowest, in breadth they occupy an intermediate position, the Moorfields being the widest, and if we omit the Moorfields females the length is the same for all three. Their cephalic indices place them on the border line between dolichocephalic and mesaticephalic skulls.

In facial measurements they are very similar, the Farringdon Street group being rather nearer to the Whitechapel than to the Moorfields. The orbital measurements give greater height to the Farringdon Street crania.

In foraminal breadth the three series are in accord, the greatest length being found in the Farringdon Street.

The agreement in the value of $A\angle$ is steady throughout, it is fairly close in $P\angle$, but $N\angle$ is definitely smallest in the Farringdon Street crania.

In capacity (Table VII) the three series resemble one another very closely, except in the case of the Moorfields females, which are considerably larger than the females of either of the other groups and agree more closely with the capacities determined on modern English heads by the Pearson and Lee formulae. The males, 17th century and modern, have very similar capacities.

TABLE VII. *Capacities.*

	Males	Females
Farringdon Street ...	1481·5 (86)	1296·5 (132)
Whitechapel	1476·9 (72)	1299·9 (80)
Moorfields	1473·8 (22)	1365·3 (31)
*British Association ...	1495	1323·5
*Bedford College Students	—	1390 (30)

The capacities deduced for the living are of course not for the general population, but for highly selected classes.

Of the characters not included in Table VI the only ones we need consider are the simotic and dacryal arcs, chords and subtenses. These measurements have been made on the Farringdon Street and Whitechapel crania only, and the agreement is close. The simotic chords are sensibly identical both in males and females, the dacryal chords are very nearly the same, while the difference in the dacryal arcs is slight but significant.

These facts, taken in conjunction with the other nasal measurements, suffice to show that the nasal organs of the two series resemble one another extremely closely.

θ_1, θ_2 are calculated from $P\angle$, $N\angle$, and $A\angle$, and the differences in the values of θ_1 and θ_2 are clearly accounted for by discrepancies in the measured angles.

The conjoint indices of the Farringdon Street and Moorfields series only differ significantly for the females. The index of the Whitechapel group is lower in both sexes.

We will now turn to the α' columns given in Table VI which give us an appreciation of the differences between the various characters, based on the standard deviations of the characters of each series. The variabilities were found to be, in most cases, slightly greater than those of the Egyptian Series E, thereby decreasing the measure of the divergence between the characters to some slight extent. The reduction is only appreciable in 3 cases, viz. *OH* for the Whitechapel females and *F* and *U* for the Moorfields females, and in 9 cases the change is in the other direction, but it is not significant and 6 of them occur in the comparison

* *Phil. Trans.* Vol. 196, pp. 252 and 257. Equations used:

$$C = \cdot 000337\,(l-11)\,(b-11)\,(h-11) + 406\cdot 01\ \male,$$
$$C = \cdot 000400\,(l-11)\,(b-11)\,(h-11) + 206\cdot 60\ \female.$$

with the Moorfields series, where the small number of skulls renders the determination of the standard deviations very unreliable. The effect will be to reduce the coefficient of racial likeness, thus between the Farringdon Street and Whitechapel it will be $3{\cdot}14 \pm {\cdot}18$ for 27 characters and between the Moorfields and Farringdon Street, $1{\cdot}77 \pm {\cdot}18$ for 26 characters.

(7) *Prediction Formulae for Cranial Capacity.* Before we turn to the other series with which we have to compare the London crania, we shall make a slight digression to consider the determination of the capacity of the skull from the length, breadth and height. The Lee and Pearson intraracial regression formulae were obtained for German, Aino and Naqada races, male and female, and the mean formulae derived from these have been used for races whose relationship to any one of the above is not sufficiently close to justify the choice of one particular equation.

Intraracial formulae were obtained by the method of least squares for ten races, European and otherwise, and Dr Isserlis has since found the intraracial regression formulae for negroes*, who were excluded in the above determinations.

Now that three English series of the same period and locality are available, it was thought worth while to determine the regression formulae from these data, and accordingly the products $L \times B \times H'$ and $L \times B \times OH$ were determined for the individual skulls whenever possible, and these products were correlated with the capacities found by mustard seed. The products are measured in cubic centimetres.

The numbers of skulls for which these products could be calculated were as follows:

Product	Sex	Farringdon Street	Whitechapel	Moorfields	Totals
$L \times B \times H'$	♂	85	66	22	173
$L \times B \times H'$	♀	124	78	29	231
$L \times B \times OH$	♂	63	66	22	151
$L \times B \times OH$	♀	62	79	29	170

The results obtained are given below:

Sex	Number	Mean Capacity	Mean LBH' in cm.^{3}	σ_C	$\sigma_{LBH'}$	$r_{C,\,LBH'}$
♂	173	1481	3500	125·53	288·19	·841
♀	231	1307	3029	110·32	256·75	·851

The regression lines are

Probable error of mean

$$\left.\begin{array}{lll} ♂ & C = {\cdot}000366\,LBH' + 198{\cdot}87 & \dfrac{45{\cdot}8}{\sqrt{n}} \\[2ex] ♀ & C = {\cdot}000366\,LBH' + 199{\cdot}43 & \dfrac{38{\cdot}9}{\sqrt{n}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

* *Biometrika*, Vol. x. p. 188.

Sex	Number	Mean Capacity	Mean $LB(OH)$ in cm^3	σ_C	$\sigma_{LB(OH)}$	$r_{C,\,LB(OH)}$
♂	151	1480	2965	122·34	248·47	·844
♀	170	1302	2588	122·34	250·41	·863

The regression lines are

Probable error of mean

$$\left.\begin{array}{lll} \male & C = \cdot000416\,LB(OH) + 247\cdot86 & \dfrac{44\cdot3}{\sqrt{n}} \\[2ex] \female & C = \cdot000422\,LB(OH) + 210\cdot83 & \dfrac{41\cdot8}{\sqrt{n}} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

To test the errors involved in the determination of capacities from these formulae, 20 male and 20 female skulls were selected at random, the formulae were applied to them and the calculated capacities were compared with those obtained by the mustard seed method. For males, the mean error of the 20 skulls as found from (1) is 47·0, from (2) 56·8, and for females, 37·3 from (1) and 38·7 from (2), corresponding to probable errors of (1) 39·7, (2) 48·0 for males and (1) 31·5, (2) 32·7 for females.

The formulae were then applied to the mean L, B, H' and OH of the three London series and the capacities thus determined are given below:

		Seed	Formula (1)	Formula (2)
Farringdon Street	♂	1481·5	1475·1	1478·2
,, ,,	♀	1296·5	1304·3	1301·9
Whitechapel	♂	1476·9	1484·3	1488·6
,,	♀	1299·9	1307·6	1304·9
Moorfields	♂	1473·8	1484·2	1500·6
,,	♀	1365·3	1341·0	1349·3

The extent of the error introduced by using the product of the means instead of the mean product has been discussed by Dr Lee*, who found an error of less than 1 per cent.

The equation to determine the error is

$$\text{Mean product} = \text{product of means}\,(1 + v_B v_{H'} r_{BH'} + v_L v_{H'} r_{LH'} + v_B v_L v_{BL}).$$

These correlations and the coefficients of variation have been found for the Whitechapel crania, and applying them to the above formula we have

$$\male \quad \text{Mean product} = 3512\cdot0\,(1 + \cdot001{,}021) = 3512\cdot0 + 3\cdot6,$$

$$\female \quad \text{Mean product} = 3027\cdot8\,(1 + \cdot001{,}485) = 3027\cdot8 + 4\cdot5,$$

showing that the errors involved are quite insignificant.

* *Phil. Trans.* Vol. 196, p. 250.

Finally we must compare our equations with the Lee and Pearson formulae to ascertain the increase in accuracy they enable us to obtain. We will consider their formulae (9)—(13) which are

Mean of three intraracial formulae for Aino, Naqada and German crania $\begin{cases} ♂ \quad C = \cdot 000337\, LB(OH) + 406\cdot 01 \\ ♀ \quad C = \cdot 000400\, LB(OH) + 206\cdot 60 \end{cases}$ ......(9),

Interracial formulae deduced from means of ten races $\begin{cases} ♂ \quad C = \cdot 000365\, LB(OH) + 359\cdot 34 \text{......(10)}, \\ ♀ \quad C = \cdot 000375\, LB(OH) + 296\cdot 40 \text{......(11)}, \end{cases}$

Interracial formulae deduced from means of ten races $\begin{cases} ♂ \quad C = \cdot 000266\, LBH' + 524\cdot 6 \text{ ....(12)}, \\ ♀ \quad C = \cdot 000156\, LBH' + 812\cdot 0 \text{ ....(13)}. \end{cases}$

	Farringdon Street		Whitechapel		Moorfields	
	♂	♀	♂	♀	♂	♀
Lee and Pearson (9)	1402·6	1243·6	1411·1	1243·7	1420·8	1285·7
" " (10) and (11)	1438·8	1268·6	1448·0	1268·7	1458·5	1308·0
" " (12) and (13)	1452·1	1282·9	1458·8	1284·3	1458·7	1298·6
English (1)	1475·1	1304·3	1484·3	1307·6	1484·2	1341·0
" (2)	1478·2	1304·9	1488·6	1304·9	1500·6	1349·3
Seed	1481·5	1296·5	1476·9	1299·9	1473·8	1365·3

A glance at the above table will show that the Lee and Pearson formulae give values for the mean capacities of the English crania, which are in all cases too low and less accurate than our new equations.

The best values they give are obtained from equations (12) and (13), i.e. the equations involving the basio-bregmatic height, whereas in applying the formulae to the races from which they were deduced the reverse is the case, better results being obtained by using the auricular height formulae. In the case of the formulae deduced from our English data, the basio-bregmatic height gives an error no greater than that involved by using the auricular height. The conclusion we can draw from these results is that already emphasised by Lee and Pearson, namely, that we must proceed with extreme caution when we apply a regression formula worked out for a special race to obtain the mean capacity of a second race, not closely allied to it *.

(8) *Comparison of London Crania with those of Hythe and Rothwell.* Reverting to the study of individual measurements and having assured ourselves of the close similarity between the Farringdon Street and the other London crania, we will now investigate their relationship to the Hythe and Rothwell skulls.

* Using the English three-arcs formula given by Lewenz and Pearson (*Biometrika*, Vol. III. p. 370) we find for males that the Farringdon Street crania have a mean capacity of 1490·3, the Whitechapel 1477·5 and the Moorfields 1480·0. These values are as close to those determined by the seed method as are those deduced from our formulae (1) and (2), and the same accord is found by using the formulae for female crania.

Table VIII gives the mean measurements of the Hythe and Farringdon Street series. The Hythe measurements are given to the nearest mm. and the indices are calculated from the mean characters.

TABLE VIII.

Mean Measurements of Hythe and Farringdon Street Crania.

Character	Males		Females	
	Hythe	Farringdon Street	Hythe	Farringdon Street
F	177 (324)	186·1 (140)	171 (230)	180·0 (188)
L	179 (319)	188·8 (139)	171 (227)	181·6 (182)
B	143 (324)	142·4 (141)	140 (230)	135·7 (180)
B'	99 (318)	96·8 (152)	96 (228)	93·3 (199)
H'	133 (307)	129·7 (118)	128 (222)	122·5 (163)
OH	120 (294)	110·0 (76)	116 (215)	105·2 (69)
B/L	{79·9}	75·5 (73)	{81·9}	74·8 (167)
H'/L	{74·3}	68·6 (115)	{74·9}	67·9 (155)
C	1441	1481·5 (86)	1206	1296·5 (132)

The most noticeable feature of this table is the extreme shortness of the Hythe crania, which distinguishes them at once from all the London series. The difference in breadth across the parietal bones is only significant in the females; in this character they are nearest to the Moorfields. Their cephalic index is high, bringing them almost into the brachycephalic class, whereas the London crania border on the dolichocephalic.

The forehead width is also greater than occurs in any of the London series.

In basio-bregmatic height the Hythe crania are markedly superior to the low vaulted Londoners; in auricular height the difference amounts to 1 cm., but this is chiefly due to the fact that on the former the measurement was taken from the centre of the ear-passage and not from the auricular points.

Parsons states that he determined the capacities of the Hythe crania by the Pearson and Lee formulae,

$$\text{Length} \times \text{Breadth} \times \text{Auricular height} \times {\cdot}000337 + 406 \text{ for } ♂ \text{ skulls,}$$

and $$\text{Length} \times \text{Breadth} \times \text{Auricular height} \times {\cdot}000400 + 206 \text{ for } ♀ \text{ skulls.}$$

Applying these to the mean measurements of the Hythe crania he gave 1441 c.c. for the mean capacity of the males and 1206 c.c. for the females. Using the male formula for the males of the Farringdon Street series, we find their mean capacity is 1402·6 c.c., but since the auricular heights were measured from different points, these results are not strictly comparable. If instead we use our formula (1),

$$C = {\cdot}000366\, L \times B \times H' + 198{\cdot}87,$$

we find the Hythe male crania have a mean capacity of 1444·9 c.c. and the Farringdon Street of 1475·1 c.c. By mustard seed, the capacity of the latter was found to be 1481·5 c.c.

Apparently Parsons did not use the auricular height formula for the female crania (or if he did made some slip), since this gives 1316·8 c.c., not 1206·0 c.c., as the mean capacity.

Using our formula $C = \cdot000366\, L \times B \times H' + 199\cdot43$ for females, we obtain 1321·0 c.c. for the Hythe crania and 1304·3 c.c. for the Farringdon Street. The capacity of the latter as determined by mustard seed was 1296·5 c.c. It would appear, then, that the capacities of the two series do not differ to any great extent.

The Hythe crania are clearly differentiated from the Farringdon Street, the difference in shape being very striking, and we are bound to conclude that the Kentish man of the 14th century represents a type quite distinct from the 17th century Londoner*.

In Table IX are given the mean measurements of the Rothwell and Farringdon Street series, and in Table X are the values of α for some of the characters.

In basio-bregmatic height the Rothwell skull is superior to the Farringdon Street, agreeing, in this character, very closely with the Whitechapel crania. It is significantly shorter than the London crania but is considerably nearer to them than to the Kentish man.

TABLE IX.

Comparative Table of Means.

F	L	B	B'	H'	$100\,B/L$	$100\,B/F$	$100\,H'/L$
186·1 (140)	188·8 (139)	142·4 (141)	96·8 (152)	129·7 (118)	75·5 (73)	{76·5} (140)	68·6 (115)
184·1 (100)	185·9 (99)	141·7 (100)	100·8 (96)	132·5 (93)	{76·2} (99)	{77·0} (100)	{71·3} (93)
180·0 (188)	181·6 (182)	135·7 (180)	93·3 (199)	122·5 (163)	74·8 (167)	{75·4} (180)	67·9 (155)
181·6 (27)	181·9 (27)	138·3 (27)	96·9 (27)	128·4 (26)	{76·0} (27)	{76·2} (27)	{70·6} (26)

TABLE X.

Values of $\dfrac{n_s n_s'}{n_s + n_s'} \left(\dfrac{M_s - M_s'}{\sigma_s}\right)^2$ *between the Rothwell and London Crania. Males.*

	$100\,B/L$	$100\,H'/L$	$100\,B/H'$	F	B	B'	H'
Farringdon Street ...	2·89	43·37	23·57	7·11	1·26	57·39	24·02
Whitechapel ...	28·31	10·24	0·26	19·23	2·53	26·56	0·80
Moorfields	2·18	22·62	14·66	7·41	2·35	10·18	10·69

* The C.R.L. between the Hythe and Farringdon Street crania is 82·50 ± ·32 for eight characters, and between the Rothwell and Farringdon Street crania is 21·80 ± ·38 for seven characters—all that are available!

In parietal breadth it is intermediate between the Farringdon Street and Whitechapel crania.

The cephalic index is higher than that found in any of the London series, placing the Rothwell crania in the mesaticephalic class.

The greatest differences between the females of the Rothwell and Farringdon Street series occur, as in the males, in the characters H', B', H'/L, B/H'.

(9) *Further Comparison of London and Lowland Scottish Crania.* The coefficients of racial likeness between the London and Scottish male crania suggested a relationship between the Lowland Scottish type and the three London series which does not diverge very widely from that which exists between the several London series themselves (see p. 25). It will be worth while, then, considering in further detail the resemblances between the English and the Lowland Scottish types. The Eastern Scottish type, on the other hand, appears for males to be well removed from them all, and as the data are not abundant, a closer study would not repay us.

Table XI gives the mean measurements of the Lowland Scottish type together with the α's between them and the Farringdon Street crania.

TABLE XI. *Mean Measurements of Lowland Scottish Group and Values of α between Lowland Scottish and Farringdon Street Series.*

Character	Lowland Scottish ♂	Lowland Scottish ♀	α ♂	α ♀
$100\,B/L$	75·3 (54)	77·0 (28)	0·05	17·57
$100\,H'/L$	70·9 (52)	70·9 (27)	21·91	25·30
$100\,B/H'$	{106·4}(52)	{108·6}(27)	21·26	5·57
$100\,NB/NH'$	44·5 (44)	44·0 (23)	26·98	36·55
$100\,O_2/O_1'$	87·3 (48)	89·2 (24)	2·29	0·22
$N\angle$	{64°·4}(38)	{65°·9}(14)	0·09	1·81
$A\angle$	{73°·5}(38)	{71°·3}(14)	0·08	7·65
L	188·8 (54)	179·6 (28)	0·00	4·27
B	142·1 (54)	138·2 (28)	0·16	7·41
B'	97·0 (53)	92·6 (28)	0·96	0·84
H'	133·6 (52)	127·3 (27)	32·34	40·06
LB	102·0 (52)	95·2 (23)	8·27	0·58
*Q	306·7 (54)	295·0 (26)	0·05	5·21
S	379·3 (51)	363·2 (27)	0·06	0·17
U	528·9 (50)	509·1 (27)	0·23	0·28
GH'	71·5 (39)	68·3 (19)	1·53	7·21
J	131·5 (43)	122·6 (20)	0·26	3·64
NH'	52·2 (46)	50·6 (24)	5·02	31·48
NB	23·1 (44)	22·2 (23)	20·65	11·22
O_1'	39·3 (48)	37·8 (24)	2·70	3·72
O_2	34·3 (48)	33·6 (24)	0·00	0·05
fml	35·7 (51)	34·9 (26)	7·01	0·05
C	1493·1 (44)	1338·2 (22)	0·30	3·87

* Turner's transverse arc is taken like Macdonell's from one "supra-auricular point" to the other "supra-auricular point," through the bregma. On 75 Farringdon Street crania, I found the ratio of the apical to the bregmatic transverse arc to be ·9942, so that this difference of method of measurement is of very small importance.

The most striking values are H'/L, B/H', NB/NH' and H'. The Lowland Scottish are considerably higher than the Londoners, in length they are identical with the Farringdon Street, and in breadth the difference is not significant. The cephalic indices are in very good accord.

The breadth of the pyriform aperture is considerably less in the Scottish skulls, while the nasal height is greater, resulting in a sensible difference in the nasal indices.

The Farringdon Street crania are shorter in length from nasion to basion, and the foraminal length is greater; but in no other character do we find a significant difference. The arcs, orbits, angles and capacities bear very strong resemblances.

The characters in which differences do occur are few, but the discordance in these is strongly marked.

In the females the agreement is not so close. Again the greater height of the Scottish skulls is brought forcibly to our notice, but here we find lack of accord in length and breadth, the Farringdon Street being both longer and narrower. The cephalic index of the Lowland Scottish is 77·0, bringing the females well into the mesaticephalic class, whereas the males are approaching the dolichocephalic.

The differences in the nasal height and breadth are even more pronounced and the Scottish skulls have a significantly greater facial height.

The resemblance between the Lowland Scottish and the Whitechapel crania is yet more marked. The calvarial measurements are almost identical, and it is only in the facial characters that significant differences are shown between the two series.

Values of α greater than 5 are found for

$$100\,NB/NH'\,(\alpha = 26{\cdot}73), \qquad 100\,O_2/O_1'\,(\alpha = 19{\cdot}24),$$
$$NB\,(\alpha = 12{\cdot}43), \qquad NH'\,(\alpha = 12{\cdot}12),$$
$$O_2R\,(\alpha = 6{\cdot}28), \qquad 100\,B/L\,(\alpha = 5{\cdot}53).$$

We are, then, led to the conclusion that the Lowland Scottish group is representative of a type distinct from any of our London series but resembling them much more closely than do either the Hythe or Rothwell crania. The outstanding difference between the Scottish and Farringdon Street skulls is in the basio-bregmatic height, the agreement between the former and the Whitechapel crania being very close in this respect.

The facial differences shown between the Scottish and Whitechapel groups do not exist between the former and the Farringdon Street series except in the breadth of the pyriform aperture.

(10) *On the Variability of the Farringdon Street Crania.* The standard deviations and coefficients of variation of the characters of the Farringdon Street crania are given in Table XII. They will be found to agree well with the variabilities of the Whitechapel and Moorfields series, which are given in *Biometrika*, Vol. v. p. 92.

In capacity, length and breadth, the differences between the coefficients of variation, both for males and females, are not significant. In height (H'), the males

TABLE XII. *Variabilities of Characters of Farringdon Street Crania.*

Character	Males No.	Males Standard Deviation	Males Coefficient of Variation	Females No.	Females Standard Deviation	Females Coefficient of Variation
C	86	130·12 ± 6·69	8·78 ± ·45	132	104·28 ± 4·33	8·04 ± ·34
F	140	6·39 ± ·26	3·43 ± ·12	188	6·35 ± ·22	3·53 ± ·12
L'	73	6·29 ± ·35	3·34 ± ·19	68	7·21 ± ·42	4·02 ± ·23
L	139	6·46 ± ·26	3·42 ± ·12	182	6·70 ± ·24	3·69 ± ·13
B	141	5·90 ± ·24	4·14 ± ·17	180	5·07 ± ·18	3·74 ± ·13
B'	152	4·58 ± ·18	4·73 ± ·18	199	3·97 ± ·13	4·26 ± ·14
H	73	5·33 ± ·30	4·09 ± ·23	66	4·83 ± ·28	3·93 ± ·23
H'	118	5·06 ± ·22	3·90 ± ·17	163	5·40 ± ·20	4·41 ± ·17
OH	76	5·17 ± ·28	4·70 ± ·26	69	5·40 ± ·31	5·13 ± ·30
LB	118	4·47 ± ·20	4·47 ± ·20	154	4·33 ± ·17	4·52 ± ·17
Q	75	12·89 ± ·71	4·21 ± ·23	67	12·16 ± ·71	4·19 ± ·24
Q'	75	13·35 ± ·74	4·32 ± ·24	67	11·11 ± ·65	3·79 ± ·22
S	128	14·20 ± ·60	3·75 ± ·16	167	12·96 ± ·48	3·58 ± ·13
S_1	153	6·47 ± ·25	5·00 ± ·19	185	6·74 ± ·24	5·43 ± ·19
S_2	147	7·99 ± ·31	6·24 ± ·25	201	7·81 ± ·26	6·31 ± ·21
S_3	135	7·84 ± ·32	6·50 ± ·27	182	7·13 ± ·25	6·21 ± ·22
S_1'	153	4·77 ± ·18	4·21 ± ·16	187	5·13 ± ·18	4·73 ± ·17
S_2'	147	6·20 ± ·24	5·41 ± ·21	201	6·15 ± ·21	5·58 ± ·19
S_3'	131	5·12 ± ·21	5·26 ± ·22	184	4·82 ± ·17	5·13 ± ·18
U	126	15·92 ± ·68	3·00 ± ·13	166	15·70 ± ·58	3·09 ± ·11
PH	80	2·79 ± ·15	14·53 ± ·79	71	2·78 ± ·16	15·27 ± ·88
$G'H$	82	4·45 ± ·23	6·31 ± ·33	73	4·41 ± ·25	6·71 ± ·38
GB	74	6·16 ± ·34	6·74 ± ·38	64	4·21 ± ·25	4·84 ± ·29
J	43	4·84 ± ·35	3·69 ± ·27	52	4·83 ± ·32	4·01 ± ·27
NH'	84	3·44 ± ·23	6·75 ± ·45	79	3·02 ± ·16	6·40 ± ·35
NH, R	86	2·89 ± ·15	5·58 ± ·29	83	2·76 ± ·14	5·74 ± ·28
NH, L	89	3·12 ± ·16	6·03 ± ·31	79	2·60 ± ·14	5·42 ± ·29
NB	83	2·01 ± ·11	8·17 ± ·43	80	1·88 ± ·10	8·00 ± ·43
DS	79	1·77 ± ·09	13·83 ± ·76	74	2·01 ± ·11	17·33 ± ·99
DC	84	2·16 ± ·11	9·73 ± ·51	81	2·38 ± ·13	11·17 ± ·60
DA	78	3·56 ± ·19	9·86 ± ·54	75	3·86 ± ·21	11·70 ± ·65
SS	82	1·10 ± ·06	23·91 ± 1·33	76	0·99 ± ·05	24·75 ± 1·43
SC	81	1·92 ± ·10	20·87 ± 1·15	80	1·75 ± ·06	19·44 ± 1·08
O_1, R	81	1·56 ± ·08	3·69 ± ·20	75	1·64 ± ·09	4·07 ± ·22
O_1, L	85	1·48 ± ·08	3·49 ± ·18	74	1·63 ± ·09	4·00 ± ·22
O_2, R	83	2·36 ± ·12	6·88 ± ·36	78	1·98 ± ·11	5·88 ± ·32
O_2, L	89	2·30 ± ·12	6·70 ± ·34	72	1·93 ± ·11	5·74 ± ·32
O_1'	81	1·67 ± ·09	4·20 ± ·22	80	1·75 ± ·09	4·55 ± ·24
EOW	77	3·88 ± ·21	3·96 ± ·22	79	3·87 ± ·21	4·11 ± ·22
G_1	67	2·78 ± ·16	5·59 ± ·33	65	3·05 ± ·18	6·52 ± ·39
G_1'	69	2·65 ± ·15	5·76 ± ·33	69	2·71 ± ·16	6·19 ± ·36
G_2	53	3·05 ± ·20	7·76 ± ·51	48	2·92 ± ·20	7·79 ± ·54
GL	64	5·34 ± ·32	5·66 ± ·34	61	5·22 ± ·32	5·75 ± ·35
EH	53	2·63 ± ·17	23·69 ± 1·64	48	2·95 ± ·20	28·64 ± 2·13
EB	53	3·04 ± ·20	7·74 ± ·51	48	2·85 ± ·20	7·60 ± ·53
fml	115	3·00 ± ·13	8·15 ± ·36	157	2·51 ± ·10	7·21 ± ·28
fmb	106	2·20 ± ·10	7·19 ± ·33	152	2·16 ± ·08	7·42 ± ·29
100 B/L'	73	3·47 ± ·19	4·60 ± ·26	66	2·79 ± ·16	3·72 ± ·22
100 H/L'	72	3·09 ± ·17	4·47 ± ·25	65	3·52 ± ·21	5·12 ± ·30
100 B/L	132	3·48 ± ·14	4·62 ± ·19	167	2·96 ± ·11	3·96 ± ·15
100 H/L	71	3·22 ± ·18	4·65 ± ·26	65	3·48 ± ·21	5·05 ± ·30
100 B/H	73	4·75 ± ·27	4·35 ± ·24	64	3·84 ± ·23	3·52 ± ·21
100 H'/L	115	3·24 ± ·14	4·72 ± ·21	155	3·45 ± ·13	5·08 ± ·20
100 B/H'	117	5·27 ± ·23	4·80 ± ·21	158	5·53 ± ·21	5·00 ± ·19
100 $(B - H')/L$	113	3·59 ± ·16	53·58 ± 3·02	149	3·60 ± ·14	51·43 ± 2·49
100 $G'H/GB$	66	6·24 ± ·37	8·09 ± ·48	56	5·85 ± ·37	7·72 ± ·49
100 NB/NH, R	81	4·32 ± ·23	9·09 ± ·49	77	4·25 ± ·23	8·67 ± ·47
100 NB/NH, L	81	4·38 ± ·23	9·22 ± ·49	76	4·49 ± ·25	9·09 ± ·50
100 O_2/O_1, R	79	6·02 ± ·32	7·43 ± ·40	75	4·42 ± ·24	5·27 ± ·29
100 O_2/O_1, L	85	5·17 ± ·27	6·39 ± ·33	72	5·61 ± ·32	6·78 ± ·38
100 O_2/O_1'	79	6·40 ± ·34	7·45 ± ·40	76	5·07 ± ·28	5·77 ± ·32
100 fmb/fml	106	5·90 ± ·27	7·07 ± ·33	149	5·59 ± ·22	6·65 ± ·26
100 DS/DC	78	9·33 ± ·50	16·06 ± ·89	74	8·19 ± ·45	15·03 ± ·85
100 SS/SC	81	12·75 ± ·68	25·15 ± 1·41	76	10·19 ± ·56	22·84 ± 1·31
100 G_2/G_1	46	5·55 ± ·39	7·04 ± ·50	48	6·69 ± ·46	8·39 ± ·58
100 EH/EB	52	6·88 ± ·46	24·14 ± 1·69	48	7·26 ± ·50	26·02 ± 1·91
$P\angle$	65	3°·15 ± ·19	3·67 ± ·22	57	2°·72 ± ·17	3·18 ± ·20
$N\angle$	64	3°·68 ± ·22	5·73 ± ·34	60	4°·05 ± ·25	6·29 ± ·39
$A\angle$	64	3°·63 ± ·22	4·95 ± ·30	60	3°·89 ± ·24	5·25 ± ·32
$B\angle$	64	3°·65 ± ·22	8·59 ± ·52	60	3°·36 ± ·21	8·12 ± ·50
θ_1	60	3°·32 ± ·20	11·14 ± ·69	53	3°·42 ± ·22	11·40 ± ·76
θ_2	60	3°·16 ± ·19	25·28 ± 1·65	53	3°·08 ± ·20	26·32 ± 1·84
$Oc. I.$	131	2·71 ± ·11	4·67 ± ·20	181	2·45 ± ·09	4·15 ± ·15

of the Moorfields series are significantly more variable, but the Farringdon Street group, male and female, show greater variability for *OH*.

The circumferences and arcs agree very closely.

The variability of *G'H* is very similar in the males and females of the Farringdon Street series, but there is a considerable sexual difference in the Whitechapel. The males are less variable than the Farringdon Street males, but the females are significantly more so.

The divergence in variability for *G'H* in the females is not significant, but in the males the value in the Farringdon Street series is just significantly greater.

The variabilities of the orbital breadths cannot be compared, but the Farringdon Street orbits are more variable in height.

The foraminal lengths of the Farringdon Street and Moorfields series are equally variable, there is a significant difference in the breadths and a marked sexual difference in the Moorfields, but it must be remembered that the Moorfields series is short.

The variabilities of the indices are in very fair agreement.

In the absolute measurements the male has a greater coefficient of variation in 18, the female in 29 characters. In the indices and angles the male has a greater standard deviation in 14, the female in 12 characters. The male is therefore not more variable than the female.

Finally, we may conclude that the variabilities of the different characters are sufficiently close in the Farringdon Street and Whitechapel series to confirm the belief that we are dealing with homogeneous material.

(11) *Comparison of Type Contours*. Transverse, horizontal and sagittal contours were drawn by means of the Klaatsch contour tracer, using the screw-on attachment first adopted by Morant*. The method described by Benington† was followed, with the additions made by later workers in the Biometric Laboratory.

(*a*) *The Transverse Vertical Section*. The method of constructing the mean transverse vertical type contours is described in *Biometrika*, Vol. XIV. p. 227, and the mean measurements from which they were drawn are given in our Table XIII. The right (*R*) and left (*L*) of Table XIII correspond with the right and left of the diagrams which show the transverse contour in Figs. I and II.

The type zone, which is a representation of the limits of variation due to random sampling, is obtained by plotting points at either side of each mean ordinate at a distance from it of twice its Probable Error. It is given by Benington‡ for 100 English skulls as 1·4 mm. at the apex tapering down to 0·8 mm. by the auricular points. The Farringdon Street transverse type contours are based on an average of 75 males and 67 females, and since the width of the type zone is inversely proportional to the square root of the number of skulls, the width of

* *Biometrika*, Vol. XVI. p. 73.

† *Biometrika*, Vol. XI. p. 129.

‡ *Biometrika*, Vol. VIII. pp. 143, 145, 147.

their type zones will be 1·6 mm. at the apex and 0·9 mm. by the auricular points for the male type and 1·7 mm. at the apex with 1·0 mm. at the auricular points for the female type contours. The type zones are not drawn on the accompanying diagrams as it would tend to lack of clearness.

Transverse vertical contours were drawn of the male Whitechapel skulls by Benington, but he did not mark *ZR, R* and *ZR, L*, the points where the zygomatic ridges are crossed, and the auricular points have been joined directly to the extremities of the parallels 2, 3, 4, etc.*, thus giving no indication of the curvature above the auricular point. Also his section shows the frontal view and must be reversed when the tracings are placed over the drawings of the posterior view of the Farringdon Street type.

TABLE XIII.

Mean Values of Transverse Vertical Contours.

Sex	No.	MA	$1R=1L$	$M\frac{1}{4}R$	$M\frac{1}{4}L$	$2R$	$2L$	$3R$	$3L$	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$
♂	75	110·0	58·6	61·5	61·0	$64{\cdot}0^{a}$	62·5	$67{\cdot}6^{b}$	$66{\cdot}0^{c}$	$69{\cdot}5^{d}$	$67{\cdot}7^{e}$	$69{\cdot}5^{a}$	$67{\cdot}4^{c}$	$68{\cdot}1^{c}$	$66{\cdot}3^{c}$
♀	67	103·0	53·9	57·7	56·3	$60{\cdot}1^{g}$	58·7	$63{\cdot}5^{h}$	$61{\cdot}6^{h}$	$65{\cdot}5^{h}$	$63{\cdot}5^{l}$	$65{\cdot}4^{h}$	$63{\cdot}5^{j}$	$63{\cdot}9^{g}$	$61{\cdot}9^{g}$

Sex	No.	$7R$	$7L$	$8R$	$8L$	$9R$	$9L$	$10R$	$10L$	$A\frac{1}{4}R$	$A\frac{1}{4}L$	ZR, R		ZR, L	
												y	x	y	x
♂	75	$65{\cdot}7^{c}$	$63{\cdot}4^{f}$	$61{\cdot}4^{c}$	$58{\cdot}8^{c}$	53·0	50·3	$38{\cdot}7^{f}$	36·3	$19{\cdot}6^{j}$	18·1	61·8	1·9	61·2	3·0
♀	67	$61{\cdot}9^{g}$	59·7	$57{\cdot}4^{g}$	55·7	$49{\cdot}7^{g}$	48·0	$36{\cdot}2^{g}$	34·0	$17{\cdot}7^{g}$	17·1	57·6	2·1	56·5	3·0

a = mean of 71 contours	f = mean of 74 contours
b = ,, 70 ,,	g = ,, 66 ,,
c = ,, 73 ,,	h = ,, 65 ,,
d = ,, 67 ,,	j = ,, 64 ,,
e = ,, 72 ,,	l = ,, 63 ,,

If this superposition is made so that the inter-auricular lines coincide, the two types are found to be very similar. There is a slight tendency on the part of the Farringdon Street contour to come inside the Whitechapel, but it is only significant in the region of the apex. Actually at A the divergence is 1·8 mm. On the left the divergence is 1 mm. at the 7th parallel and increases slightly up to the apex, but nowhere on the right is it as great until $A\frac{1}{4}$, and it is only in these regions that the type zones fall outside one another. On the right side they are almost coincident.

The type contours, then, confirm the difference in height suggested by the mean measurements.

The greatest breadth shown by the contours is 137·2 mm., which is 5·2 mm. less than the mean parietal breadth of the Farringdon Street crania and 3·5 mm.

* *Biometrika*, Vol. XIV. p. 231.

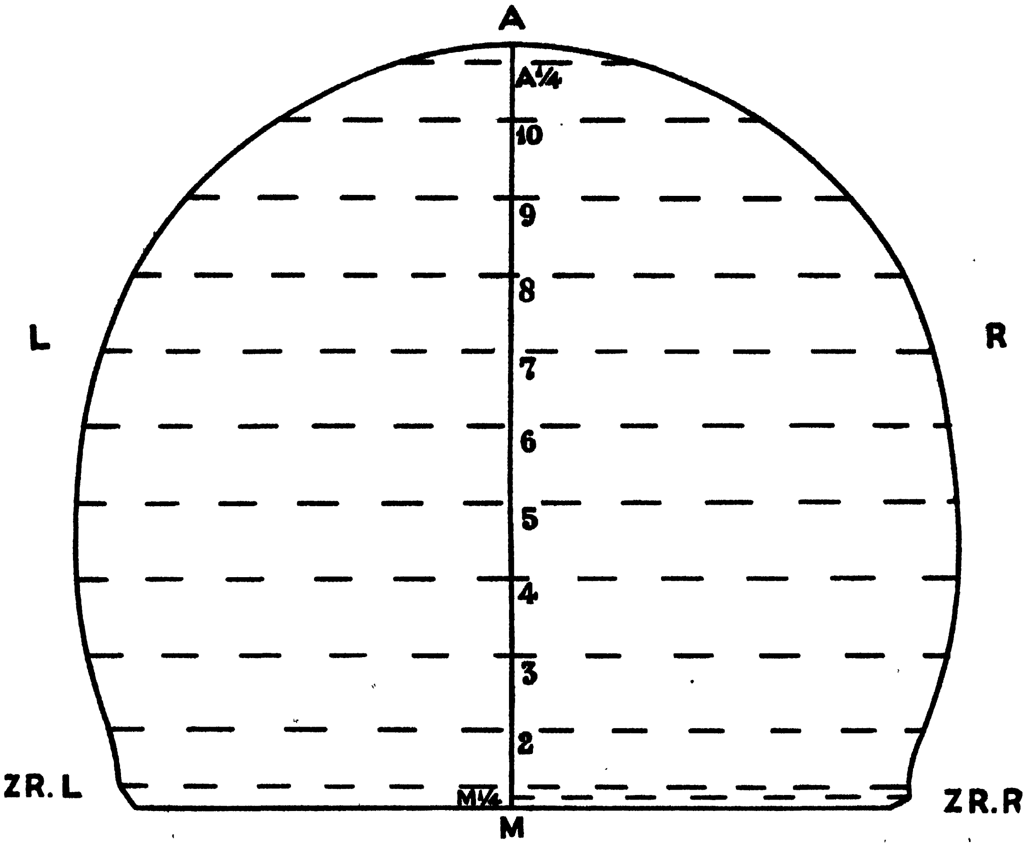


FIG. I 17th Century Londoners ♂ Transverse Type Contour.

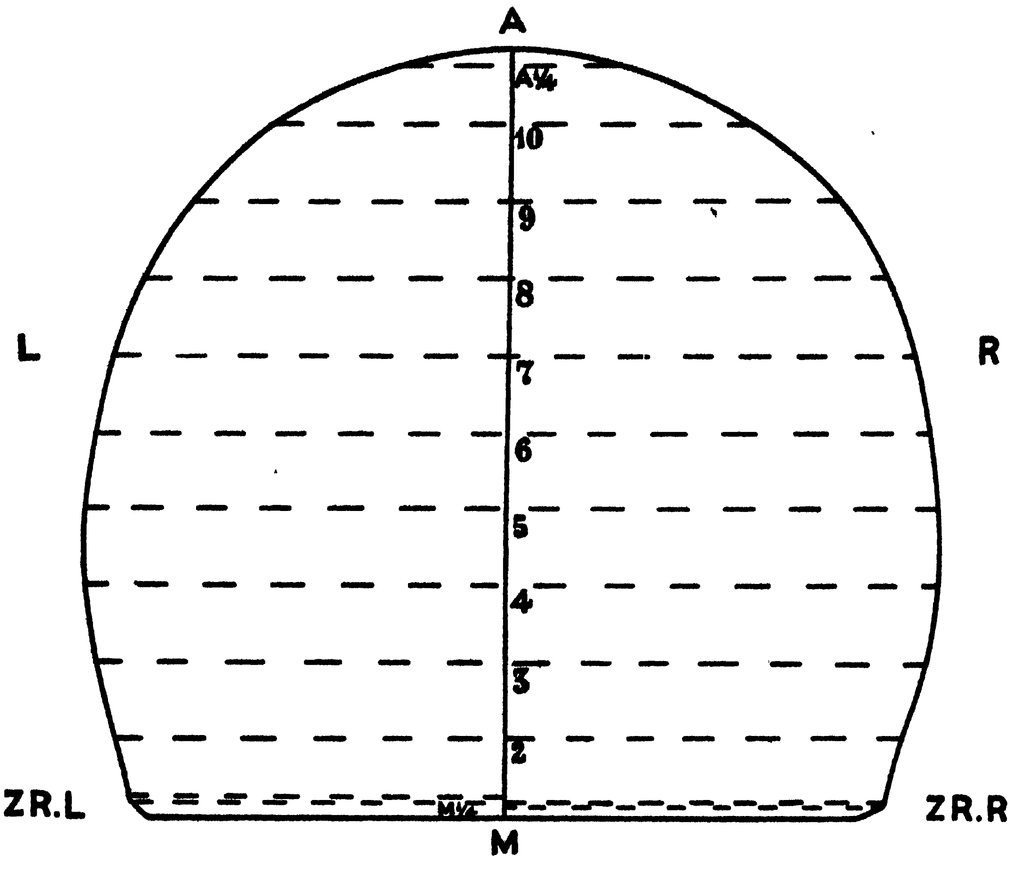


Fig. II 17th Century Londoners ♀ Transverse Type Contour.

less than the Whitechapel. Such a defect is just what we should expect, since the vertical section through the auricular points is usually in front of that through the greatest parietal width. The range of this defect has been observed in other series to be from 3·5 to 6·0 mm.

The auricular height, MA, in both types is identical with the OH of the mean direct measurements, although the A of the individual contours might, as we have drawn MA as the perpendicular bisector of the inter-auricular length, be not the same point as the "apex" to which the OH measurement is taken.

Unfortunately no contours have been drawn of the females of the Whitechapel series.

If the Farringdon Street female type contour is placed on the male, so that the 6th parallels are coincident, it will be observed that the two are very much alike in shape.

The difference between them is merely one of size.

The greatest breadth of the female type contour is 130 mm., that is, 5·7 mm. less than the mean direct measurement.

All three contours show a slightly greater development on the right side, a characteristic more marked in the Farringdon Street than in the Whitechapel series.

Table XIV gives the inter-auricular length and auricular height, together with the index, for the Farringdon Street series. The Whitechapel contour is not drawn into the ear passage, so that an accurate determination of the inter-auricular length is not possible.

TABLE XIV.

Measurements on Transverse Type Section.

	Sex	Inter-auricular Length	Auricular Height	Index
Farringdon Street	♂	117·2	110·0	93·8
" "	♀	107·8	103·0	95·6

(*b*) *The Glabella Horizontal Section.* The method of drawing the horizontal type contour is described in *Biometrika*, Vol. XIV. p. 234; the additional parallels $2\frac{1}{2}R$ and $2\frac{1}{2}L$*, introduced later, were drawn. These enable us to determine the curvature behind the temporal lines more accurately. The mean measurements from which the type contours were constructed are given in Table XV.

The diagrams (Figs. III and IV) show the vertical aspect and their right and left are the right and left of Table XV.

Benington's horizontal type zone for 100 skulls has an approximate width of 1·2 mm. all round. In the Farringdon Street series 72 male and 65 female contours were drawn, so that their type zones should have widths of 1·4 and 1·5 mm. respectively.

* *Biometrika*, Vol. XIV. p. 82.

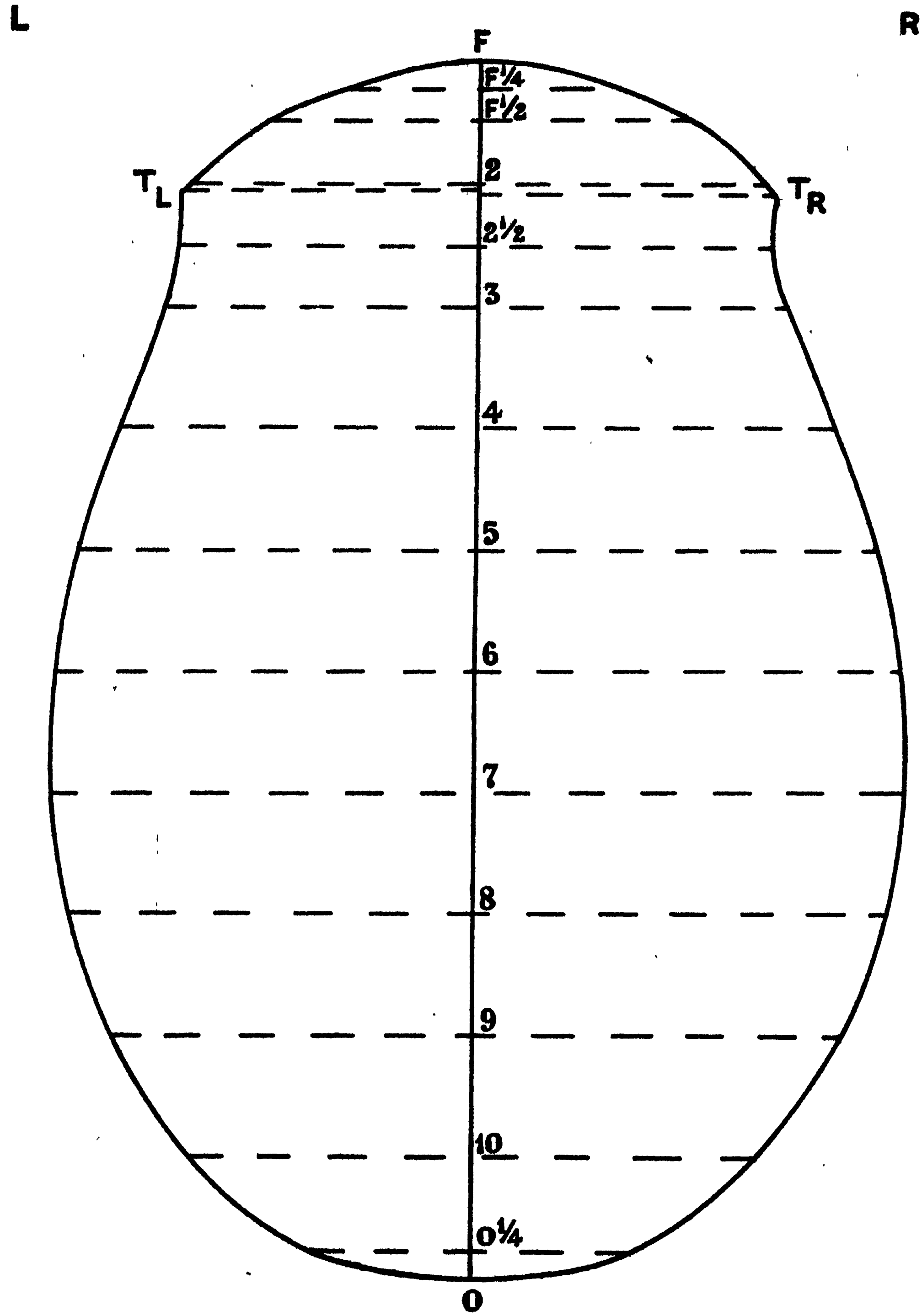


FIG. III 17th Century Londoners ♂ Horizontal Type Contour.

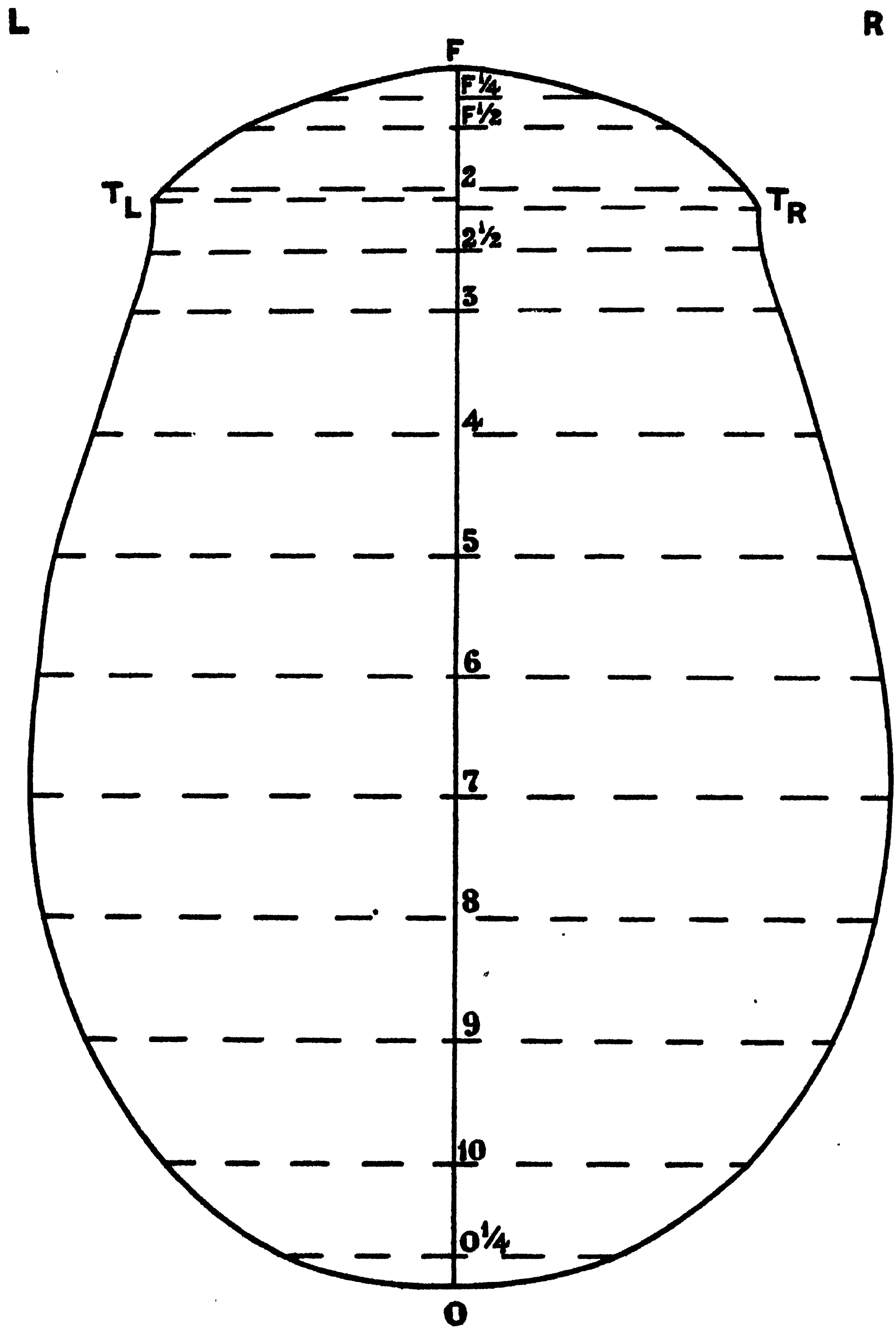


FIG. IV 17th Century Londoners ♀ Horizontal Type Contour.

The points T, R and T, L, where the temporal lines are crossed, were not marked on the Whitechapel horizontal contours, thus giving the frontal region a rounded appearance and making a comparison with that part less valuable.

If the Whitechapel horizontal contour is superposed on the Farringdon Street so that the axes FO and the points F coincide, it will be noticed that the lengths are practically identical. The width from the 5th to the 8th parallels is also the same, and then the Whitechapel curves in more rapidly to the occipital region, this being more pronounced on the right side than on the left. From the 8th

TABLE XV.

Mean Values of Horizontal Contours.

Sex	No.	FO	$F\frac{1}{4}R$	$F\frac{1}{4}L$	$F\frac{1}{2}R$	$F\frac{1}{2}L$	$2R$	$2L$	$2\frac{1}{2}R$	$2\frac{1}{2}L$	$3R$	$3L$
♂	72	185·8	22·6	21·6	35·4	34·6	46·5	47·0	47·7	49·1[a]	50·7[b]	51·2[c]
♀	65	174·8	22·1	21·4	33·3	33·1	44·3	44·9	46·5	46·9	49·4[h]	50·0[k]

Sex	No.	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$	$7R$	$7L$	$8R$	$8L$	$9R$
♂	72	58·6[d]	58·4[c]	65·9[e]	65·0[c]	69·8[f]	68·4[g]	70·5[b]	69·2[g]	67·9[a]	66·4	60·6[a]
♀	65	55·5[h]	55·7[l]	61·0[k]	61·7[l]	65·3[h]	64·0[h]	66·4	65·3[k]	64·1	62·9	57·7

Sex	No.	$9L$	$10R$	$10L$	$O\frac{1}{2}R$	$O\frac{1}{2}L$	T, R		T, L	
							x	y	x	y
♂	72	59·6	46·7	46·4	26·2	27·2	20·4	48·9	19·9	48·7
♀	65	56·6	44·8[h]	44·6	24·8	25·5	20·6	46·2	18·7	46·4

a = mean of 71 contours	f = mean of 67 contours
b = " 69 "	g = " 70 "
c = " 68 "	h = " 64 "
d = " 66 "	k = " 63 "
e = " 65 "	l = " 62 "

parallel to the 10th on the right and at the 10th on the left, the type zones fail to overlap. This confirms our deductions from the mean direct measurements, namely, that the parietal width is greater in the Farringdon Street series.

In front of the 5th parallels the Farringdon Street type narrows more rapidly than the Whitechapel and at the 3rd parallels the divergence is 4·5 mm. on the right and 3·5 mm. on the left.

In front of the temporal lines, the coincidence is again marked.

The differences between the two series are more clearly indicated in these contours than in the transverse vertical type.

The female type may be compared with the male as before, by placing the tracing over the drawing so that the *FO* axes and the 6th parallels are made to coincide, and again the marked similarity in shape is noted.

If the ordinates of the points where the temporal lines are crossed are superposed as nearly as possible, it will be observed that the curvatures of the foreheads are very much the same.

The Temporal Indices and Indices of Frontal Flattening are given in Tables XVI and XVII.

TABLE XVI.

Measurements on Horizontal Type Section.

	Sex	Temporal Index	$100 \frac{\text{Ordinate 3}}{\text{Minimum Forehead Breadth}}$
Whitechapel ...	♂	58·8	111·6
Farringdon Street	♂	54·8	104·1
	♀	56·8	106·9

The temporal index is formed by expressing the total length of ordinate 3 as a percentage of the length *FO*. The Whitechapel series are better developed in the fronto-temporal regions, and the females of the Farringdon Street series more so than the males.

TABLE XVII.

Index of Frontal Flattening of Farringdon Street Crania.

	$\frac{1}{2}[TR(x)+TL(x)]$	Length of Section (FO)	$\frac{100 \times \frac{1}{2}[TR(x)+TL(x)]}{FO}$
Males	$\frac{1}{2}(20{\cdot}4+19{\cdot}9)=20{\cdot}15$	185·8	10·8
Females	$\frac{1}{2}(20{\cdot}6+18{\cdot}7)=19{\cdot}65$	174·8	11·2

(*c*) *The Sagittal or Median Section.* The construction of the sagittal type contour is given in *Biometrika*, Vol. XIV. pp. 239 and 240. The actual measurements used are given in Table XVIII. The abbreviations used in the table and diagrams are:

N = Nasion. G = Glabella. *Bas.* or B = Basion. β = Bregma. V = Vertex. A or *Alv.* = Alveolar point. *Sub-Orb.* = Left infra-orbital point. *Aur.* = Auricular point. I = Inion. *Op.* = Opisthion. *Sp.* = Sphenoidal point, i.e. the point of intersection of the median plane and the suture between the sphenoid and basioccipital bones. P = point of intersection of the palatine sutures. P' = Extremity of *spina nasalis posterior*. NS = Extremity of anterior nasal spine. L = Tip of nasal bone. L' = Point at which NL first meets the outline of the nasal bone.

Benington's Whitechapel section has a type zone of 1·6 mm. from the glabella to λ tapering to 0·7 mm. at gamma and 0·9 at the nasion. This section was only

TABLE XVIII. *Means of Contour Measurements used in plotting Sagittal Type Contours.*

Sex	No.	$N\gamma$	Ordinates above $N\gamma$													
			$O=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$	$\gamma=10$
♂	71	184·3	20·0	33·0	53·6	66·2	74·5	78·9	80·2	80·6	77·6	68·7	52·5	23·6	14·9	4·3
♀	64	174·9	17·2	32·4	50·6[b]	61·5[b]	68·3[b]	72·1[b]	74·5	75·5[b]	72·6	65·0	47·2	22·0	15·7	5·1

Sex	No.	Ordinates below $N\gamma$								Vertex		β		Glabella		Occipital Point		λ		Sub-Orb.	
		$O=N$	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from N	y	x from N	y	x from N	y	x from γ	y	x from γ	y	x from N	y
♂	71	64·4[a]	63·8[d]	56·6[f]	54·2[g]	48·2	40·0	31·3	21·6	102·1	81·5	79·3	79·5	2·6	9·8	0·4	1·2	5·7	24·3	10·0	30·3
♀	64	59·6[c]	57·2[e]	52·1[g]	50·4[h]	45·8	36·7	28·1	21·0	97·3[b]	76·0[b]	75·8	73·0	1·1	10·9	0·3	0·9	4·5	22·0	8·3	29·1

Sex	No.	Aur.		Op.		Inion		Basion		Alv. Pt.		Nose					Palate			
															Max. Sub.		P'		P	
		x from γ	y	x from γ	y	x from γ	y	from N	from γ	from N	from Bas.	NL	NL'	$\angle LN\gamma$	x from N	y	x from N	y	x from A	y
♂	71	96·8	30·1	61·8	53·5	18·1	41·6	99·6[k]	109·7[k]	70·8[l]	93·3[p]	24·2[r]	21·3[r]	124°·8[r]	3·0[r]	9·3[r]	46·3[s]	53·0[s]	15·6[m]	34·3[m]
♀	64	92·0	29·1	58·4[a]	52·4[a]	18·0[a]	37·9[a]	94·3[a]	104·0[a]	65·7[m]	88·6[q]	21·3[s]	19·6[s]	122°·2[s]	2·8[s]	8·8[s]	41·2[t]	49·3[t]	32·9[m]	14·6[m]

Sex	No.	Frontal Max. Sub. to $N\beta$		Occipital Max. Sub. to λ Op.		Max. Sub. to $N\lambda$		Max. Sub. to GI		Sp.		Sub. from ½ Bas. Sp. Chord		N. S.		Max. Sub. of Alv. N. S. Arc	
		x from N	y	x from λ	y	x from N	y	x from G	y	x from N	y	x from Bas.	y	x from N	y	x from N	y
♂	71	46·7[k]	25·5[k]	50·1	30·1	86·5	68·6	105·4	103·6	65·4[k]	32·7[k]	12·9[k]	1·1[k]	6·1[l]	51·6[l]	3·1[l]	57·8[l]
♀	64	42·7[b]	24·9[b]	42·4[a]	26·5[a]	83·1[a]	63·9[a]	99·9[a]	96·2[a]	62·0[a]	32·8[a]	11·4[a]	0·6[a]	6·0[p]	47·5[p]	3·7[v]	52·9[b]

a = mean of 62 contours; b = " 63 "; c = " 60 "; d = " 67 "
e = mean of 59 contours; f = " 69 "; g = " 61 "; h = " 57 "
k = mean of 70 contours; l = " 54 "; m = " 52 "
p = mean of 53 contours; q = " 50 "; r = " 16 "
s = mean of 26 contours; t = " 56 "; v = " 58 "

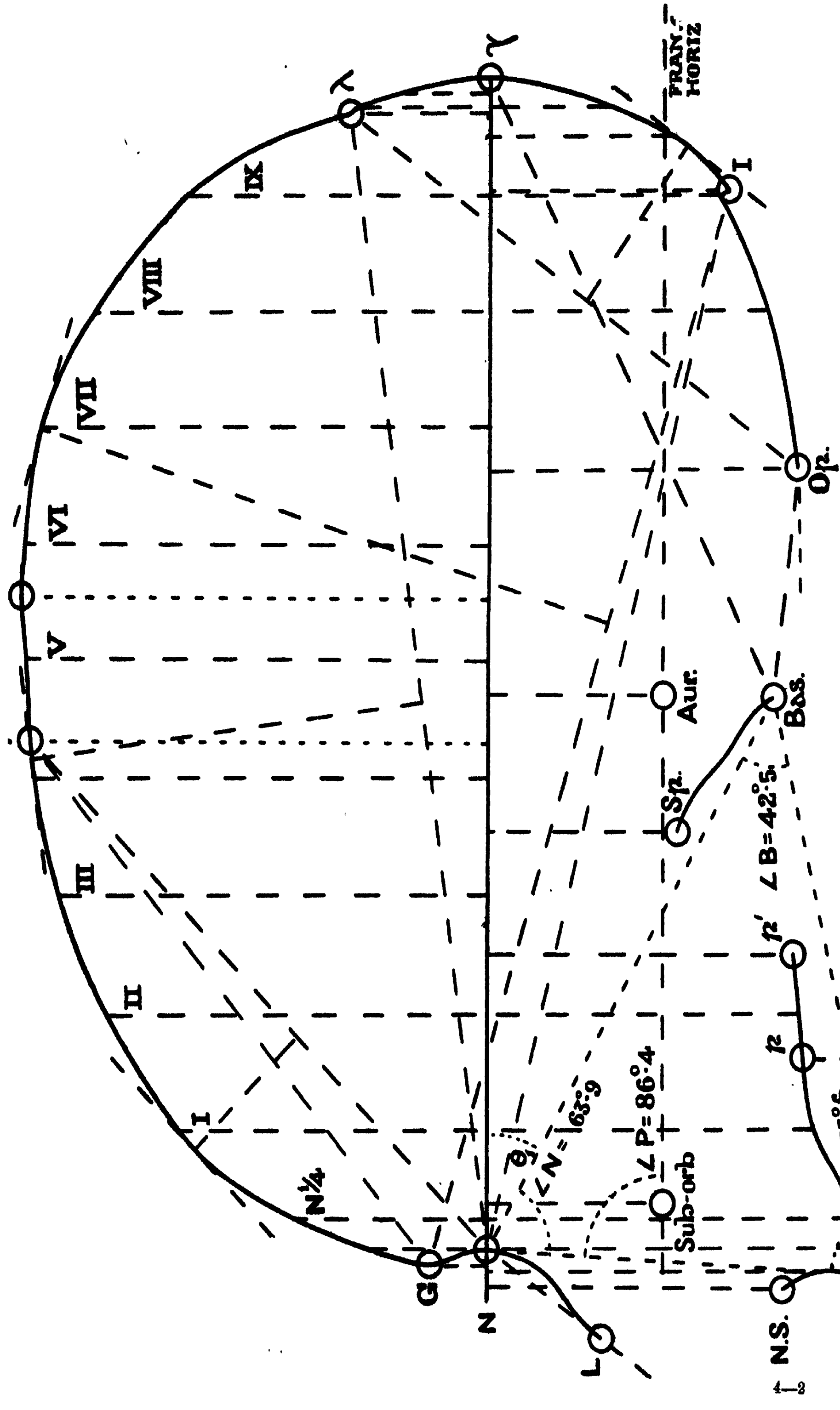


FIG. V 17th Century Londoners ♂ Sagittal Type Contour.

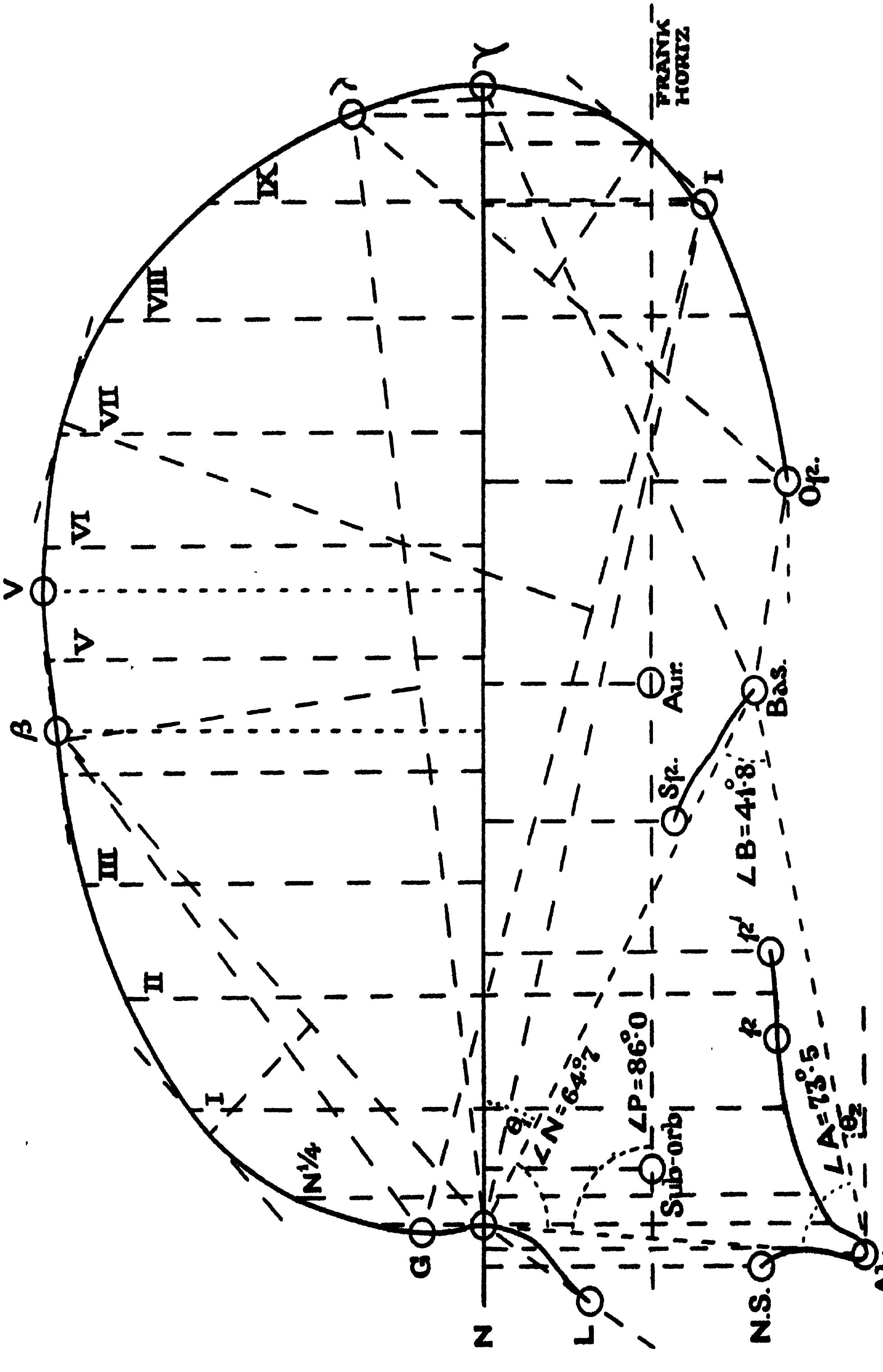


FIG. VI 17th Century Londoners ♀ Sagittal Type Contour

drawn from the nasion to the gamma. The widths for the Farringdon Street type zones for 71 males and 64 females are tabled below.

Width of Type Zone in mm.

Sex	Nasion	Vault	Gamma
♂	1·1	1·9	0·8
♀	1·1	2·0	0·9

From the sagittal section we see that the Farringdon Street crania are characterised by rather low, receding forehead, a distinct glabella, well-developed occiputs, and pronounced inions.

If the Whitechapel type contour be superposed on the Farringdon Street so that the $N\gamma$ axes coincide, the equality in length observed in the horizontal contours is again evident, and the discrepancies in height indicated by the transverse vertical type are also confirmed. The bregma of the Whitechapel type is slightly in advance of the Farringdon Street β, and both are between the 4th and 5th parallels, rather nearer to the 4th as has been observed in other cranial types. The vertex of the Whitechapel contour is directly above that of the Farringdon Street, both being slightly nearer to the 6th than to the 5th parallel.

In the frontal area the two types are identical, the Whitechapel beginning to rise above the Farringdon Street at the parallel $N\frac{1}{4}$.

The type zones just fail to overlap from the 3rd to the 8th parallels, the divergence being somewhat greater behind the vertex.

While the tracings are in this position, the distance of the Whitechapel bregma from the Farringdon Street basion is 132·8 mm. The mean value of the direct measurements give the Whitechapel crania a basio-bregmatic height of 132·7. Again the length from nasion to basion is 100 mm. for the Farringdon Street and 101·6 mm. for the Whitechapel. This suggests that the basion of the Whitechapel series would be on a level with and slightly behind that of the Farringdon Street. The height of the Whitechapel vertex above the sub-orbital-auricular line is 114·5 mm. as compared with 112·1 mm., the mean of the direct measurements. We may suppose, then, that the corresponding line on the Whitechapel type contour would lie from 2 to 3 mm. higher than the same line on the Farringdon Street sagittal section.

If now the tracing is turned so that the $N\beta$ lines coincide, the similarity between the two contours is even more striking. Not only are N and β similarly placed on both types but it will be observed that the λ of the Whitechapel type is extremely close to the λ of the Farringdon Street.

The sagittal arcs were not measured in the former series, but we may safely conclude that S_1, S_2, S_1' and S_2' do not differ significantly in the two series. The mean values of the complete sagittal arcs are 378·8 mm. for the Farringdon Street

and 377·1 mm. for the Whitechapel crania, so we may infer that the arc from λ to opisthion is slightly greater in the Farringdon Street crania.

A discrepancy is noted in the relative positions of γ, the difference between them being 6 mm.

It is very unfortunate that a comparison of the occipital regions and of the palates cannot be made.

When the contours are placed with the $N\beta$ line as base, there is no portion which is not covered by the type zone.

The female sagittal type contour shows the same low receding forehead exhibited by the male. They can be compared by placing the tracing of the female type over the drawing of the male, with the $N\gamma$ line as base and 4·7 mm. between the two nasions, since the total difference in length is 9·4 mm., and as in the other sections the two are seen to be very similar in shape.

The male is more developed immediately in front of the bregma and also in the occipital region, as would be expected. The glabella of the female section is less marked than is the case in the male.

The palates, and indeed the whole of the contours from NS to P', are extremely alike in shape and size, the palate length of the female being a little shorter.

In Table XIX we give the physiognomic angle of flatness of the three types. The values are very close and are low compared with other races for which this measurement has been taken.

TABLE XIX.

	Sex	Angle of Frontal Bone Flatness	Bregmatic Angle	Physiognomic Angle of Flatness
Farringdon Street	♂	28·9	45·2	74·1
" "	♀	30·5	44·0	74·5
Whitechapel ...	♂	26·6	47·1	73·7

Conclusion.

The characteristics exhibited by the Farringdon Street crania confirm those already attributed by Dr Macdonell to the Londoner of the 17th century, as we might have expected from a knowledge of the history of the skulls.

All three series are remarkable for their low retreating foreheads, the Farringdon Street crania having even lower vaults than the Whitechapel, and in this respect they are farther removed from the Long Barrow skulls to which Macdonell traced resemblances in the series he investigated.

In Scotland we find somewhat the same type reproduced in the Lowland Scottish crania, more closely, indeed, than in the mediaeval English skulls as represented by the Hythe and Rothwell series, both of which diverge markedly from the London series.

To find the predecessors of our type we must go back to the early invaders of this country, the people of the Iron Age period, rather than to the Anglo-Saxons*. A discussion of these relationships is given elsewhere in the present issue of *Biometrika*.

I should like here to acknowledge my thanks to Mr E. S. Pearson for photographing the crania.

DESCRIPTION OF PLATES.

Plate I.	*FA* 44.	Wormian bone (? Ossicle of Asterion).
,, II.	*FA* 89.	Styliform process on temporal squama.
,, III.	*FA* 136.	Lower occipital flattening.
,, IV.	*FA* 114.	Partially healed injury on right frontal bone.
,, V.	*FA* 264.	Transverse occipital groove.
,, VI (*a*).	*FA* 1.	Anomalous process on Pterygoid plate.
,, VI (*b*).	*FA* 88.	Porus crotaphitico-buccinatorius divided into two externally.

[* Professor F. G. Parsons at the Cardiff Meeting of the British Association asserted that Macdonell's association of the Whitechapel crania with the Long Barrow crania of Schuster was wholly unjustifiable. He has since stressed the point again in a paper in the *Journal of the Royal Anthropological Institute*, Vol. LI. p. 55 *et seq*. No reply was possible until a series of Anglo-Saxon crania had been measured by biometric workers. The following series of Coefficients of Racial Likeness will speak for themselves:

Whitechapel and Schuster's Long Barrow	$3{\cdot}71 \pm {\cdot}18$ (28).
Farringdon Street and Schuster's Long Barrow	$5{\cdot}28 \pm {\cdot}18$ (27).
Farringdon Street and Whitechapel from Table I	$4{\cdot}15 \pm {\cdot}18$ (27).
Anglo-Saxon and Farringdon Street	$5{\cdot}27 \pm {\cdot}18$ (30).
Anglo-Saxon and Whitechapel	$2{\cdot}98 \pm {\cdot}18$ (27).
Anglo-Saxon and Moorfields	$4{\cdot}88 \pm {\cdot}20$ (25).
Anglo-Saxon and Parsons' Hythe crania	$72{\cdot}53 \pm {\cdot}32$ (8).
Anglo-Saxon and Parsons' Rothwell crania	$10{\cdot}33 \pm {\cdot}32$ (8).

From these results it appears that whatever the Hythe and Rothwell crania represent their measurements are not Anglo-Saxon, and that the Whitechapel and Schuster's Long Barrow skulls are as near to each other as the Anglo-Saxons to Farringdon Street, Whitechapel and Moorfields crania or indeed as the Whitechapel to the Farringdon Street crania. The general result seems to be that Schuster's series (Mean C.R.L. $= 4{\cdot}49$) is as closely related as the Anglo-Saxons (Mean C.R.L. $= 4{\cdot}37$) to the Londoners measured in the Biometric Laboratory. Where to place the Hythe and Rothwell crania measured by Professor Parsons, it is impossible to say. ED.]

A FIRST STUDY OF THE CRANIOLOGY OF ENGLAND AND SCOTLAND FROM NEOLITHIC TO EARLY HISTORIC TIMES, WITH SPECIAL REFERENCE TO THE ANGLO-SAXON SKULLS IN LONDON MUSEUMS.

By G. M. MORANT, M.Sc.

CONTENTS.

(1) *Introduction.*

The material available for a biometric study of British ethnology is miserably scanty. From the early papers of Macdonell and the recent one of Hooke on the skull, and the classical memoir of Pearson and Bell on the femur, the physical type of 17th century Londoners is sufficiently well known, but data for earlier periods and other localities are, for statistical purposes, almost wholly inadequate. The present paper partly fills that gap by providing full individual measurements and type contours of all the Anglo-Saxon skulls in the British Museum (Natural History), the Museum of the Royal College of Surgeons and the London Museum. The writer acknowledges his great indebtedness to the curators of those collections for freely giving him permission to study the material in their charge. An attempt has also been made to compile the mean measurements of racial types extant in Britain in earlier periods, but, in most cases, both the number of measurements provided and the number of skulls dealt with are too few to provide a definitive determination of the single types. The present study, then, is essentially nothing more than a first contribution. For no single one of the populations dealt with is the material plentiful enough to furnish more than crude first approximations to the statistical constants. The method of the Coefficient of Racial Likeness,

first suggested by Professor Karl Pearson*, has been found an indispensable aid. To all the populations dealt with the following conditions were applied:

(*a*) That the distributions of characters were fitted reasonably well by normal curves;

(*b*) That the sexual differences between the ♂ and ♀ mean measurements were of the same order as those of a standard population; and

(*c*) That the standard deviations of characters were of the same order as those of that population.

The standard population with which comparison has been made is that of the long series of 17th century skulls found in a single pit in Whitechapel and now preserved in the Biometric Laboratory. They were sexed by Sir George Thane and Professor Pearson and fulfilled all the statistical conditions demanded from a homogeneous population.

The following are the main conclusions which the available material suggests, but they may be substantially modified by later and more comprehensive studies.

(i) The population of England and Scotland in late Neolithic times was racially homogeneous. The skull has a peculiarly great length, a low cephalic index and an average height.

(ii) The invading Bronze Age people were of a markedly contrasted brachycephalic type and the English and Scottish populations of that period known to us are similar, but not identical. Some skulls of the pure Neolithic type have been found associated with the typical Bronze Age artifacts, and there appears to have been a certain amount of intermixture between the two races, but the invading one easily predominated. The earlier type was probably extinct before the end of the Bronze Age.

(iii) An entirely different form of skull is found in Iron Age settlements. It is just dolichocephalic and is above all characterised by its low calvarial height. Crania of that type, representing a homogeneous population, have been found in numerous scattered English sites and in the Lowlands of Scotland, and there was apparently no modification of that population until at least as late as the close of the Roman era. For the Iron Age period there are no recorded skeletal remains which conform to the type of one or other of the two earlier peoples, so we assume that the latter were either entirely exterminated or that remnants of them sought refuge in the more inaccessible parts of the country.

(iv) All the Anglo-Saxon skulls dealt with in this paper can be dated between the 5th and early 10th centuries. They, again, form a perfectly homogeneous population, and the type is clearly distinguished from that of the British Iron Age by its greater calvarial height, though the lengths, breadths and cephalic indices of the two are almost identical. The ♀ Anglo-Saxon skulls are of precisely the same types as the ♂, suggesting that the earlier and later invaders lived side by side without intermixture for some centuries. From other evidence we know that the former were not exterminated, but that they were, in all probability, far more numerous than the Anglo-Saxons during that period.

* See *Biometrika*, Vol. XVI. 1924, pp. 11—14.

The question of the relationships of these four racial types to those of the populations found in Britain in later historic times is discussed in another paper in this issue.

(2) *English and Scottish Skulls of Neolithic and Bronze Age Date.*

It is now generally recognised that no enquiry into the origin of the present day populations of England and Wales can be complete which does not take cognisance of the races that occupied the countries in late prehistoric times. The extent, if any, to which those earlier populations have participated in the make-up of existing types is a matter which has frequently been discussed without leading to any unanimity of opinion and the lack of agreement has evidently been due to a large extent to a lack of adequate descriptions of sufficiently large numbers of the skeletal remains of the people concerned. The accumulation of archaeological evidence within the last hundred years has resulted in a reliable classification of the sequences of culture that followed Palaeolithic times. The vast majority of the late prehistoric skulls found in England were interred in tumuli which are generally classified as either long or round barrows. In early Neolithic times the dead were not covered by tumuli and few human remains of that period have been preserved. The long barrows are the characteristic erections of the late Neolithic period. In general they are found in isolated positions—being in that respect unlike the Bronze Age barrows which were almost invariably grouped in cemeteries—and the majority of them appear to be analogous to family vaults. It is usual to find in the central chamber the bodies of a number of people who did not die at the same time. Inhumation was the invariable form of disposal of the dead. The different varieties in structure of these long barrows are not generally supposed to have any chronological or racial significance. The stone chambered tumuli found plentifully in Gloucestershire and North Wiltshire are not essentially different from the unchambered barrows of Yorkshire and Derbyshire; the different form in the south being merely occasioned, it is supposed, by the presence of suitable large blocks of stone. Towards the end of the Neolithic period an alien race invaded England, arriving on the east coast and gradually spreading westwards. Their artifacts and funerary customs were distinctly different from those of the autochthonous population. The invaders covered their dead with round barrows which are found clustered together in graveyards. Each mound was intended to cover a single body. Cremation was common. Either on their first appearance in this country or very shortly afterwards, the alien people introduced bronze and a characteristic pottery of the so-called "beaker" type. These Bronze Age people buried more objects with their dead than the Neolithic people they supplanted, but a number of round barrows have been found containing no artifacts and hence it is impossible to arrange the available skeletal material in any approach to a chronological sequence. As early as 1863 Thurnam (2, p. 158) made the statement: "Long barrows, long skulls; round barrows, round or short skulls," and our present purpose is to test the truth of that statement in the light of all the available material; to isolate the racial types represented and to obtain, if possible, the mean cranial measurements of the homogeneous components.

In the earlier descriptions of the contents of barrows the use of the terms primary and secondary interments often led to a slight confusion. Following Windle (1), the term "primary interment" will be used in the present paper to designate all burials of the same class although they may not be absolutely contemporaneous, while burials in the same barrow but of a different class will be called "alien interments." Thus, in general, the primary interment in a long barrow consists of several bodies buried at different times, though probably all within a comparatively few years, while the primary interment in a round barrow is a single body.

Measurements of skulls of Neolithic Age found in England have been given by the following writers: Thurnam (2 and 3), Davis and Thurnam* (4, Tables I and II), Rolleston (5), Schuster (6, Table I), Pitt-Rivers (8 and 9), Garson (10 and 11) and Keith (7). From these sources all skulls were selected which were undoubtedly primary interments in Neolithic long barrows and great care was taken to exclude all that were alien, or possibly alien, interments in such barrows. All accepted were barrow skulls, except those described by the last writer, which had been found in a megalithic monument at Coldrum in Kent, which was possibly of early Neolithic date. The others would, according to the general accepted chronology, be late Neolithic. Schuster (6, Table I) has given measurements of 37 long barrow skulls contained in the Museum of Anatomy at Oxford. One of them (No. 3) appears to be that described by Canon Greenwell (5, p. 485) as an alien interment, and 12 from a long barrow at Crawley in Oxfordshire are certainly not Neolithic. Bronze artifacts were found with the primary interments (see Ackerman, *Archaeologia*, Vol. XXXVII. p. 432, 1857). The mean cephalic index of the Crawley skulls is 79·3. The long barrow is probably of Bronze Age date, its occurrence in that period being quite exceptional.

The geographical distribution of the available 144 skulls of Neolithic Age is: Wiltshire 48, Gloucestershire 25, Dorsetshire 5, Staffordshire 47, Derbyshire 3, Yorkshire 9, Kent 7. Neolithic long barrows have also been found in small numbers in Somersetshire, Hampshire, Oxfordshire, Westmorland, Cumberland and Durham, but no measurements at all of skulls from those counties appear to have been given. The settled habitations of the people were only scattered over a relatively small part of the whole of England. Unfortunately very few measurements have been given even for the majority of the skulls described, and many have to be rejected because they are insufficiently defined. The standard deviations of all the ♂ skulls available are:

	F†	B	$100\,B/F$	U
Standard Deviation	$6·50 \pm ·34$	$4·85 \pm ·25$	$3·07 \pm ·16$	$14·16 \pm ·82$
Number of Skulls	85	85	80	67

* Measurements of several of the Neolithic skulls in the *Crania Britannica* had previously been given by Thurnam (2).

† The letters used to indicate skull measurements in this paper are identical with those given by Miss Hooke in her memoir in this issue of *Biometrika*, see pp. 15—16.

Judged by these characters, the variability of the Neolithic English population is no greater than that of the Whitechapel English series of 17th century Londoners (cf. Table XIV), so we may provisionally assume that it is racially homogeneous.

The evidence relating to the earliest remains of man in Scotland has been comprehensively dealt with by Sir William Turner (12). The suggested evidences of Palaeolithic man's habitation in that country, earlier than Azilian times, have not been universally accepted, but remains of Neolithic date have been found widely dispersed and as far north as Caithness and the Orkney Islands. The culture of the more northernly people was not identical with that of the contemporary English population, but the differences were apparently only occasioned by local conditions, and, from such evidence, we have no reason to believe that the peoples in the two countries were not of the same race. Turner (12) gives measurements of 10 Neolithic skulls in Table I, and 2 in Table VI. Two other skulls (Skerrabrae) had previously been dealt with by Garson (13)*. The geographical distribution of these crania is: Caithness 1, Orkney Islands 4, Argyll 2, Arran 5†, Fife 1 and Midlothian 1. It is a noteworthy fact that all were found close to the sea.

Considering the ♂ series, we find Coefficients of Racial Likeness between the English (20·4) and Scottish (7·1) Neolithic populations of $-0{\cdot}12 \pm {\cdot}23$ for 16 characters and $-0{\cdot}44 \pm {\cdot}52$ for 3 indices‡. Low C.R.L.'s are to be expected when such small numbers are being dealt with, but in this case we may feel tolerably certain that the two samples represent the same race §. Combining the two series we have a population with the following standard deviations and it will be seen that the addition of the Scottish skulls does not sensibly affect the variability of the English group.

	F	B	$100\,B/F$	U
Standard Deviation	$6{\cdot}28 \pm {\cdot}32$	$4{\cdot}90 \pm {\cdot}23$	$3{\cdot}22 \pm {\cdot}17$	$14{\cdot}46 \pm {\cdot}79$
Number of Skulls	88	97	83	75

* The skeletons discovered by Laing in Caithness and described by Huxley (14) were thought at that time to be of Neolithic date but they are now attributed to a later period.

† Some measurements of these 5 Neolithic skulls from Arran had been given by Bryce (23) before Turner (12).

‡ The following are the more reliable ♂ mean measurements that can be compared:

	$100\,B/L$	L	B	B'	S	U	H'	NB	O_2
English skulls of Neolithic date	71·7 (17)	193·3 (17)	138·6 (86)	98·6 (13)	395·6 (33)	536·7 (67)	133·5 (7)	23·0 (12)	32·5 (10)
Scottish skulls of Neolithic date	72·7 (11)	192·7 (11)	140·1 (11)	98·6 (7)	381·7 (6)	535·0 (8)	134·4 (6)	23·3 (7)	32·7 (7)

§ In calculating all the Coefficients of Racial Likeness given in the present paper, the standard deviations of the long Egyptian E series (*Biometrika*, Vol. xvi. 1924, p. 338) were used. They are nearly all smaller than those of the Whitechapel English and of the Farringdon Street English, so the C.R.L.'s are probably rather greater than they ought to be. The Farringdon Street standard deviations could now be more appropriately used when dealing with British material.

Fitting the combined English and Scottish distributions of characters for Neolithic skulls with normal curves gives for goodness of fit:

	F	B	$100\,B/F$
Probability P ...	·513	·870	·541
Number of Groups*	11	9	8
Number of Skulls	88	97	83

Judging from the statistical constants derived from the distributions of skull measurements, we are justified in concluding that a single homogeneous race inhabited England and Scotland in late Neolithic times.

Owing to the common practice of cremation, the number of measured Bronze Age English skulls is hardly greater than that of those of Neolithic date, although round barrows are far more numerous than long barrows. The sources for the later period are: Schuster (6, Tables II, III and IV), Wright (15), Thurnam (2, Tables I and II, and 3, Tables I and II), Davis and Thurnam (4, Tables I and II), Pitt-Rivers (9, Tables following pp. 26 and 50), Garson (11, Wor Barrow and Handley Hill barrows other than Wor Barrow), Horton-Smith (16, Table II), Garson (17, Table facing p. 20), and Rolleston (5 and 18, Vol. I. p. 453). The skulls which could safely be considered to be of Bronze Age date were:

(1) primary interments in Bronze Age round barrows, possibly not associated with bronze or pottery,

(2) alien interments in long barrows associated with Bronze Age artifacts, and

(3) a few other interments associated with Bronze Age artifacts, and not covered by barrows.

In collecting the measurements all specimens of doubtful age were excluded and no attention was paid to cranial characters. By far the greater number of specimens had been found as primary interments in round barrows. The geographical distribution of the skulls selected in this way is: Yorkshire 157, Derbyshire 44, Wiltshire 34, Staffordshire 13, Dorsetshire 4, Gloucestershire 4, Northumberland 2, Cumberland 1 and Westmorland 1. As in earlier times, the forests of the Midlands and of the Weald of Surrey, Sussex and Kent appear to have been unpopulated. The three main centres of population in both Neolithic and Bronze Age times were Wiltshire and Gloucestershire in the south, Yorkshire in the north and Derbyshire and Staffordshire†. It is clear that all the skulls of Bronze Age date

* When possible it is well to divide a distribution into about 20 groups when testing the goodness of fit of a curve, but in the present case a smaller number had to be taken as many measurements had been given in tenths of inches, which led to an artificial grouping when they were converted into mm. With a small population the values of P may vary considerably according as the limits of the groups are changed. The values given above are higher than for some other groupings that were tried, but it is quite possible that they could be raised still more by re-adjusting the limits.

† They correspond closely with the districts where "beakers" have been found most plentifully (see Abercromby's map in *Journal of the Royal Anthropological Institute*, Vol. XXXII. 1902, facing p. 396).

do not belong to the same race. The standard deviations of the distributions of characters for ♂ skulls are:

	F	B	$100\,B/F$	U	$G'H$
Bronze Age Standard Deviation	7·17 ± ·28	7·48 ± ·32	5·42 ± ·21	15·79 ± ·62	4·02 ± ·26
Bronze Age No. of Skulls ...	148	156	151	146	56
Bronze Age σ / Whitechapel English σ	1·162 ± ·061	1·417 ± ·043	1·663 ± ·077	1·051 ± ·059	1·041 ± ·089

The Bronze Age population was undoubtedly not homogeneous and Thurnam's dictum, "Round barrows, round skulls," cannot be accepted without some qualification. As a working hypothesis we may assume that not more than two pure racial elements were concerned in the make-up of the population found associated with objects of Bronze Age date in England. Thus that population will, in all probability, comprise:

(1) a pure Neolithic type element,

(2) a pure Bronze Age type element representing the invading race that introduced bronze and the "beaker" pottery into England, and

(3) a hybrid population resulting from the crossing of (1) and (2).

It is generally supposed that the groups (1) and (3) were small in proportion to (2). Our present purpose is to separate the three groups, if possible, by disintegrating the available distributions of cranial measurements. From the values of the standard deviations it would appear that the pure Neolithic and the pure Bronze Age types differ hardly at all for the characters U and $G'H$; the mean lengths (F) must be rather more distinctive, but it is between the breadths and the cephalic indices ($100\,B/F$) that we may expect to find the most significant differences. The splitting up of the distribution of those indices—shown in Table I—is most likely to give satisfactory results.

TABLE I.

Distribution of the Cephalic Indices ($100\,B/F$) *of Male Bronze Age English Skulls.*

$100\,B/F$	64·5—66·5	66·5—68·5	68·5—70·5	70·5—72·5	72·5—74·5	74·5—76·5	76·5—78·5	78·5—80·5	80·5—82·5	82·5—84·5	84·5—86·5	86·5—88·5	88·5—90·5	90·5—92·5	Total
Frequency	1	4·5	6	8·5	15	14	20·5	21·5	23	12	12	10	2	1	151

The range of the cephalic indices in Table I is almost as great as for all modern races of man. For the constants of the postulated pure Neolithic component of the distribution we may accept the values found for the homogeneous population of English and Scottish skulls of Neolithic date. The mean index was found to be

71·3 and the standard deviation 3·223. Of the skulls of Bronze Age date there are 15 with an index less than 71·3 and provisionally we may make the assumptions: (1) that those 15 skulls are all of pure Neolithic type*, and (2) that in all there are 30 of the Bronze Age skulls of pure Neolithic type. By accepting the constants previously found and supposing that the population is normally distributed, we can now find the distribution of the cephalic indices of those 30 skulls and subtract it from the distribution in Table I. The standard deviation of the remainder is $4·23 \pm ·18$ and that value is higher than that of a homogeneous population owing, as we suppose, to the presence of a small hybrid element added to the pure Bronze Age type. To separate the two we may consider the form of the right end of the curve. Assuming that all skulls with a cephalic index greater than 80·5 are of pure Bronze Age type, we may determine the constants of the normally distributed population of which the distribution to the right of 80·5 is a truncated portion†. The tail is found to represent more than half of a normal curve having a mean of 82·086, a standard deviation of 3·8136 and a total population of 91 individuals. The S.D. is rather higher than that of a homogeneous population, but we may reasonably accept it. This rough analysis of the distribution of the cephalic indices of 151 skulls of Bronze Age date has led to the conclusion that 30 of them are of pure Neolithic type, 91 of pure Bronze Age type and 30 of hybrid descent. A similar treatment applied to the distributions of B and F leads to no reasonable results owing, evidently, to the greater overlapping of the component curves. We shall adopt the divisions given by the cephalic indices. In doing this we are accepting as correct a statement which may be no more than a rough approximation to the truth, but it is well to remember that a statistical analysis of the data would seem to give quite the most reasonable solution of the problem and that no partitioning of a population by statistical methods can give more than a first approximation when the total number of individuals is no greater than 150. It is only our good fortune in finding a character as greatly contrasted for the two types as the mean cephalic indices are that makes such a method of approach possible. It will be shown later that the procedure leads to results which are in every way reasonable when tested by other methods.

Supposing that the total number of skulls in each of the three component groups of the Bronze Age date population to be known, some method of determining to which group each particular skull belongs is yet required before the mean measurements can be found. That distributing of the single skulls was carried out in the following manner. There are only 6 characters (viz. F, B, J, U, S and $100\,B/F$) which are available for all, or nearly all, the skulls dealt with, and the Neolithic means for those characters may be considered known. Taking each of the Bronze Age date skulls in turn, the differences of its 6 measurements, or as many of the 6 as were available, from the mean Neolithic values were divided by

* The following consideration provides some justification for this assumption. For the 15 skulls of Bronze Age date with cephalic indices less than 71·3 the mean B is 134·6 and the mean F is 196·0. For 48 skulls of Neolithic date with indices less than 71·3 the corresponding means are 135·8 and 196·9.

† By the method of Table XI in *Tables for Statisticians and Biometricians* (1914).

the standard deviations* of the characters and the mean of these quantities irrespective of sign—λ say—was calculated. The value of λ is a rough measure of the resemblance of the skull to the mean Neolithic type. Of the skulls for which the index $100\,B/F$ is given we have already considered all having values less than 71·3 to be of the pure Neolithic type. Apart from these there are 15 with values of λ less than 0·90 and of the skulls for which the cephalic index is not given there are 5 with values of λ less than 0·90. So in all there are 35 Bronze Age skulls which may be supposed to conform to the unhybridised Neolithic type. There are 31 individuals having cephalic indices and values of λ greater than 0·89 and less than 1·31 and 6 others with values of λ between the same limits for which $100\,B/F$ is not given. Those 37 may be taken to belong to a hybridised population resulting from the crossing of the two pure types†. All other Bronze Age date skulls are supposed to be of the pure Bronze Age type.

Sir William Turner (12, Tables II, IV and V) has collected together the measurements of a considerable number of Scottish Bronze Age skulls. Very few tumuli precisely similar to the round barrows of England have been found in Scotland. The usual form of burial there in the Bronze Age was in short cists covered by cairns or tumuli, or often unmarked by any mound. But the funerary customs were in other respects very similar and similar artifacts are found in the two countries. In both, inhumation—always with the body in a contracted position —and cremation were practised and each grave was prepared for the reception of a single individual. Measurements of Scottish Bronze Age skulls other than those dealt with by Sir William Turner are given by Barnard Davis (19, pp. 10 to 13), Davis and Thurnam (4, Table II), Busk (21), Garson (22) and Bryce (23). The total number of ♂ and ♀ skulls is 56. Bronze Age interments have been found in all parts of Scotland, though they are far more plentiful on the east side than the west. The distribution of the measured skulls is: Orkneys 18, North-East (Caithness, Sutherlandshire, Elgin, Banff and Aberdeen) 24, South-East (Lothians, Roxburgh and Berwick) 16, North-West (Ross) 1, Hebrides (Bernera and Benbecula) 2, South-West (Lanark and Ayr) 2, Central (Perth and Forfarshire) 4, and Arran 2. We may expect to find a few Scottish Bronze Age skulls of pure Neolithic type. Considering the ♂ skulls only, there are none with a cephalic index less than 71·3. Applying the method of analysis which was used to distinguish the various types among the English Bronze Age skulls, we find only 4 which may be supposed pure Neolithic (λ less than 0·90), 3 of hybrid descent (λ greater than 0·89 and less than 1·31) and 31 of pure Bronze Age type. But it must not be assumed that the latter represent identically the same population as the English Bronze Age type skulls.

Having disintegrated the Bronze Age date populations of England and Scotland, though by a method which is admittedly crude, it will be well to check, as far as possible, the accuracy of the results. A certain number of skulls were selected

* The standard deviations used in computing the Coefficients of Racial Likeness were used for this purpose, i.e. those of the long Egyptian E series (*Biometrika*, Vol. XVI. 1924, p. 338).

† Geographically the accepted pure Neolithic and hybrid groups of skulls seem to be randomly selected from the total Bronze Age date population.

which could be considered identically the same in type as the homogeneous population that inhabited England and Scotland in late Neolithic times. From the method of selection it follows that there will be a close similarity for the characters F, B, J, U, S and $100\,B/F$. If the process has performed what it was intended to, then the similarity of all other cranial characters should be as close as for those six. Between the English and Scottish Neolithic data means (30·7) and the supposed Neolithic skulls of the Bronze Age (15·6), C.R.L.'s are found of $0{\cdot}23 \pm {\cdot}21$ for 20 characters and $0{\cdot}50 \pm {\cdot}36$ for 6 indices and angles and as neither of those coefficients differs significantly from zero we have sufficient statistical justification for considering that the two samples represent the same population. A significant difference is not shown for any single character. The pooled means of the two series are given in column 2 of Table XII. The standard deviations of that population are in column 3 of Table XIV.

Fitting the distributions of characters with normal curves gives for goodness of fit:

	F	B	$100\,B/F$
Probability P ...	·716	·309	·600
Number of Groups	9	8	8
Number of Skulls	116	128	116

If these constants are compared with those of the Neolithic date population it will be seen that with one exception—the goodness of fit for the breadth B—they are not significantly changed, so the addition of the Bronze Age skulls has not sensibly affected the homogeneity of the population. The means of the British Neolithic race that we have finally arrived at are unfortunately only reliable for a few characters and the majority of them are based on small numbers of crania. The type is distinguished from that of almost all modern races by its great length and low cephalic index. The counterpart of its other single characters can readily be found though the small nasal breadth and low nasal index are rather distinctive.

We may suppose that the majority of the skulls of Bronze Age date that have been found in England represent the racial type of the people who invaded the country at the beginning of that epoch—the pure Bronze Age type. The distributions of characters of the ♂ skulls that have been distinguished conform to the type of the normal curve.

	F	B	$100\,B/F$
Probability P ...	·923	·757	·951
Number of Groups	8	9	8
Number of Skulls	90	89	89

The standard deviations of characters (Table XIV) are, with the exception of that of the skull breadth (B), not significantly greater than those of the Whitechapel

English population, and for B the difference is only just significant. It is reasonable to accept the population we have separated from the total available English skulls of Bronze Age date as representing a homogeneous race. That brachycephalic type (32·6) is widely different from the type of the Neolithic people (38·0). Between the two series of means the extraordinarily high C.R.L.'s of 56·55 ± ·18 for 26 characters and 91·93 ± ·30 for 9 indices and angles are found, and yet comparatively few characters show markedly significant differences. It must be remembered that for both populations the means of the calvarial characters in general—and in particular F, B, U and $100\,B/F$—are based on fairly large numbers of crania while the facial measurements are poorly represented. For that reason alone the former are more likely to show markedly significant differences than the latter. Values of α* greater than 5 are given in Table V. All the greatest differences between the Neolithic and English Bronze Age types are associated with the marked differences between the lengths and breadths of the brain-box; the latter type being shorter and at the same time much broader than the former. In spite of those differences the forehead breadths, nasio-basion lengths, and basio-bregmatic heights are practically identical for the two types. The only facial direct measurement which is differentiated is the nasal breadth, the small Neolithic value of that character being distinctive among European races. With more ample material we might be able to distinguish more significant differences, but it is at least surprising to find that two forms of cranium which differ so profoundly in some ways should show a close similarity for the greater number of characters. In spite of the large difference when compared with the British Neolithic skull, the distinguishing feature of the English Bronze Age type is its considerable length associated with brachycephaly. Hardly any other modern or prehistoric European race has a skull length greater than 184 mm. and a cephalic index greater than 80.

We cannot suppose that the samples of the English Bronze Age male type (33·6) and the Scottish Bronze Age male type (21·8) were drawn from identically the same population as Coefficients of Racial Likeness are found of 2·88 ± ·20 for 21 characters and 3·70 ± ·33 for 7 indices and angles. Significant differences (cf. the mean measurements given in Table XII) are shown by the characters $100\,H'/L$, U, LB and $100\,B/L$. The two are quite similar variants of the same racial type.

(3) *The Population of England and Scotland from late Bronze Age to Anglo-Saxon times.*

Very few skeletal remains of the population of England from the end of the Bronze Age to the Saxon invasion have been preserved and measured. That an alien people invaded the country at the beginning of that period, bringing iron implements and many other evidences of a new culture, is sufficiently well established. The practice of interring the dead in round barrows died out. Some Iron

* With the usual notation (see *Biometrika*, Vol. XVI. 1924, p. 12)

$$\alpha = \frac{(M_s - M_s')^2}{\sigma_s^2} \times \frac{n_s n_s'}{n_s + n_s'}.$$

Age barrows have been found in England, but we can get no determination of the physical type of their builders from the few skeletons for which measurements have been given. Cremation was common. In 1894 an ancient cemetery was discovered near Brandon in Suffolk and by the following year fragments of 121 skeletons had been disinterred. An account of the skulls with measurements of 57 of them has been given by Prof. C. S. Myers (26). No bones were found at a greater depth than 4 ft. and the absence of ornaments and pottery and the non-observance of any definite orientation of the bodies made any exact determination of the age of the burials impossible. Large pieces of iron were dug up but they had so far decayed that their former use could not be ascertained. The remains are almost certainly of pre-Saxon date. Myers proceeded to sub-divide the series into several groups which were supposed racially distinct; the Romano-British, the Long-Barrow and "the type of the slaves introduced into Britain at the time of or subsequent to the Roman invasion." No statistical justification whatever for such a procedure can be found. The standard deviation of the cephalic index of all the adult skulls—excluding the distorted specimens—is $2{\cdot}93 \pm {\cdot}20$, a value lower than that of the Whitechapel ♂ series; and the other indices show variabilities that are not at all greater than those of a homogeneous population. The standard deviations of all the direct measurements, too, are not significantly greater than those of the Whitechapel English ♂ skulls, although ♂ and ♀ skulls were said to be found in almost equal numbers at Brandon. By supposing that all the skulls are of the same race and accepting the sexing given we arrive at the following mean values:

	L	B	H'
Males	186·3 (28)	141·0 (28)	130·8 (24)
Females	182·5 (22)	136·0 (21)	127·7 (18)

The differences between these supposed ♂ and ♀ means are much smaller than the corresponding differences usually found for recent races of man and we are obliged to question the accuracy of the sexing. The sample being too small to be treated by the method of the nonic*, the following rough method of mathematical sexing was applied†. Taking three characters—L, B and H'—each distribution was divided as nearly as possible into five histograms having equal bases and covering the entire range. For each particular character a skull falling in the central histogram was assigned the mark 0; -1 and $+1$ for the histograms below and above the central one respectively and -2 for the extreme left and $+2$ for the extreme right histogram. Individuals with a negative total of marks for the three characters were supposed ♀ and others ♂. In this way 24 ♂'s and 26 ♀'s were distinguished. But the sexes determined by the above method were in many cases different from those given by Myers; 18 of his 28 ♂'s remaining ♂ and

* See Karl Pearson, Osteometric Sexing, *Biometrika*, Vol. x. p. 479.

† This method is similar to that used by Pearson and Bell, *A Study of the Long Bones of the English Skeleton*, Part I. The Femur, p. 46.

10 becoming ♀, while 16 of his 22 ♀'s remain ♀ and 6 become ♂ *. The mean measurements become:

	F	B	H'
Males	188·4 (24)	143·1 (23)	133·7 (16)
Females	181·1 (26)	135·2 (26)	126·4 (24)

and the sexual differences are now almost of the same order as for the Whitechapel English and other long series of modern crania†. If Professor Myers divided the whole series into three supposed racially different groups before sexing then the numbers in each were so small that the difficulty of sexing must have been greatly enhanced. The mathematical method is admittedly a rough one which is not unlikely to give the wrong sex to particular skulls, but we may suppose that the mean measurements derived in that way are more nearly accurate than those originally given. Professor Myers, like the greater number of English craniometricians writing at the end of the 19th century, does not define the measurements he has taken or give a reference to any paper in which the definitions may be found. As in other cases we may assume that the methods of measurement adopted were those given by Flower in his *Osteological Catalogue* of 1879 and not modified in the 2nd edition of 1907. Then the skull length is our F, the nasal height NH'. The orbital width is less than Broca's taken from the dacryon (O_1') and cannot be compared with it. The palatal measurements (defined by Flower, *Journal of the Royal Anthropological Institute*, Vol. x. 1880, p. 161) are not comparable with the Frankfurt lengths as measured in the Biometric Laboratory.

Although several thousand skeletons of the inhabitants of Roman Britain must have been disinterred, the number for which measurements have been provided hardly exceeds 200. The majority of the skulls termed Romano-British were found in coffins or with artifacts thought to be of the date of the Roman occupation and bearing signs of Roman workmanship or design. Skeletons not found with some such objects cannot be assigned to the period with any certainty. It has generally been assumed that the physical type of the Romano-Britons was far from pure; heterogeneous elements having been introduced by the Romans themselves and the non-Roman soldiers and slaves. The pure foreign types are seldom found because of the almost invariable custom of cremation and no *à priori* evidence to support the theory that the admixture of foreign blood was large enough to modify the physical type of the conquered people has ever been adduced. Measurements of so-called Romano-British crania are found in the following sources. It is doubtful in some cases whether they were actually contemporaneous with the Roman occupation, but all are almost certainly representatives of the

* Numbers 678, 682, 688, 692, 714, 735, 737, 742, 756, 761 were changed from ♂ to ♀ and Numbers 676, 689, 696, 759, 760 and 685 from ♀ to ♂.

† The Whitechapel English sexual differences are 7·3 for F, 6·0 for B and 7·4 for H'.

Iron Age population of England if that period be supposed to extend down to the time of the Anglo-Saxon invasion. Barnard Davis and Davis and Thurnam 44 ♂ and 11 ♀ (4, pp. 248—251; 19, Nos. 58, 59, 1309, 216; 20, Nos. 1483 and 1542); Flower (27, pp. 64—68), 16 ♂ and 8 ♀; Pitt-Rivers and Garson (8, following p. 243; 9, following Plate 142; 10, p. xiii and pp. 286 and 288; 11, p. 65) 45 ♂ and 17 ♀ and Rolleston (18, Vol. II. p. 676) has given measurements of 1 ♀. The Romano-British skulls of which measurements have been given came principally from the southern counties, but some were found in Yorkshire. A comparison of the measurements suggested that all the skulls were not of the same racial type. The C.R.L.'s in Table II are found between five groups of the ♂ skulls. The clubbing together of the Yorkshire and Shropshire skulls with those from Gloucestershire and Bath is justified by the fact that the northern and southern series are statistically identical.

TABLE II.

Coefficients of Racial Likeness between Various Series of English Romano-British Male Skulls.*

	Wiltshire (Pitt-Rivers) (34·6)	London and Kent (9·5)	Dorset, Somerset and Wilts. (Flower and Davis) (5·8)	Yorkshire, Shropshire, Gloucestershire and Bath (17·5)	Berkshire (10·3)
Wiltshire (Pitt-Rivers) (34·6)	—	0·50 ± ·25 {13}*	0·69 ± ·25 {13}	9·44 ± ·45 {4}	17·48 ± ·45 {4}
London and Kent (9·5)	0·50 ± ·25 {13}	—	0·33 ± ·24 {14}	5·13 ± ·38 {5}	10·67 ± ·38 {5}
Dorsetshire, Somersetshire and Wiltshire (Flower and Davis) (5·8)	0·69 ± ·25 {13}	0·33 ± ·24 {14}	—	0·39 ± ·36 {6}	2·69 ± ·38 {5}
Yorkshire, Shropshire, Gloucestershire and Bath (17·5)	9·44 ± ·45 {4}	5·13 ± ·38 {5}	0·39 ± ·36 {6}	—	4·59 ± ·36 {6}
Berkshire (10·3)	17·48 ± ·45 {4}	10·67 ± ·38 {5}	2·69 ± ·38 {5}	4·59 ± ·36 {6}	—

The available evidence, which is admittedly of the slenderest kind, suggests that round about the time of the Roman occupation the population of the south of England—as judged from the Wiltshire, Dorsetshire, Somersetshire, London and Kent skulls, but excluding the specimens from Gloucestershire and Bath—was racially homogeneous. Clubbing those groups together we find that that population (40·5) is identical in type with the supposed contemporaneous one from

* The numbers in curled brackets following the C.R.L.'s are the numbers of characters on which each was based.

Brandon in Suffolk (14·9)*. The C.R.L.'s between the ♂ means are 0·50 ± ·23 for 16 characters and 0·41 ± ·30 for the 6 available indices and angles. It is found that the combined Brandon and southern English Romano-British populations fulfil sufficiently well the other conditions of racial homogeneity. The P's for goodness of fit of normal curves to three of the distributions are:

	F	B	$100\,B/F$
Probability P ...	·488	·701	·420
Number of Groups	14	13	16
Number of Skulls	89	86	86

and the standard deviations of characters are all of the same order and actually all slightly smaller than the corresponding values for the Whitechapel English series. The one short series from Berkshire, and the series from Yorkshire, Shropshire, Gloucestershire and Bath, are distinctly different from the southern type and from one another.

Measurements of the Iron Age skulls of Scotland were collected together by Sir William Turner in his classical paper (12, Tables VI, VII and VIII). Some were possibly of Anglo-Saxon date, but the majority were undoubtedly earlier. There again the number of skulls is too small to give a reliable determination of type. The means are remarkably similar to those of the southern English Romano-British type and the C.R.L.'s show that there is full justification for considering the two populations to be identical. Between the Scottish (10·8) and English (52·7) ♂ series the C.R.L.'s are 0·25 ± ·21 for 19 characters and − 0·44 ± ·33 for the 7 indices and angles. The combined means are shown in Table XII, and they will be referred to as those of the British Iron Age type. As far as we are able to tell, the population of the two countries in that epoch was very nearly homogeneous. The aberrant types of the skulls from the Midland counties of England are not sufficiently well established to afford evidence of the existence of another racial type. The differences—which are nearly all ones of size—may be due to the selection of merely local variations or possibly to alien elements. A comparison of all the characters that show significant differences is made in Table III.

TABLE III.

Mean Male Measurements of Iron Age Series of Skulls.

	F	L	B	$100\,B/F$	U	S	$100\,B/L$
Southern English Counties	186·8 (65)	188·4 (43)	141·4 (63)	75·7 (63)	528·1 (62)	384·5 (8)	75·3 (43)
Brandon	188·4 (24)	—	143·1 (23)	76·1 (23)	—	—	—
Scotland... ...	—	184·2 (18)	138·8 (10)	—	529·3 (11)	376·3 (12)	74·5 (17)
Berkshire ...	193·9 (11)	—	149·6 (11)	77·2 (11)	552·2 (11)	390·7 (9)	—
Yorkshire, Shropshire, Gloucestershire and Bath	186·3 (19)	—	146·5 (17)	78·6 (17)	538·8 (19)	376·5 (19)	—

* Using the re-sexed means.

The standard deviations of characters of the British Iron Age population are shown in Table XIV and for the goodness of fit of normal curves to the ♂ distributions the following probabilities are found:

	L	B	$100\ B/L$
Probability P ...	·459	·994	·880
Number of Groups	13	14	14
Number of Skulls	61	102	57

These criteria bear out the assumption made that the population is racially homogeneous.

Having obtained the mean measurements of the type extant in England and Scotland in the Iron Age we may compare them with those of the two earlier prehistoric races. The C.R.L.'s between the three series are given in Table IV.

TABLE IV.

Coefficients of Racial Likeness between Early Series of British Crania.*
Male Means.

		British Neolithic (38·0)	English Bronze Age (32·6)	British Iron Age (53·4)
British Neolithic (38·0)	All Characters ...	—	56·55 ± ·18 {26}	10·94 ± ·20 {22}
	Indices and Angles	—	91·93 ± ·30 {9}	19·65 ± ·33 {7}
English Bronze Age (32·6)	All Characters ...	56·55 ± ·18 {26}	—	25·28 ± ·20 {22}
	Indices and Angles	91·93 ± ·30 {9}	—	38·95 ± ·33 {7}
British Iron Age (53·4)	All Characters ...	10·94 ± ·20 {22}	25·28 ± ·20 {22}	—
	Indices and Angles	19·65 ± ·33 {7}	38·95 ± ·33 {7}	

The striking feature common to all the C.R.L.'s shown in Table IV is their high value. We are evidently dealing not with local and closely related types, but with distinct racial populations which are as markedly contrasted as almost any that are to be found in Europe to-day. The Iron Age series, containing more skulls than the other two, is likely to show larger coefficients with them than they would with one another if all three were equally related, and yet its coefficients are markedly smaller than those between the Bronze Age and Neolithic types. The Iron Age type thus takes up a position intermediate between the other two. In comparing individual characters it must be remembered that

* The numbers in round brackets, following the designations of the types, are the mean numbers of skulls available for each type for the characters used in computing the C.R.L.'s. The numbers in curled brackets following the coefficients are the numbers of characters on which each coefficient is based.

all the means of the facial measurements of the three types are based on small numbers of crania and they are, for that reason alone, less likely to show significant differences than the more adequately determined measurements of the brain-box. Between the Neolithic and English Bronze Age populations, 4 characters which are not available for the Iron Age series can be compared. They are $100\,G'H/GB$, $100\,fmb/fml$, G_2 and fmb, and they show no significant differences. Of the 22 characters available for all three series, 8—B', LB, $G'H$, NH', O_1', fml, $N\angle$ and $A\angle$—show no significant differences whatever. The constancy of the forehead breadth (B') and nasio-basion length (LB) is all the more remarkable because the greatest length and greatest breadth of the calvaria show more markedly significant differences than any other direct characters.

The values of α* greater than 5 shown for the 14 characters are given in Table V. A comparison of the values of α between the three series emphasises the extreme importance of the length and breadth of the calvaria and the resulting cephalic index as the characters which provide the clearest racial differentiation. In each case the combined values of α for those three characters are many times greater than the sum of the α's for all the other characters, and the α for $100\,B/F$ is very significantly greater than the α for any other single character. This distinctive feature of the cephalic index is very frequently found in comparing members of the same family of allied races and the importance which Anders Retzius attached to the character is entirely justified by statistical comparison. Apart from the dimensions and shapes in *norma verticalis* of the brain-boxes—affecting the characters $100\,B/F$, B, F, S, $100\,B/H'$ and $100\,H'/F$—and the zygomatic breadths, the three British types are not markedly dissimilar. The greater zygomatic and nasal breadths of the English Bronze Age type distinguish it clearly from the other two, but no other clear differences between the conformations of the facial regions of the Neolithic and Iron Age types can be detected. The material is not ample enough to provide justification for the statement that in reality there are no such differences. Considering the actual mean measurements (Table XII) it may be noted that for all the characters, except J, which show the greatest differences between the three prehistoric types—viz. $100\,B/F$, B, F, S, $100\,B/H'$, $100\,H'/F$, NB and $100\,NB/NH'$—the Iron Age type has mean measurements intermediate between those of the Neolithic and Bronze Age populations.

The English Iron Age skulls that have as yet been considered were not found definitely associated with the peculiar artifacts to which the name "Late Celtic" is now usually applied. That new culture is supposed to have been introduced into England and Scotland by an immigrant race which came in large numbers, took up its settled habitation in the country and eventually formed an important element in the jumbling of races from which the population of mediaeval England is supposed to have been made up. It is very frequently stated that this Celtic people was brachycephalic and of a type differing from that of the earlier Bronze Age invaders, but it appears that the only evidence in support of that statement

* See footnote, p. 66.

is the fact that the carriers of the same culture in neighbouring countries—notably France—were of a distinct brachycephalic type. Very few remains of the people who introduced or adopted the new culture in England are extant as cremation was their almost invariable practice. Apart from isolated skulls, all the available skeletal material has been obtained from a single cemetery near Driffield in the East Riding of Yorkshire. Remains of chariots, weapons, personal ornaments and pottery were found in small mounds or barrows and all could be assigned to the late Celtic period, though known locally as the "Danes' Graves." Measurements of 22 of the skulls were published by William Wright in 1903 (24) and the length, breadth, cephalic index and, in some cases, the forehead breadth of 37 additional specimens were furnished by the same writer in an appendix to a paper by Canon Greenwell three years later (25). The mean measurements of that combined material are given in Table VI, with the combined English and Scottish Iron Age ♂ means. Wright distinguished 5 types, all of which were dolichocephalic, but since the means do not, as far as they go, indicate a heterogeneous population we need not suppose that the assembly was such a cosmopolitan one.

The "Danes' Graves" skulls have lower cephalic indices than the English and Scottish Iron Age type, so there is not the slightest suggestion of their bearing any close blood relationship to the brachycephalic peoples associated with the "Late Celtic" culture on the continent. The ♂ means of the direct measurements are nearly all smaller than those of the Iron Age population, and in many cases the differences would be quite significant. But we may suspect that the discordance there is not due to a real difference between the types. The differences between the ♂ and ♀ Danes' Graves means are decidedly smaller than those ordinarily found. The sexual differences compared with those for the Whitechapel English series are:

	Whitechapel English	"Danes' Graves"
L	8·6	6 8
B	6·0	0·9
B'	4·9	1·5

The numbers are far too small to warrant any dogmatic statement, but we may take it to be at least probable that the supposed "Danes' Graves" ♂ skulls included several ♀'s. If the difference in absolute size from the Iron Age population may be supposed due to that cause then the two would seem to be more or less alike. The only characters showing significant differences would be the basio-bregmatic height (H') and the indices dependent on it ($100\,H'/L$ and $100\,B/H'$). It may be that the differences there are indicative of a real racial distinction, but no proof of that can possibly be given which depends on a mean based on no more than six skulls. It cannot be dogmatically asserted that the Danes' Graves skulls belong to a population which is different from that which was widely spread over England and Scotland in the late Celtic and Romano-British periods. We have, however, thought it best not to include them in our Iron Age series.

TABLE V.

Values of α between Early Series of British Male Crania.

	100 B/F	B	F	J	S	100 B/H'	100 H'/F	NB	100 NB/NH'	Bregma Q	U	H'	100 O_2/O_1'	O_2
Neolithic and English Bronze Age	754·87	334·42	181·56	64·31	51·25	51·01	16·72	9·61	9·46	8·83	0·19	0·18	0·56	0·04
Neolithic and Iron Age	118·64	15·72	63·34	0·05	7·02	14·78	2·51	0·07	0·04	1·22	10·19	5·04	5·46	5·23
English Bronze Age and Iron Age	235·75	151·24	26·40	69·14	1·25	24·50	10·96	8·99	1·99	17·49	10·79	2·98	1·62	1·44

TABLE VI.

Comparison of "Danes' Graves" and other Iron Age Skulls.*

	L	F	B	H'	B'	S_1	S_2	S_3	S	U
"Danes' Graves" ♂	185·2 (45)	182·3 (15)	134·3 (43)	135·3 (6)	96·1 (23)	125·6 (15)	131·3 (15)	112·4 (14)	369·8 (14)	514·4 (11)
"Danes' Graves" ♀	178·4 (14)	175·8 (6)	133·4 (14)	125·0 (2)	94·6 (11)	122·2 (6)	118·7 (6)	124·0 (4)	366·3 (4)	502·0 (6)
English and Scottish Iron Age Type ♂	187·4 (61)	187·2 (89)	141·4 (102)	132·9 (77)	98·0 (45)	127·6 (20)	129·9 (20)	120·6 (20)	379·6 (20)	528·3 (73)
English and Scottish Iron Age Type ♀	179·7 (22)	180·8 (58)	136·2 (62)	126·5 (42)	97·1 (13)	123·7 (11)	123·0 (10)	115·7 (10)	362·0 (10)	510·7 (31)

	$G'H$	J	'NH''	NB	O_2	100 B/L	100 H'/L	100 B/H'	100 NB/NH'	100 $G'H/GB$
"Danes' Graves" ♂	66·6 (10)	125·6 (5)	49·6 (11)	23·6 (8)	34·9 (7)	{72·5 (43)}	74·0 (6)	{99·3 (6)}	46·9 (8)	{74·0 (5)}
"Danes' Graves" ♀	64·8 (5)	117·7 (3)	48·7 (6)	23·0 (6)	33·8 (6)	{74·8 (14)}	70·5 (2)	—	47·2 (5)	{73·6 (4)}
English and Scottish Iron Age Type ♂	69·1 (30)	130·6 (55)	50·6 (99)	23·7 (67)	33·6 (65)	{75·4 (61)}	{70·9 (61)}	{106·3 (77)}	{46·8 (67)}	—
English and Scottish Iron Age Type ♀	63·6 (18)	123·8 (24)	47·9 (30)	23·5 (28)	33·1 (29)	{75·8 (22)}	{70·4 (22)}	{107·7 (42)}	{49·1 (28)}	—

* Some of the available "Danes' Graves" means which are based on five or fewer crania have been omitted from this table. The indices in curled brackets were calculated from the means of the component lengths.

(4) *English Anglo-Saxon Crania and their Relationships to Earlier British Types.*

It is customary to apply the term Anglo-Saxon to that period of English history which followed the departure of the Roman legions and preceded the Norman Conquest. But it is only for the first half of that epoch—roughly until the union of the several kingdoms under Alfred in 886—that we have any evidence relating to the physical type of the invaders. The characteristic Anglo-Saxon cemeteries in which weapons, ornaments and numerous other distinctive artifacts are found alongside the bodies, all belong to the earlier period and the majority are prior to the conversion of the inhabitants to Christianity at the beginning of the 7th century. Graveyards of the 10th and 11th centuries have frequently been found, but they were of little interest to archaeologists as the practice of interring artifacts with the body had died out by that time and no skulls of the period appear to have been preserved. Records of the excavations of a large number of cemeteries which can be assigned to the earlier centuries are to be found in *Archaeologia* and in the *Proceedings* of various county archaeological societies, but unfortunately few of the skeletons are at present available for study, the usual practice having been to re-inter them. All the material is in a fragile and incomplete condition, the shallow inhumations, usually in wooden coffins or without coffins at all, being ill-adapted to preserve the skeletons, and in that respect they are contrasted with the prehistoric remains which were protected by stone cists and tumuli.

The individual measurements of all the Anglo-Saxon crania in the Museum of the Royal College of Surgeons, the London Museum and the British Museum (Natural History) will be found in Appendix II, together with some particulars relating to the various graveyards. A qualitative comparison of all that material suggested that not more than one racial type was represented—a muscular dolichocephalic type which at first sight only appeared to differ from that of the 17th century English, as represented by the Whitechapel, Moorfields and Farringdon Street series preserved in the Biometric Laboratory, in having a greater calvarial height. No examples of the type of skull with a markedly retreating frontal bone, which is so peculiar to the modern English skulls, were found among the earlier populations.

To supplement the measurements of the crania in London museums, all previously published measurements of other Anglo-Saxon crania were collected together and pooled with them*. The whole material is still very meagre—particularly for facial measurements—and there is an urgent need for more Anglo-Saxon skeletal remains in our museums†. The only writers who have given measurements of Anglo-Saxon skulls other than those in London museums are

* Measurements of the majority of the skulls in the Royal College of Surgeons had previously been published, but they were always few in number (see Appendix II). The Mitcham skulls in the London Museum and the Sleaford in the British Museum (Natural History) had not previously been dealt with.

† The measurements of the long series discovered at Bideford-on-Avon in 1923 would be a welcome addition to our knowledge.

Horton-Smith (16), Duckworth (28), Pitt-Rivers (9), Schmidt (29) and Davis and Thurnam (4).

The division of the invaders into Angli, Saxons and Jutes which Bede made, though it was not observed by other early writers, is still occasionally cited as evidence of their racial heterogeneity. Archaeological research has shown that in customs and cultural practices the Jutes are distinguished from the Angles and Saxons, but no such differences are found between the two latter*. Until the present, no attempt to investigate the question of racial admixture which is of the slightest biometric value has been made. The whole available material is so meagre that it cannot be answered yet with any finality. Division was made into the following four groups:

(1) *West Saxons:* chiefly from Wiltshire and Berkshire, but some from Oxfordshire, Gloucestershire, Buckinghamshire, Dorsetshire and Somersetshire.

(2) *South Saxons:* chiefly from London district (Mitcham) but some from Sussex and Essex.

(3) *Angles:* chiefly from Lincolnshire and Cambridgeshire but some from Yorkshire, Norfolk, Nottinghamshire and Northamptonshire.

(4) *Jutes:* from Kent.

The most important mean measurements of the ♂ skulls in each group are given in Table VII and the C.R.L.'s between those means are in Table VIII.

TABLE VII.

Mean Measurements of Various Groups of Male Anglo-Saxon Crania.

Character	West Saxons	South Saxons	Jutes	Angles
F	188·4 (34)	188·3 (13)	188·0 (15)	189·9 (32)
B	141·0 (36)	143·0 (22)	142·6 (15)	140·9 (30)
B'	97·4 (15)	98·1 (12)	97·4 (14)	96·5 (18)
H'	135·5 (9)	137·8 (8)	135·8 (8)	135·1 (6)
LB	103·8 (7)	105·9 (8)	103·7 (9)	103·1 (7)
S	378·0 (25)	382·6 (8)	382·5 (12)	379·1 (19)
U	529·8 (32)	534·7 (13)	535·8 (11)	531·6 (17)
$G'H$	69·6 (4)	72·2 (7)	74·1 (4)	70·9 (7)
J	133·3 (17)	129·7 (4)	136·8 (8)	130·8 (5)
NH'	52·7 (7)	52·8 (9)	55·1 (5)	50·7 (7)
NB	24·7 (8)	24·3 (7)	24·5 (7)	24·6 (6)
O_2	32·8 (8)	34·4 (7)	33·0 (7)	34·7 (7)
$100\,B/F$	75·8 (45)	74·0 (16)	76·1 (11)	74·0 (40)
$100\,H'/F$	74·0 (20)	71·1 (13)	72·9 (7)	70·4 (20)
$100\,B/H'$	104·3 (21)	105·0 (14)	105·2 (7)	105·5 (19)
$100\,NB/NH'$	49·5 (17)	46·1 (14)	45·7 (5)	47·3 (22)
Oc. I.	58·7 (5)	58·7 (6)	57·7 (9)	58·0 (15)
$100\,G'H/GB$	71·0 (11)	73·8 (10)	77·8 (3)	77·9 (18)

* Cf. Chadwick, *The Origin of the English Nation*, 1907, p. 81: "The evidence of the social systems confirms in a striking manner Bede's statement that the inhabitants of Kent were of a different nationality from those of the surrounding kingdoms. We have seen that the historical evidence gives no confirmation of this statement, while the linguistic evidence is worthless. In the light of the facts pointed out above, however, there can be no doubt as to its accuracy....On the other hand, the evidence of the social systems has totally failed to substantiate the distinction drawn by Bede between the Saxons and the Angles."

TABLE VIII.

Coefficients of Racial Likeness between Groups of Anglo-Saxon Male Crania.*

	West Saxons (17·7)	South Saxons (10·6)	Angles (13·9)	Jutes (8·4)
West Saxons (17·7)	—	0·71 ± ·22 {18}	1·87 ± ·21 {19}	−0·20 ± ·22 {17}
South Saxons (10·6)	0·71 ± ·22 {18}	—	−0·09 ± ·21 {20}	−0·28 ± ·22 {17}
Angles (13·9)	1·87 ± ·21 {19}	−0·09 ± ·21 {20}	—	0·53 ± ·20 {22}
Jutes (8·4)	−0·20 ± ·22 {17}	−0·28 ± ·22 {17}	0·53 ± ·20 {22}	—

If the reader will compare the actual mean measurements of tee four groups of Anglo-Saxon skulls he will be persuaded that they represent populations which are extremely similar if not absolutely identical. The C.R.L.'s confirm that conclusion. Only one—that between the Angles and West Saxons—suggests any real difference of type. There were 26 characters which could be used in computing the coefficients: 20 of them show no significant differences whatever. All the values of α greater than 5 are given below:

	100 B/F	100 H'/F	100 $G'H/GB$	100 NB/NH'	J	NH'
West Saxons and South Saxons	5·36	7·66	1·67	6·07	—	0·18
West Saxons and Angles ...	9·61	15·00	13·21	3·18	1·16	3·19
Angles and Jutes	5·33	3·75	—	0·71	5·30	6·61

Before deciding whether the observed differences must be considered to indicate different racial types or not, it will be well to compare the ♀ means. They are (excluding the Jutish means which are quite unreliable because only based on 5 skulls):

	100 B/F	100 H'/F	100 $G'H/GB$	100 NB/NH'	J	NH'
West Saxons	76·5 (33)	72·1 (17)	72·4 (17)	50·1 (17)	126·1 (9)	48·5 (12)
South Saxons	73·4 (11)	72·5 (6)	70·8 (4)	50·7 (6)	126·0 (3)	49·3 (7)
Angles ...	74·8 (27)	70·6 (13)	71·9 (17)	50·9 (13)	121·8 (5)	47·7 (9)

The ♀ means show fewer significant differences than the ♂. For the latter almost all the highest values of α were occasioned by the high West Saxon means

* The figures in round brackets give, as usual, the mean numbers of skulls available for the characters used in computing the C.R.L.'s. Means based on fewer than five crania were neglected. The numbers in curled brackets are the numbers of characters used in computing the C.R.L.'s.

for the indices $100\,B/F$, $100\,H'/F$ and $100\,NB/NH'$ and the zygomatic breadth (J). But the only ♀ mean of the West Saxons that is distinguished in the same way is that for $100\,B/F$. We may reasonably conclude that the differences in the case of the other characters were merely due to chance causes and no significance need be attached to them. The distinctiveness of the high facial index ($100\,G'H/GB$) of the ♂ Angles is not confirmed by a comparison of the ♀ means and it, too, may be considered of no importance. But the case of the higher cephalic indices of both ♂ and ♀ series of West Saxon skulls cannot be dismissed in the same way. It may be noted that that type does not differ from the others by having a smaller length and greater breadth, such as might have resulted from admixture with a brachycephalic type like that of the English Bronze Age population. The question whether it will be legitimate to consider the pooled measurements of all the Anglo-Saxon skulls to represent a single homogeneous type may be largely decided by combining the several groups and examining the statistical constants of that population. The ♂ and ♀ standard deviations are given in Table XIV and it will be seen that they are of the same order as the Whitechapel English values. The variability of the ♀ skull breadth (B) is the sole distinctively great one*. For goodness of fit of normal curves to the distributions of all the Anglo-Saxon skulls the following probabilities are found:

	B		F		$100\,B/F$	
	♂	♀	♂	♀	♂	♀
Probability P ...	·705	·759	·566	·988	·589	·831
Number of Groups	15	8	12	7	15	9
Number of Skulls	103	67	93	63	112	77

We may conclude then that, *from the evidence at present available*, there is sufficient statistical justification for considering that all skulls found in England associated with Anglo-Saxon artifacts belong to a single homogeneous racial type. Some of the invaders may have been modified by admixture with the indigenous population, but the modification of type effected in that way must in any case have been slight and the remains that have been preserved are insufficient in number to enable us to tell whether there was any such crossing or not. It should be remembered that hardly any of the Anglo-Saxons with which we are dealing were interred later than the 6th century. Provisionally then we may pool the West Saxons and the Jutes with the other groups to give the mean measurements of the accepted Anglo-Saxon type. The ♂ mean measurements are given in Table XII and the ♀ in Table XIII. The majority of the specimens were sexed by the present writer (see Appendix II) and a comparison of the sexual differences between the means with the differences for the sufficiently long Whitechapel

* A possible cause of large standard deviation of B, though one we should be loath to accept, is to be found in the fact that after restoration the breadths are more likely to be incorrect than other measurements.

series suggests that that was done adequately. For the characters below both ♂ and ♀ means were based on more than 50 skulls.

	Anglo-Saxon Sexual Difference	Whitechapel Sexual Difference
F ...	8·0	7·3
B ...	6·1	6·0
B' ...	3·0	4·9
U ...	21·1	20·5

There is a satisfactory agreement between the ♂ and ♀ Anglo-Saxon mean indices. The Coefficients of Racial Likeness given in Table IX are found with the series of ♂ means.

TABLE IX.

Coefficients of Racial Likeness with Anglo-Saxon Male Means.*

		British Iron Age (53·4)	British Neolithic (38·0)	English Bronze Age (32·6)
Anglo-Saxons (41·2)	All Characters ...	3·67 ± ·20 {22}	6·82 ± ·18 {26}	24·19 ± ·18 {27}
	Indices and Angles	3·25 ± ·33 {7}	10·81 ± ·30 {9}	40·51 ± ·29 {10}

Of all the C.R.L.'s between the four racial types compared in Tables IV and IX that between the Anglo-Saxons and Iron Age British is the only one low enough to suggest a close relationship. These two later types take up positions which are intermediate between those of the extremely dissimilar Neolithic and English Bronze Age populations, being closer to the former than to the latter. It cannot be said that either the one or the other of the intermediate types, i.e. Anglo-Saxon and Iron Age, stands closer than its fellow to either of the extremes, namely Neolithic and Bronze Age. In comparing the Anglo-Saxons with each of the other three types it is found that the characters $100\,NB/NH'$, $100\,O_2/O_1'$, U, $G'H$, O_1', O_2, fml, $A\angle$, fmb, $100 fmb/fml$ and G_2—the last three not being available for the Iron Age population—show no significant differences whatever. The values of α between the other measurements compared are shown in Table X.

With the Neolithic and English Bronze Age types the most marked differences are almost precisely the same as those between the Iron Age type and the same two populations (see Tables V and X); the cephalic indices differ most profoundly and then, in order, the characters B, F, $100\,B/H'$ and J. The basio-bregmatic heights, nasio-basion lengths (LB), frontal breadths (B') and facial characters other than the zygomatic breadth (J) are almost identical for the three types. For the characters showing greatest differences between the Neolithic and

* The numbers in curled brackets following the C.R.L.'s are the numbers of characters on which each is based.

TABLE X.

Values of α between the Anglo-Saxon and other British Male Series.

	100 B/F	B	F	J	100 B/H'	100 H'/F	Bregma Q	S	H'	LB	B'	NH'	NB	$N\angle$	$P\angle$
Anglo-Saxon and British Iron Age	5·50	0·20	4·02	7·12	3·57	2·89	11·13	0·00	8·40	8·41	0·76	12·72	12·72	11·68	—
Anglo-Saxon and British Neolithic	82·51	19·83	36·26	7·55	5·50	8·17	0·53	9·81	0·14	0·51	2·90	0·91	0·91	3·21	—
Anglo-Saxon and English Bronze Age	350·63	141·32	52·14	23·78	37·91	4·01	17·39	2·08	0·50	1·31	8·73	1·21	1·21	4·30	6·19

TABLE XI.

Mean Male Measurements distinguishing the Neolithic and Bronze Age Populations.

	100 B/F	B	F	J	S	100 B/H'	100 H'/F	NB	100 NB/NH'	Bregma Q
British Neolithic ...	71·7	138·9	193·6	130·4	389·0	102·5	70·0	23·6	45·4	308·5
Anglo-Saxon ...	74·9	141·7	188·9	133·3	379·8	104·9	72·0	24·5	47·5	316·1
British Iron Age ...	75·8	141·4	187·2	130·6	379·6	106·3	71·1	23·7	46·8	303·6
English Bronze Age	82·0	149·9	182·8	138·5	376·6	111·2	73·4	25·0	48·3	320·5

Bronze Age populations the differences between the Anglo-Saxon and British Iron Age skulls are small, if significant at all, while the two later types show significant differences for several characters which do not at all differentiate the Anglo-Saxon, Neolithic and English Bronze Age types. The position will be made more clear if we consider only the characters which show significant differences between the Neolithic and English Bronze Age types (see Table V). The means for all such characters are given in Table XI.

A striking fact is brought out by a comparison of the mean measurements given in Table XI. For every single character showing a significant difference between the Neolithic and English Bronze Age populations, the Anglo-Saxon means are intermediate in value between the contrasted values for the two earlier populations. The suggestiveness of that relationship is greatly enhanced when we observe that, with one exception, all the other measurements that may be compared are statistically identical for the three types. The exception is for the frontal breadth (B'), the Anglo-Saxon mean of 97·3 being smaller than the Neolithic (98·7) and significantly less than the English Bronze Age value (99·7). That discordance, however, may not be of importance*. There may be something to be said for the hypothesis that the Anglo-Saxon invaders of our shores were originally related to two contrasted racial types which were akin to the ones found in England in the late Neolithic and Bronze Age respectively, the Neolithic element predominating. A comprehensive study of the craniology of Europe in late prehistoric times could alone prove or disprove that theory with any definitiveness. What now of the British Iron Age type? Being very similar to the Anglo-Saxon, it has many important characters intermediate between the English Bronze Age and Neolithic values (see Table XI), but three shown in the table—viz. J, NB, and $100\,NB/NH'$—are not differentiated from the Neolithic values, and the small basio-bregmatic height and transverse arc of the Iron Age skulls differentiate them, though not very markedly, from the other three types and give a close bond with the 17th century London skull which the others lack. It is extremely unlikely that such a type was compounded from the earlier populations of Britain which are known to us.

(5) *Mean Female Measurements of Early British Series of Crania.*

For all the series of which the ♂ mean measurements have been dealt with in earlier sections of this paper, the ♀ data are less plentiful than the ♂. The Neolithic means given in the second column of Table XIII are those of English and Scottish skulls of Neolithic date. The ♀ Bronze Age population is too small to make any approximate disintegration by mathematical methods, similar to that used in the case of the ♂ skulls, at all worth while. The only ♀ Neolithic means based on numbers that are at all adequate indicate a type that is distinguished from all later ones by its great length and low cephalic index, and the agreement between the ♂ and ♀ indices is quite satisfactory. For the Iron Age

* It may be noticed that the Anglo-Saxon sexual differences (p. 79) are almost precisely the same as for the Whitechapel English except for B'. The Anglo-Saxon ♀ B' does not differ at all from the Neolithic and Bronze Age values.

TABLE XII.

Comparative Table of English and Scottish Mean Skull Measurements for Various Periods. Male Skulls.

Character	English and Scottish Neolithic	English Bronze Age	Scottish Bronze Age	English and Scottish Iron Age	Anglo-Saxon	Whitechapel English‡ (17th Century)
Capacity*	[1533·2 (25)]	[1564·4 (25)]	[1560·9 (30)]	[1487·8 (61)]	[1543·3 (31)]	1476·9 (72)
L†	193·7 (53)	184·5 (48)	180·7 (34)	187·4 (61)	190·6 (58)	189·1 (137)
F	193·6 (116)	182·8 (90)	181·6 (5)	187·2 (89)	188·9 (94)	187·4 (138)
L'	191·9 (7)	185·3 (7)	—	—	190·0 (16)	187·8 (72)
B	138·9 (128)	149·9 (89)	150·1 (33)	141·4 (102)	141·7 (103)	140·7 (135)
B'	98·7 (41)	99·7 (43)	—	98·0 (45)	97·3 (59)	98·0 (132)
H	136·3 (7)	141·3 (3)	—	—	137·7 (16)	—
H'	135·5 (25)	134·9 (25)	137·2 (30)	132·9 (77)	136·0 (31)	132·0 (122)
OH	117·6 (8)?	121·1 (7)?	—	—	114·9 (17)	112·1 (135)
LB	103·3 (21)	102·9 (27)	99·4 (21)	101·6 (67)	104·1 (31)	101·6 (119)
Q'	313·7 (15)	333·9 (7)	—	—	316·1 (23)	—
Bregmatic Q	308·5 (10)	320·5 (15)	318·5 (17)	303·6 (10)	—	307·9 (115)
S	389·0 (31)	376·6 (62)	375·4 (24)	379·6 (20)	379·8 (64)	377·1 (131)
S_1	132·5 (27)	131·1 (61)	128·4 (30)	127·6 (20)	129·3 (83)	—
S_2	133·9 (26)	128·1 (57)	129·6 (31)	129·9 (20)	129·0 (87)	—
S_3	123·3 (20)	117·9 (42)	116·7 (25)	120·6 (20)	121·7 (71)	—
U	536·6 (106)	535·6 (81)	525·0 (26)	528·3 (73)	532·0 (73)	524·3 (131)
$G'H$	70·8 (32)	69·1 (30)	68·5 (15)	69·1 (30)	71·7 (22)	70·2 (75)
GB	95·3 (11)	98·0 (11)	—	—	95·0 (19)	90·9 (55)
J	130·4 (41)	138·5 (40)	135·1 (14)	130·6 (55)	133·3 (34)	130·1 (43)
NH, R	50·6 (15)	49·1 (13)	—	—	52·2 (22)	51·2 (79)
NH, L			—	—	52·3 (22)	
NB	23·6 (34)	25·0 (28)	24·5 (17)	23·7 (67)	24·5 (28)	24·3 (70)
NH'	52·0 (21)	51·8 (16)	50·2 (19)	50·6 (99)	52·7 (28)	50·3 (76)‡
O_1R	42·3 (3)	44·0 (14)	—	—	42·9 (19)	43·0 (68)
O_1L			—	—	42·2 (16)	43·1 (61)
O_2R	32·6 (27)	33·0 (30)	32·2 (16)	33·6 (65)	33·6 (29)	33·4 (69)
O_2L					33·5 (27)	33·5 (67)
O_1' §	39·8 (22)	39·7 (17)	40·6 (16)	39·3 (21)	40·3 (16)	40·5 (67)‡
G_1		—	—	—	50·1 (20)	48·3 (69)
G_2	40·2 (7)	12·4 (10)	—	—	41·3 (27)	39·6 (32)‡
GL	96·8 (16)	96·4 (20)	93·6 (16)	96·3 (63)	96·0 (22)	95·9 (73)
fml	36·3 (17)	36·1 (20)	35·9 (20)	35·8 (11)	37·5 (20)	35·9 (117)‡
fmb	30·4 (7)	31·1 (14)	—	—	31·1 (18)	30·3 (112)‡
$100\,B/F$	{71·7 (116)}‖	{82·0 (89)}	{82·7 (5)}	{75·8 (86)}	74·9 (112)	{75·1 (135)}
$100\,B/L$	{71·7 (53)}	{81·3 (48)}	{83·1 (33)}	{75·4 (61)}	74·7 (52)	74·3 (131)
$100\,H'/F$	{70·0 (25)}	{73·4 (25)}	{75·6 (5)}	{71·1 (63)}	72·0 (60)	{70·4 (122)}
$100\,H'/L$	{70·0 (25)}	{72·8 (25)}	{75·9 (30)}	{70·9 (61)}	71·2 (25)	70·0 (120)
$100\,B/H'$	{102·5 (25)}	{111·2 (25)}	{109·4 (35)}	{106·3 (77)}	104·9 (61)	106·6 (122)
$100\,(B-H')/L$	{+1·8 (25)}	{+8·1 (25)}	{+7·1 (30)}	{+4·5 (61)}	+3·8 (24)	{+4·6 (122)}
$100\,G'H/GB$	{74·3 (11)}	70·4 (13)	—	—	75·3 (43)	76·5 (53)
$100\,NB/NH, R$	46·6 (15)	{50·9 (13)}	—	—	48·1 (19)	47·6 (70)
$100\,NB/NH, L$			—	—	47·9 (19)	
$100\,NB/NH'$	{45·4 (21)}	{48·3 (16)}	{48·8 (17)}	{46·8 (67)}	47·5 (58)	{48·3 (70)}‡
$100\,O_2/O_1, R$	{76·8 (3)}	{75·2 (14)}	—	—	77·9 (19)	77·7 (68)
$100\,O_2/O_1, L$			—	—	79·4 (16)	77·9 (63)
$100\,O_2/O_1'$	{81·9 (22)}	{83·4 (17)}	{79·3 (16)}	{85·5 (21)}	83·3 (46)	{82·5 (67)}‡
$100\,fmb/fml$	{83·7 (7)}	{86·1 (14)}	80·6 (1)	—	82·3 (18)	84·5 (112)‡
$100\,G_2/G_1$	—	—	—	—	81·5 (18)	{82·0 (32)}‡
P∠	83°·3 (4)	84°·2 (5)	—	—	88°·1 (16)	86°·1 (63)
N∠	{64°·2 (16)}	{64°·4 (20)}	{64°·3 (15)}	{65°·6 (30)}	62°·1 (16)	65°·2 (69)
A∠	{74°·7 (16)}	{75°·3 (20)}	{74°·2 (15)}	{73°·5 (30)}	75°·5 (16)	73°·4 (69)
B∠	{41°·1 (16)}	{40°·3 (20)}	{41°·5 (15)}	{40°·9 (30)}	42°·4 (16)	41°·4 (69)
θ_1∠	—	—	—	—	30°·6 (13)	28°·7 (59)
θ_2∠	—	—	—	—	11°·9 (13)	12°·9 (59)

the numbers are large enough to give a good first approximation to the ♀ type and in shape there is a close resemblance to that of the contemporary ♂ skulls. Both show characteristically low calvarial heights and small $100\,H'/L$ indices. The nasio-basion lengths are also small and for both sexes the means are hardly distinguishable from those of the 17th century Whitechapel skulls. The distinctiveness of the Anglo-Saxon type is confirmed by the ♀ measurements. Some writers have suggested that the invading Anglo-Saxons came without their women

Notes to Table XII.

* The capacities in square brackets were reconstructed from the mean measurements by Hooke's formula for ♂ English skulls $C = \cdot 000366 \times L \times B \times H' + 198\cdot 9$, using mean measurements. See the present issue of *Biometrika*, p. 83.

† It is usual to find for European races that the ♂ mean glabellar-occipital length (L) exceeds the mean ophryo-occipital length (F) by about 2 mm. when both are based on the same skulls. Some of the types for which measurements are given in the table do not show differences of that order because the means are not based on the same numbers of skulls. The divergences are not greater than those which may be expected to arise as the result of random sampling.

‡ The measurements O_1', NH', fml and fmb of the Whitechapel English were taken by the present writer with the kind permission of Professor Karl Pearson. Macdonell had not provided them in his paper (30). They give the additional indices $100\,O_2/O_1'$, $100\,NB/NH'$ and $100\,fmb/fml$. Macdonell had given the palatal width (G_2) of 66 of the ♂ skulls as 36·8, which is a peculiarly small value. I examined the whole series and only found 32 specimens with alveolar borders sufficiently well preserved to give unquestionably correct measurements. Their mean G_2 is 39·6 and I have ventured to use that value in place of Macdonell's for the direct measurement and for the palatal indices.

§ Almost all the individual dacryal widths on which these means are based are of the right orbit only, but some may have been of the left orbit in place of the right.

|| The indices and angles in curled brackets were calculated from the means of the component lengths.

The following additional male mean measurements of the Anglo-Saxons and Whitechapel English are available.

	G_1'	$100\,G_2/G_1'$	DS	DC	DA	SS
Anglo-Saxons ...	46·1 (21)	88·2 (19)	12·6 (14)	22·2 (17)	31·7 (14)	4·4 (19)
Whitechapel English	44·7 (69)	*88·6 (32)	12·1 (50)	21·4 (50)	34·6 (50)	4·72 (50)

	SC	$100\,DS/DC$	$100\,SS/SC$	PH	S_3'	$Oc.\ I.$
Anglo-Saxons ...	9·2 (21)	58·0 (14)	47·3 (19)	19·8 (27)	98·5 (37)	58·2 (35)
Whitechapel English	9·43 (50)	57·4 (50)	51·2 (50)	—	—	—

	Q	G_1''	$EB = G_2$	EH	$100\,EH/EB$	$100\,H/L$
Anglo-Saxons ...	314·4 (15)	53·9 (17)	41·3 (27)	12·4 (16)	30·4 (16)	72·6 (15)
Whitechapel English	—	—	*39·6 (32)	—	—	—

The dacryal and simotic measurements of 50 ♂ Whitechapel skulls were first provided by Ryley and Bell (31, p. 397).

* See note ‡ above.

TABLE XIII. *Comparative Table of British Mean Female Skull Measurements for Various Periods.*

Character	British Neolithic	British Iron Age	Anglo-Saxon	Whitechapel English (17th Century)
Capacity	{1452·5 (61)}*	{1332·6 (22)}*	{1370·0 (28)}*	1299·9 (80)
F	186·5 (41)	180·8 (58)	180·9 (62)	180·1 (143)
L'	185·3 (3)	—	180·0 (23)	180·1 (57)
L	183·5 (13)	179·7 (22)	182·0 (55)	180·4 (140)
B	135·7 (42)	136·2 (62)	135·6 (67)	134·7 (140)
B'	94·2 (11)	97·1 (13)	94·3 (58)	93·1 (147)
H	138·8 (3)	—	130·3 (12)	—
H'	137·5 (6)	126·5 (42)	129·6 (28)	124·6 (124)
OH	115·5 (3)	—	110·5 (19)	109·2 (143)
LB	98·2 (6)	95·7 (34)	97·4 (26)	95·3 (122)
Q'	309·0 (4)	—	301·7 (18)	—
Bregmatic Q	—	—	—	294·0 (122)
S	374·0 (18)	362·0 (10)	366·5 (38)	362·8 (130)
S_1	125·4 (14)	123·7 (11)	124·4 (63)	—
S_2	128·4 (14)	123·0 (10)	123·2 (57)	—
S_3	117·3 (13)	115·7 (10)	115·2 (46)	—
U	520·4 (30)	510·7 (31)	510·9 (54)	503·8 (136)
$G'H$	63·0 (4)	63·6 (18)	65·9 (30)	65·9 (62)
GB	94·2 (3)	—	90·2 (26)	84·9 (58)
J	125·0 (11)	123·8 (24)	125·6 (26)	120·3 (33)
NH, R	46·9 (5)	—	48·4 (29)	48·7 (67)
NH, L		—	48·0 (27)	
NH'	49·0 (1)	47·9 (30)	47·1 (32)	—
NB	23·0 (6)	23·5 (28)	24·2 (24)	23·2 (64)
O_2R	30·8 (4)	33·1 (29)	32·8 (27)	33·7 (64)
O_2L			32·9 (26)	33·6 (64)
O_1R	—	—	41·6 (23)	41·0 (62)
O_1'	38·8 (4)	37·0 (2)	38·9 (20)	—
G_1	—	—	51·2 (13)	45·1 (57)
G_2	36·0 (1)	—	39·9 (23)	35·2 (58)
GL	90·8 (3)	91·8 (28)	94·0 (19)	90·4 (58)
fml	34·5 (4)	—	35·5 (19)	—
fmb	30·1 (4)	—	28·9 (19)	—
100 B/F	{72·8 (41)}†	{75·3 (58)}	75·2 (77)	{74·8 (140)}
100 B/L	{74·0 (13)}	{75·8 (22)}	74·4 (51)	74·7 (130)
100 H'/F	{73·7 (6)}	{69·9 (42)}	71·6 (37)	{69·2 (124)}
100 H'/L	74·4 (5)	{70·4 (22)}	71·5 (27)	69·1 (117)
100 B/H'	{98·7 (6)}	{107·7 (42)}	105·7 (42)	108·5 (115)
100 $(B-H')/L$	{−1·0 (6)}	{+5·4 (22)}	+2·6 (21)	{+5·6 (124)}
100 $G'H/GB$	{66·9 (3)}	—	72·1 (41)	77·9 (54)
100 $NB/NH, R$	49·3 (4)	—	51·0 (20)	47·8 (64)
100 $NB/NH, L$		—	51·1 (20)	
100 NB/NH'	51·0 (1)	{49·1 (28)}	50·2 (38)	—
100 O_2/O_1, R	—	—	79·6 (23)	82·5 (62)
100 O_2/O_1, L	—	—	79·4 (24)	81·7 (57)
100 O_2/O_1'	78·0 (3)	96·0 (2)	84·5 (37)	—
100 G_2/G_1	—	—	76·1 (12)	77·7 (51)
100 fmb/fml	{87·2 (4)}	—	81·7 (18)	—
$P\angle$	—	—	83°·9 (20)	87°·1 (52)
$N\angle$	{64°·0 (3)}	{67°·0 (18)}	67°·2 (15)	64°·7 (57)
$A\angle$	{77°·3 (3)}	{73°·4 (18)}	72°·6 (15)	73°·9 (57)
$B\angle$	{38°·7 (3)}	{39°·6 (18)}	40°·1 (15)	41°·4 (57)
$\theta_1\angle$	—	—	29°·0 (11)	28°·1 (50)
$\theta_2\angle$	—	—	10°·2 (11)	13°·1 (50)

* These capacities are those given by Hooke's reconstruction formula $C = \cdot000366 \times L \times B \times H' + 199\cdot4$ for English ♀ skulls, using mean measurements. See the present issue of *Biometrika*, p. 88.

† The indices and angles in curled brackets { } were calculated from the means of the component lengths.

folk and took wives from the natives they had conquered, but the direct evidence does not support that contention *. The ♂ and ♀ Anglo-Saxon skulls are clearly of the same racial type, which is chiefly distinguished from that of the British Iron Age people by its greater basio-bregmatic height and nasio-basion length. The agreement between the two sexes is less satisfactory for the mean facial measurements, all of which are based on smaller numbers than the calvarial, and the profile angles and angles of the fundamental triangle are particularly discordant. We may reasonably expect that lack of accordance to disappear when the means represent larger numbers of skulls.

The following additional mean measurements are available for the female Anglo-Saxons: $S_3' = 97{\cdot}1$ (33), $Oc.\ I. = 59{\cdot}7$ (32), $PH = 17{\cdot}2$ (32), $DS = 12{\cdot}1$ (16), $DC = 22{\cdot}0$ (21), $DA = 32{\cdot}6$ (16), $100\ DS/DC = 57{\cdot}3$ (16), $SS = 3{\cdot}8$ (19), $SC = 8{\cdot}7$ (23), $100\ SS/SC = 45{\cdot}4$ (19), $EB = G_2 = 39{\cdot}9$ (23), $EH = 10{\cdot}4$ (16), $100\ EH/EB = 26{\cdot}6$ (16), $G_1'' = 53{\cdot}6$ (12), $G_1' = 47{\cdot}0$ (13), (Whitechapel $G_1' = 41{\cdot}5$ (58)), $100\ G_2/G_1' = 82{\cdot}9$ (12), (Whitechapel $100\ G_2/G_1' = \{84{\cdot}9\ (58)\}$) †, $Q = 299{\cdot}2$ (16), $O_1L = 41{\cdot}2$ (25), (Whitechapel $O_1L = 41{\cdot}2$ (57)), $100\ H/L = 72{\cdot}5$ (12), $100\ B/H = 104{\cdot}2$ (12).

(6) *The Standard Deviations of Early British Series of Crania.*

In Table XIV are given the standard deviations of 6 characters, and the numbers of skulls on which each is based, for the series of which the mean measure-

TABLE XIV. *The Standard Deviations of Characters for Various British Series of Crania.*

Character	MALES					FEMALES	
	Whitechapel English (17th Century)	British Neolithic	English Bronze Age	British Iron Age	Anglo-Saxon	Whitechapel English	Anglo-Saxon
$100\ B/L$ or $100\ B/F$ ‡	$3{\cdot}26 \pm {\cdot}14$ 131	$3{\cdot}02 \pm {\cdot}13$ 116	$3{\cdot}14 \pm {\cdot}16$ 89	$3{\cdot}17 \pm {\cdot}16$ 86	$3{\cdot}12 \pm {\cdot}14$ 112	$2{\cdot}98 \pm {\cdot}12$ 130	$3{\cdot}23 \pm {\cdot}18$ 77
L or F ‡	$6{\cdot}27 \pm {\cdot}25$ 137	$6{\cdot}36 \pm {\cdot}28$ 116	$5{\cdot}88 \pm {\cdot}30$ 90	$5{\cdot}71 \pm {\cdot}29$ 89	$5{\cdot}49 \pm {\cdot}27$ 94	$6{\cdot}22 \pm {\cdot}25$ 140	$5{\cdot}93 \pm {\cdot}36$ 63
B	$5{\cdot}28 \pm {\cdot}22$ 135	$4{\cdot}65 \pm {\cdot}16$ 128	$6{\cdot}67 \pm {\cdot}34$ 89	$5{\cdot}08 \pm {\cdot}24$ 102	$5{\cdot}79 \pm {\cdot}27$ 103	$4{\cdot}77 \pm {\cdot}19$ 190	$6{\cdot}11 \pm {\cdot}35$ 67
H'	$5{\cdot}56 \pm {\cdot}24$ 122	— —	— —	$5{\cdot}55 \pm {\cdot}30$ 77	$5{\cdot}81 \pm {\cdot}39$ 31	$4{\cdot}93 \pm {\cdot}21$ 124	$4{\cdot}97 \pm {\cdot}48$ 24
U	$15{\cdot}02 \pm {\cdot}63$ 131	$14{\cdot}46 \pm {\cdot}67$ 106	$17{\cdot}72 \pm {\cdot}93$ 82	$13{\cdot}80 \pm {\cdot}76$ 73	$11{\cdot}79 \pm {\cdot}66$ 73	$14{\cdot}70 \pm {\cdot}60$ 136	$13{\cdot}62 \pm {\cdot}88$ 54
$100\ NB/NH, R$ or $100\ NB/NH'$	$4{\cdot}58 \pm {\cdot}26$ 70	— —	— —	$4{\cdot}82 \pm {\cdot}29$ 62	$3{\cdot}95 \pm {\cdot}27$ 58	$3{\cdot}90 \pm {\cdot}23$ 64	$5{\cdot}01 \pm {\cdot}39$ 38

* Some standard deviations of the Anglo-Saxon ♀ measurements are given in Table XIV, and it will be seen that they indicate a population which is not more variable than that of the Whitechapel English from a single pit.

† The indices and angles in curled brackets { } were calculated from the means of the component lengths.

‡ The Whitechapel English standard deviations are for $100\ B/L$ and L while for all the other series in the table they are for $100\ B/F$ and F.

ments are in Tables XII and XIII. Where fewer than 30 crania were available the standard deviations were not calculated except in the case of the H' of the ♀ Anglo-Saxons, which is of special interest. The Whitechapel skulls came from a single pit and may be supposed contemporaneous. Each of the other series was made up by clubbing together the skulls of the same epoch from different, and often widely scattered, localities, and a period of several hundred years must be allowed between the dates of death of the earliest and latest individuals representing the same type. In spite of those differences there is on the whole a surprising parity between the variabilities of the modern series and the others. For the ♂ skulls there are only two standard deviations which differ significantly—by more than 2·5 times the probable error of the difference say—from the Whitechapel values. The standard deviation of the skull breadths of the Bronze Age skulls is significantly greater and the Anglo-Saxon variability of the horizontal circumference (U) is significantly less. In computing C.R.L.'s the assumption is made that all the series compared have the same variabilities for all characters as the standard Egyptian series—the E series of 26th to 30th Dynasty skulls from Gizeh. It is necessary to make that assumption in the present state of our knowledge, but it is probably only an approximation to the truth. It is more probable that the variabilities of a homogeneous series of the same race are peculiar to that race. It is at least suggestive that the Bronze Age skulls, which are the broadest and most brachycephalic, should have the greatest standard deviation of skull breadth while the most dolichocephalic and narrowest skulls—the Neolithic—have the least. The inter-racial coefficients of variation are probably more constant than the standard deviations. The numbers of skulls on which the constants in Table XIV are based are obviously too small to furnish really distinctive racial values, but we may rest assured that the Anglo-Saxon and prehistoric series compiled from various sources are as homogeneous as the series usually accepted by the craniometrician as racially unique.

(7) *Anglo-Saxon Type Contours.*

Male and female type contours (Figs. I—VI) of the Anglo-Saxon skulls in London museums were constructed from the measurements of contours of individual skulls by the methods first used by Benington and described in detail in *Biometrika*, Vol. XIV. 1923, pp. 227—240. In dealing with a cranial series in that way it has been customary to exclude all specimens with defective facial bones for which the Frankfurt horizontal plane cannot be determined, but, as a large proportion of the small total population of Anglo-Saxon skulls would have been excluded by following that practice, it was thought better to use the contours of the imperfect skulls. That was done by supposing that the vertex of such a one was 10·5 mm. behind the bregma in the case of a ♂ skull and 12·9 mm. behind in the case of a ♀, while the gamma was taken 30·8 mm. below the lambda for a ♂ skull and 29·0 mm. below for a ♀, all the points being located, of course, in the median sagittal plane. Those distances are the means given by the complete skulls. The mean contour measurements of the complete and incomplete

series were then calculated separately and found to agree closely. The following, given by the ♂ horizontal contours, are typical.

		Complete Skulls	Incomplete Skulls (Gamma taken 30·8 from lambda)
	FO	191·7 (19)	190·4 (21)
Ordinates	3*R*	51·2 (19)	49·7 (19)
	5*R*	65·6 (19)	63·6 (14)
	7*R*	70·9 (18)	70·2 (19)
	9*R*	60·1 (17)	59·4 (23)

The pooled mean measurements from which the types were constructed are given in Tables XV, XVI and XVII, and they show a satisfactory agreement with the means of the direct measurements with which comparison can be made. The numbers of skulls are large enough to give good first approximations to the type outlines except in the case of the facial section of the sagittal contours. The most significant differences between the Anglo-Saxon skull and that of the 17th century Londoners can be clearly appreciated by superposing the profile views—whether ♂ on ♂ or ♀ on ♀—with the aid of the tracings provided for both in this number of *Biometrika*. In place of the low calvaria and markedly retreating forehead of the more modern type, the Anglo-Saxons had a reasonably high brain-box and quite a prominent frontal bone.

(8) *Measurements of Anglo-Saxon Mandibles.*

Measurements of all the Anglo-Saxon mandibles in the Museum of the Royal College of Surgeons, the British Museum (Natural History) and the London Museum were taken by the methods described in detail in *Biometrika*, Vol. XIV. 1923, pp. 253—260. All old, senile and immature bones were excluded, as were those of which the form might possibly have been modified by the loss of a considerable number of teeth. Of the remaining 132 specimens, the sex of 98 (49 ♂ and 49 ♀) could be judged from the skull and the isolated mandibles were supposed ♂? or ♀? after a careful comparison with the ones for which the complete skulls were available. Without some such control the sex determined from the lower jaw alone is certainly more uncertain than that determined from the skull without the mandible. The series is longer than any previous one measured by the same methods, but, owing to the defective state of many of the specimens, the means (Table XVIII) are not based on sufficiently large numbers to make it worth while to calculate the constants of variation and correlation. The proportional sexual difference in size appears to be of the same order as that found for calvarial and facial measurements and the ♂ and ♀ indices and angles are so similar that, for the small numbers dealt with, the differences would almost certainly not be significant.

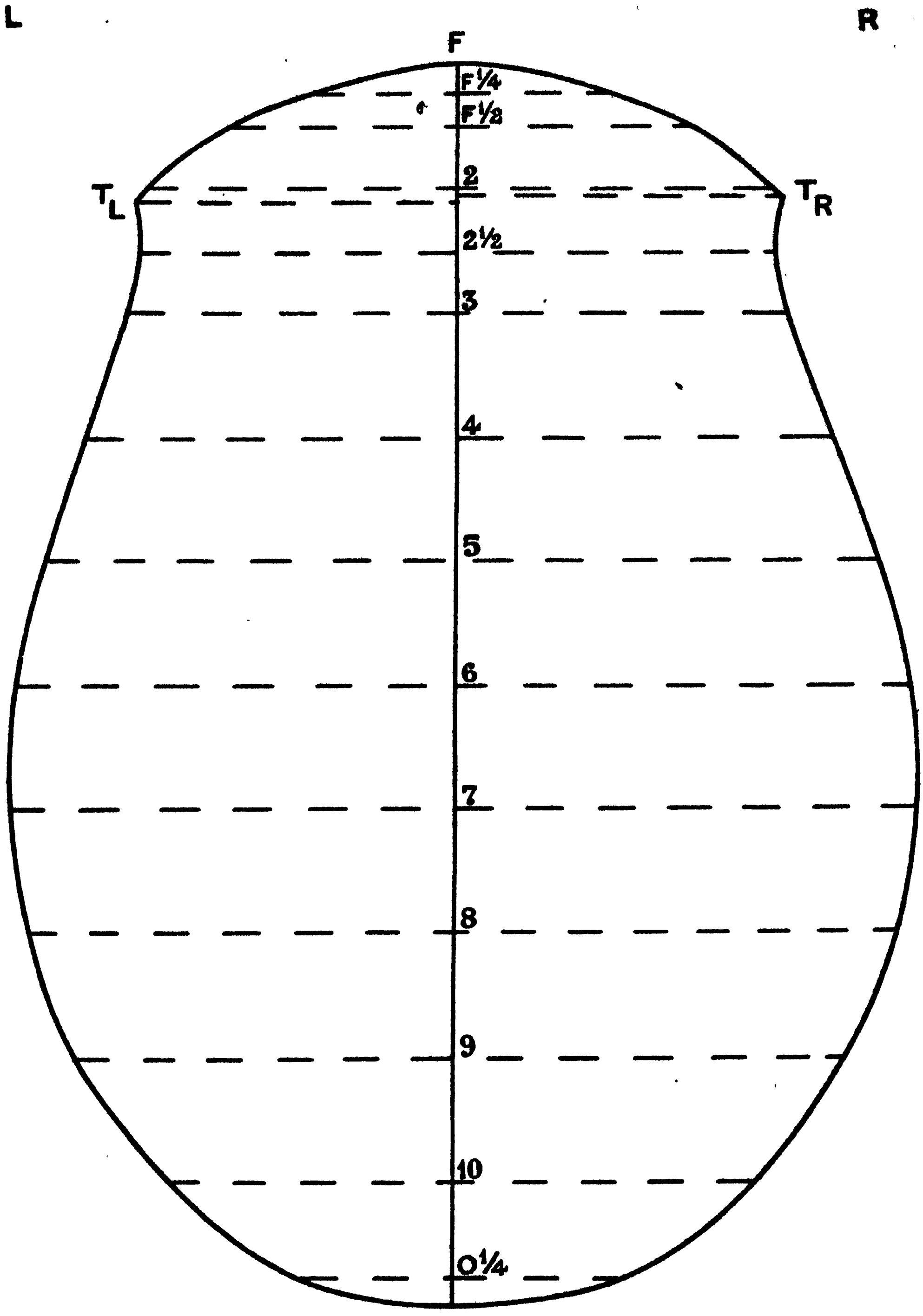


FIG. I Anglo-Saxon ♂ Horizontal Type Contour.

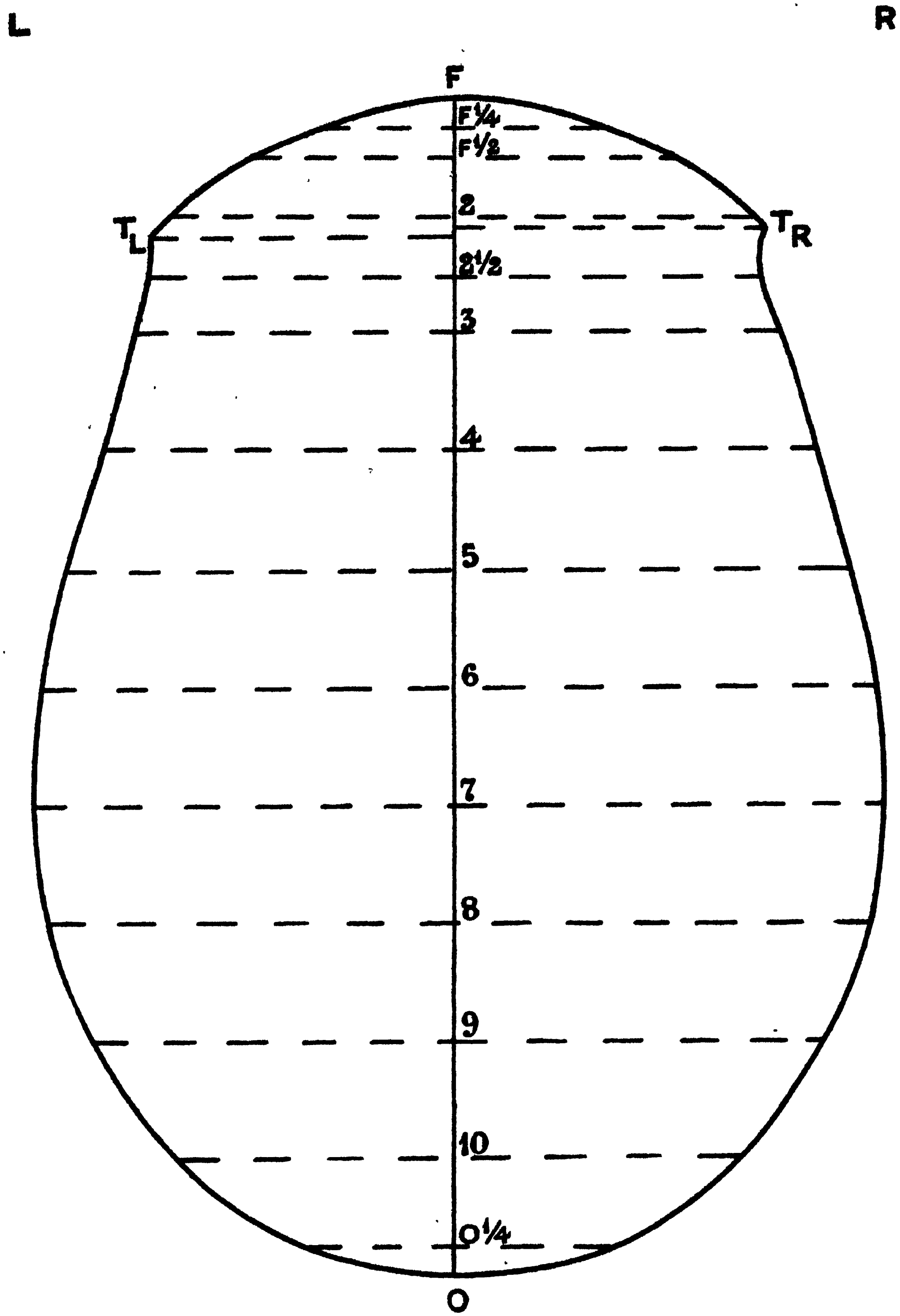


Fig. II Anglo-Saxon ♀ Horizontal Type Contour.

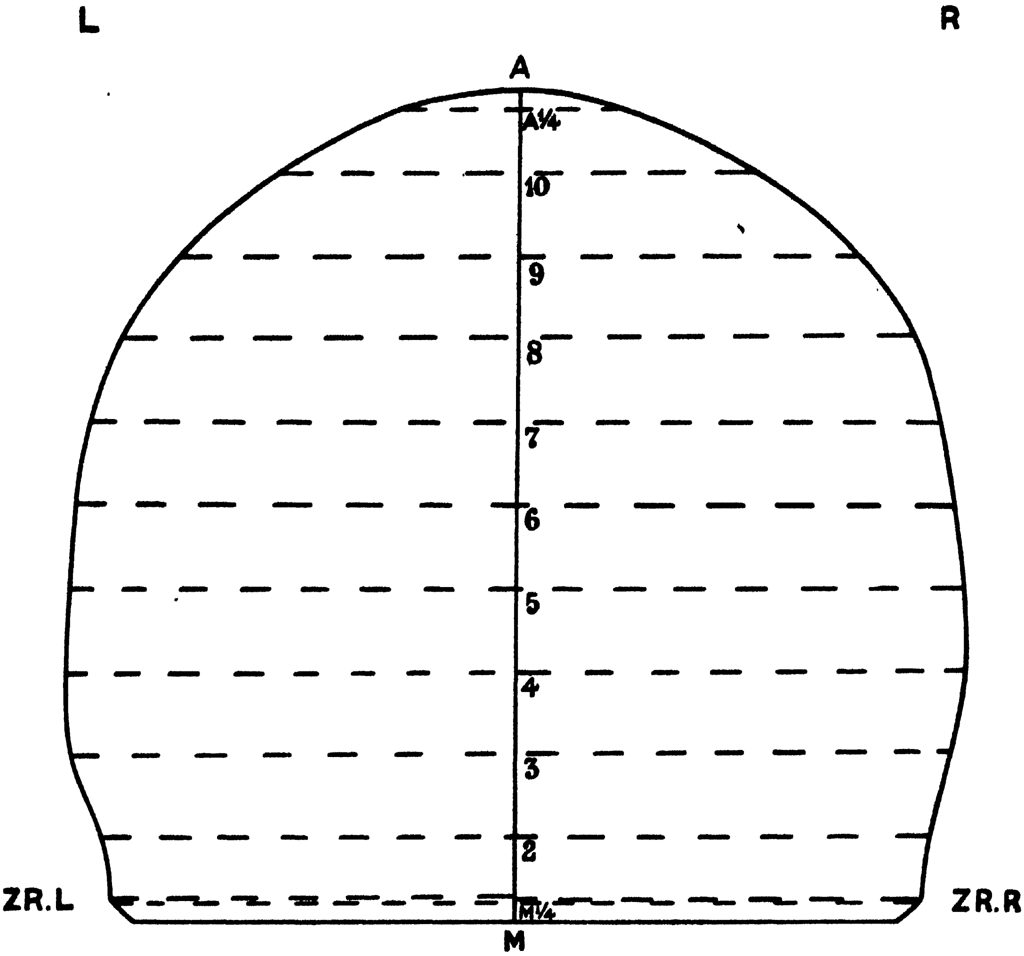


FIG. III Anglo-Saxon ♂ Transverse Type Contour.

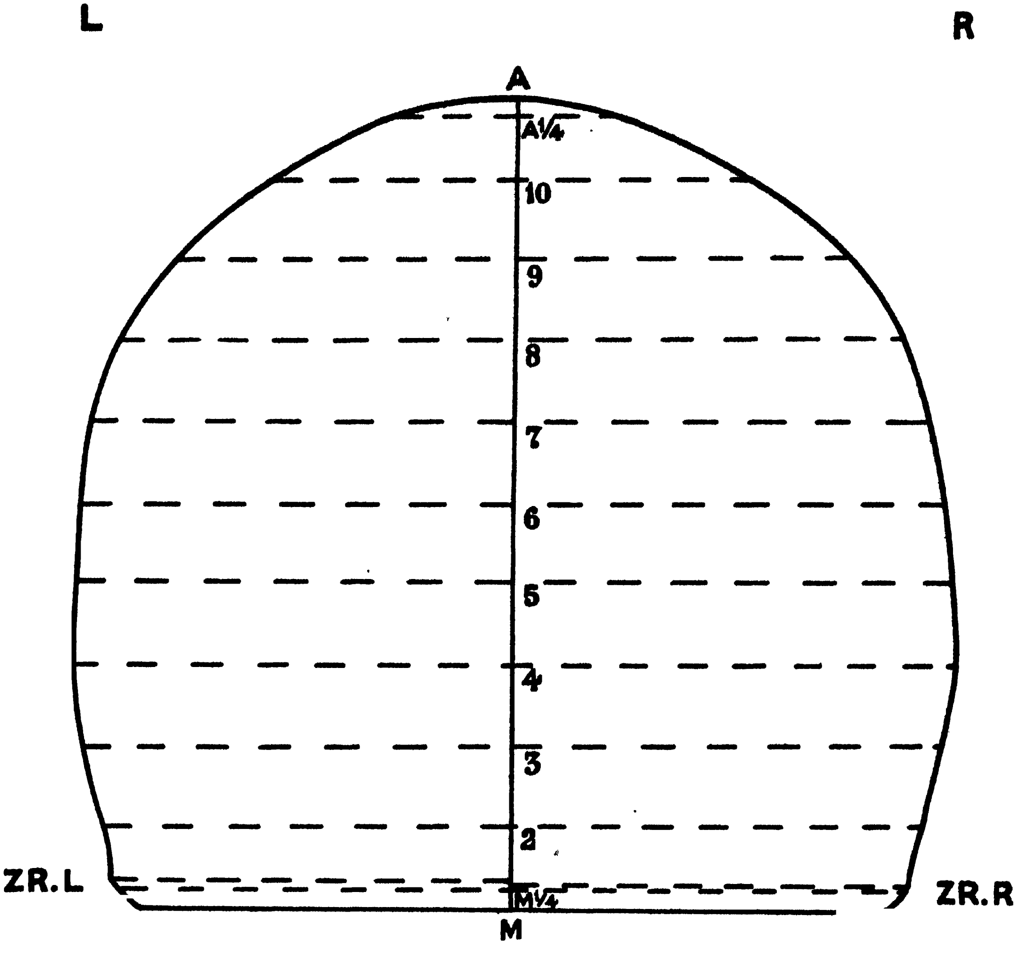


FIG. IV Anglo-Saxon ♀ Transverse Type Contour.

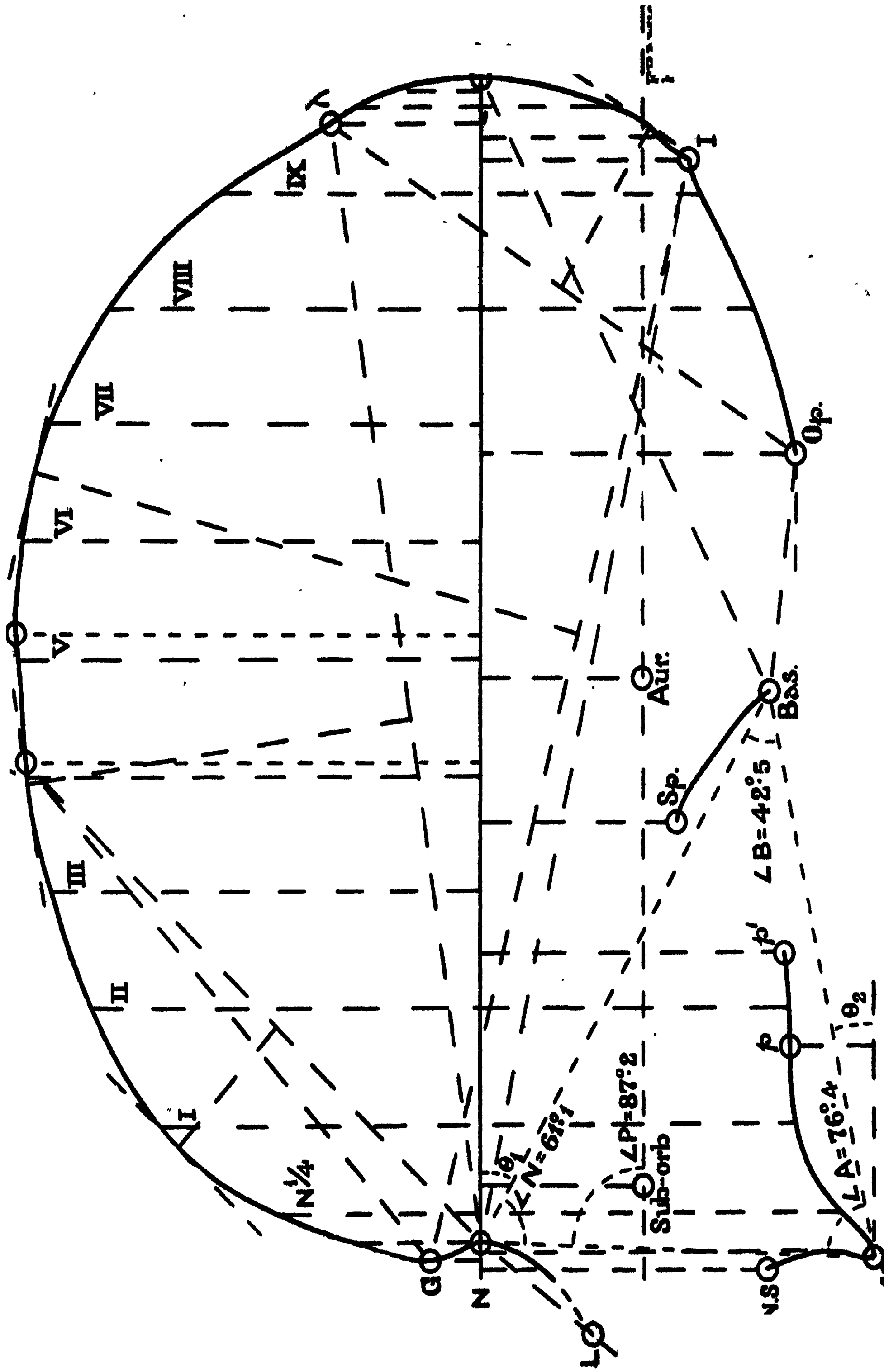


FIG. V Anglo-Saxon ♂ Sagittal Type Contour.

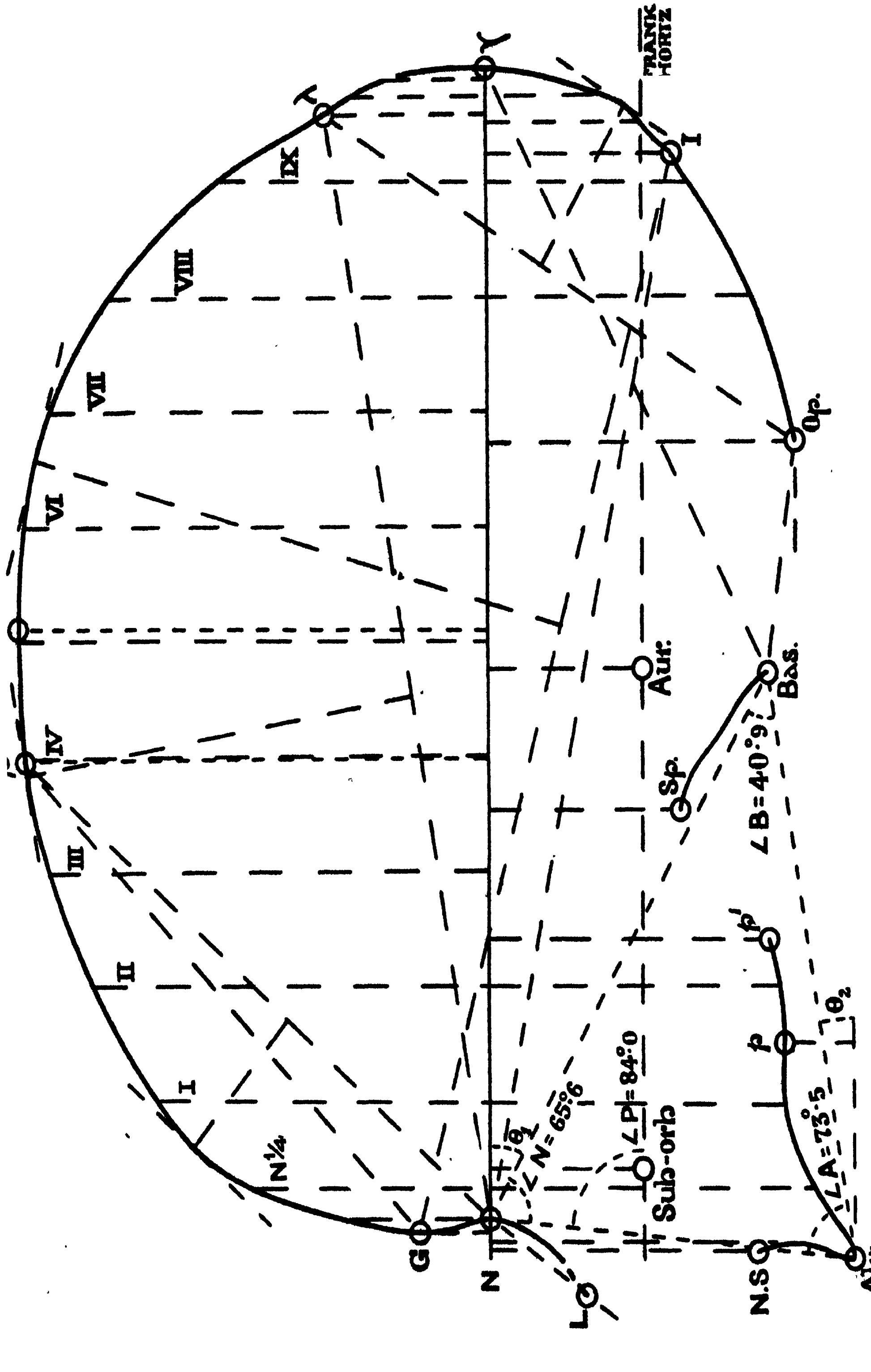


FIG. VI Anglo-Saxon ♀ Sagittal Type Contour.

TABLE XV.

Measurements of Anglo-Saxon Horizontal Type Contours.

	FO	F¼R	F¼L	F½R	F½L	2R	2L	2½R	2½L	3R	3L	4R	4L	5R	5L
♂	191·0 (43)	23·0 (42)	21·9 (42)	36·0 (42)	35·0 (39)	47·8 (41)	47·1 (39)	45·4 (41)	48·2 (38)	50·4 (38)	50·5 (36)	57·4 (29)	56·4 (29)	64·6 (33)	62·6 (31)
♀	178·2 (43)	23·3 (41)	20·7 (42)	34·8 (42)	31·9 (40)	46·1 (40)	44·3 (39)	48·2 (38)	46·9 (40)	50·6 (37)	49·4 (39)	56·0 (33)	54·5 (34)	61·1 (36)	60·0 (37)

	6R	6L	7R	7L	8R	8L	9R	9L	10R	10L	O¼R	O¼L	TR		TL	
													x	y	x	y
♂	69·6 (34)	67·1 (34)	70·5 (37)	67·8 (37)	67·4 (36)	65·0 (40)	59·7 (40)	57·9 (38)	46·0 (43)	43·3 (41)	26·5 (43)	23·4 (43)	20·1 (42)	49·7 (42)	21·3 (39)	49·2 (39)
♀	65·5 (39)	63·9 (39)	66·0 (39)	65·0 (39)	64·9 (41)	62·7 (39)	57·2 (41)	55·4 (39)	44·4 (42)	42·3 (40)	25·7 (42)	23·1 (41)	19·7 (41)	48·1 (41)	20·9 (40)	47·0 (40)

TABLE XVI.

Measurements of Anglo-Saxon Transverse Type Contours.

	MA	1R=1L	M¼R	M¼L	2R	2L	3R	3L	4R	4L	5R	5L	6R	6L
♂	114·0 (32)	58·4 (32)	61·9 (32)	61·5 (32)	63·4 (30)	63·6 (29)	66·4 (22)	68·4 (21)	68·8 (23)	69·2 (25)	68·4 (31)	68·5 (28)	67·2 (32)	67·4 (31)
♀	111·4 (31)	55·7 (31)	58·8 (31)	58·7 (31)	61·4 (31)	61·1 (31)	63·9 (28)	64·6 (25)	66·2 (27)	66·1 (28)	65·4 (29)	65·6 (30)	64·2 (30)	65·1 (30)

	7R	7L	8R	8L	9R	9L	10R	10L	A¼R	A¼L	ZR, R		ZR, L	
											y	x	y	x
♂	64·9 (32)	65·6 (31)	60·8 (32)	60·9 (30)	52·3 (32)	52·2 (30)	36·8 (32)	37·1 (32)	15·7 (32)	18·5 (32)	62·2 (32)	3·0 (32)	62·0 (32)	3·5 (32)
♀	62·2 (30)	63·8 (31)	58·2 (30)	59·2 (31)	49·9 (31)	51·0 (30)	34·9 (31)	36·5 (31)	15·1 (30)	18·8 (31)	59·2 (31)	3·3 (31)	59·9 (31)	3·7 (31)

TABLE XVII. *Measurements of Anglo-Saxon Sagittal Type Contours.*

	$N\gamma$	Ordinates above $N\gamma$												
		$0=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$
♂	186·4 (42)	22·5 (42)	37·3 (42)	58·9 (41)	71·3 (41)	78·4 (41)	82·7 (41)	84·2 (40)	83·4 (41)	79·1 (42)	68·3 (41)	47·8 (42)	23·1 (42)	16·6 (42)
♀	176·1 (44)	27·0 (43)	43·6 (43)	59·3 (43)	71·3 (44)	77·8 (43)	81·7 (42)	83·4 (42)	82·6 (43)	78·0 (43)	67·5 (42)	48·2 (43)	24·5 (44)	18·6 (44)

	Ordinates below $N\gamma$							Vertex		β		Glabella	
	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from N	y	x from N	y	x from N	y
♂	65·4 (19)	57·9 (20)	56·4 (17)	49·9 (41)	40·1 (41)	32·5 (41)	23·8 (41)	97·0 (40)	85·2 (40)	76·9 (41)	82·9 (41)	2·9 (42)	9·3 (42)
♀	56·7 (26)	52·0 (25)	51·5 (13)	46·9 (40)	35·6 (41)	27·2 (41)	20·3 (42)	89·5 (41)	83·9 (41)	70·2 (44)	82·1 (44)	2·3 (43)	12·4 (43)

	Occipital Pt.		λ		Sub-Orb.		Aur. Pt.		Opisthion		Inion		Basion	
	x from γ	y	x from γ	y	x from N	y	x from γ	y	x from γ	y	x from γ	y	from γ	from N
♂	0·5 (40)	−0·9 (40)*	7·2 (41)	27·2 (41)	9·0 (20)	29·7 (20)	95·9 (35)	29·3 (35)	60·0 (21)	58·2 (21)	13·3 (41)	37·8 (41)	111·4 (23)	102·7 (23)
♀	0·5 (43)	2·0 (43)	6·8 (44)	28·4 (44)	7·3 (28)	26·8 (28)	92·6 (33)	27·2 (33)	57·8 (30)	54·3 (30)	13·4 (41)	32·7 (41)	106·4 (18)	95·6 (18)

	Alv. Pt.		Nose					Palate				Frontal	
								P'		P		Max. Sub. to $N\beta$	
	from N	from Bas.	(i)	(ii)	(iii)	$\angle LN\gamma$	NL	x from N	y	x from Alv.	y	x from N	y
♂	71·8 (15)	92·8 (15)	0·8 (18)	3·1 (15)	6·3 (7)	125°·9 (2)	25·3 (2)	46·1 (16)	55·2 (16)	33·8 (15)	15·4 (15)	50·8 (41)	26·5 (41)
♀	65·2 (27)	91·0 (13)	1·0 (23)	3·4 (21)	6·3 (14)	124°·8 (6)	20·3 (6)	42·1 (13)	49·5 (13)	33·0 (21)	12·6 (21)	47·0 (42)	26·1 (42)

	Occipital Max. Sub. to λ Op.		Max. Sub. to $N\lambda$		Max. Sub. to GI		Sp.		Sub. from ½ Bas. Sp. Chord		N. S.		Vert. Tang. to Alv. N. S. Arc	
	x from λ	y	x from N	y	x from G	y	x from N	y	x from Bas.	y	x from N	y	x from N	y
♂	49·6 (21)	30·4 (21)	85·0 (38)	71·1 (38)	104·0 (41)	102·6 (41)	67·1 (22)	35·6 (22)	13·5 (22)	1·7 (22)	4·5 (19)	52·0 (19)	1·8 (17)	62·6 (17)
♀	44·6 (29)	28·4 (29)	80·9 (44)	69·8 (44)	96·0 (38)	97·2 (38)	62·0 (22)	33·7 (22)	12·4 (18)	1·2 (18)	5·2 (27)	47·2 (27)	4·0 (25)	52·9 (25)

* The negative sign indicates that the occipital point is below the $N\gamma$ line.

TABLE XVIII. *Anglo-Saxon Mean Mandibular Measurements.*

	w_1	w_2	h_1	zz	$c_r c_r$	rb	rb'	G_2'	$c_y c_r$
♂	123·7 (25)	103·2 (45)	33·1 (40)	45·3 (57)	100·3 (27)	36·4 (58)	33·2 (61)	48·7 (43)	33·9 (40)
♀	116·6 (22)	96·0 (38)	30·5 (31)	44·1 (50)	93·2 (28)	34·6 (53)	31·0 (56)	47·5 (46)	33·0 (37)

	$g_o g_o$	$g_n g_o$ (l)	$g_n g_o$ (r)	$c_y l$	$c_y b$	$m_2 p_1$	$p_a d_t$	$p_a g_n$	$p_a d_f$
♂	100·4 (33)	87·9 (38)	89·9 (41)	21·7 (38)	9·5 (42)	28·1 (59)	28·2 (38)	7·1 (59)	25·3 (41)
♀	92·9 (35)	83·3 (38)	83·4 (40)	19·1 (35)	8·5 (42)	27·6 (57)	25·0 (30)	6·9 (55)	22·5 (33)

	$g_n d_f$	$p_a p_b$	$g_o p_a g_o$	ih	ih'	$c_r h$	$c_y h$	$d_t h$	$m_2 h$
♂	30·0 (41)	3·1 (6)	198·8 (39)	47·9 (51)	13·6 (35)	65·7 (48)	59·4 (44)	36·3 (40)	27·2 (51)
♀	27·5 (31)	4·2 (11)	187·2 (33)	43·0 (51)	12·7 (36)	59·2 (47)	54·7 (47)	33·0 (29)	24·4 (52)

	$p_1 h$	$c_p l$	rl	ml	$100\, c_r h/ml$	$100 c_r c_r/ml$	$100\, g_o g_o/c_p l$	$100\, rb'/rl$
♂	30·9 (54)	77·7 (42)	64·0 (45)	107·2 (31)	60·9 (27)	94·4 (15)	129·0 (32)	51·5 (45)
♀	27·8 (47)	74·6 (49)	59·1 (45)	104·2 (45)	58·3 (38)	91·7 (26)	126·2 (35)	53·0 (43)

	$100\, c_y b/c_y l$	$100\, g_o g_o/c_r c_r$	$100\, c_y h/c_r h$	$100\, ih'/c_y c_r$	$100 d_t h/c_r h$	$M \angle$	$R \angle$	$C \angle$
♂	44·0 (38)	99·3 (19)	89·9 (40)	40·4 (35)	55·2 (30)	120°·3 (47)	72°·0 (36)	68°·2 (32)
♀	43·7 (36)	99·3 (23)	89·6 (40)	39·0 (35)	56·4 (25)	122°·5 (49)	68°·2 (36)	70°·0 (25)

	$C' \angle$	$L \angle$	$L' \angle$	$F \angle$	$S \angle$	$G \angle$
♂	68°·3 (35)	82°·6 (6)	77°·9 (6)	87°·0 (6)	11°·5 (6)	68°·6 (31)
♀	70°·6 (32)	88°·0 (7)	87°·2 (7)	87°·3 (11)	14°·7 (6)	68°·5 (34)

APPENDIX I.

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APPENDIX II.

Individual Measurements of Anglo-Saxon Skulls in London Museums.

Individual measurements of all the adult and undeformed Anglo-Saxon skulls in the Museum of the Royal College of Surgeons (R.C.S. in tables), the British Museum (Natural History) (B.M.) and the London Museum (L.M.) are given in the following tables. A considerable number that had been posthumously distorted by earth pressure were not measured. Some measurements of nearly all the Royal College skulls have been given by Barnard Davis (19), Flower (27), Parsons (32 and 33) and Peake and Hooton (34). The sex was indicated in many cases by the artifacts interred with the body, and, where such evidence was available, the anatomical characters confirmed that estimate. Where there was no grave furniture the sex was judged by the present writer after assembling all the material in the particular museum. The divisions in the tables (Column 4) only correspond roughly with the Saxon kingdoms (see p. 76 of text). A. = Angle, J. = Jute, W.S. = West Saxon, S.S. = South Saxon. The abbreviations used in the references are: *J.R.A.I.* = Journal of the Royal Anthropological Institute, *Arch.* = Archaeologia: or Miscellaneous Tracts relating to Antiquity published by the Society of Antiquaries of London. The museum numbers preceded by B.D. are those of Barnard Davis' catalogue (19). The measurements and methods of measurement employed are precisely those of which a list is given by Hooke in the present issue of *Biometrika*, pp. 15—16, with the single addition of G_1'', a palate length measured from the alveolar point to the tip of the posterior nasal spine. The greater number of the remarks relating to cranial anomalies usually given by workers in the Biometric Laboratory have had to be omitted as nearly all the skulls are defective. Many of the more common anomalies were, however, found. Unless otherwise stated a specimen is fully adult, showing no signs of aging. The teeth, often considerably worn, gave the impression of being better preserved than those of 17th century Londoners. There were several cases of carious teeth. Out of 127 skulls there were 2 evident and 2 rather doubtful cases of trepanation, all 4 being on ♂ skulls. One ♀ skull from West Harnham, Wilts., now in the Royal College of Surgeons, appears to be a clear case of artificial deformation, though the only one found among Anglo-Saxon skulls. It is figured by Barnard Davis (19, p. 30). Several skulls bore bluish-green stains evidently made by the copper in metal artifacts buried with the body.

THE PRESENT STATE OF OUR KNOWLEDGE OF BRITISH CRANIOLOGY IN LATE PREHISTORIC AND HISTORIC TIMES.

By BEATRIX G. E. HOOKE AND GEOFFREY M. MORANT, M.Sc.

Two contributions to the study of British craniology are provided in this number of *Biometrika*: that giving individual measurements of 350 skulls of the 17th century Londoners compared with other modern material furnished by one of the present writers, and the measurements of Anglo-Saxon skulls in London museums compared with other series of earlier date furnished by the other. Both are biometric studies continuing the survey of British material which was inaugurated by the two classical papers of Macdonell. Free use has been made of the method of the Coefficient of Racial Likeness which has proved an additional aid of extreme value since those pioneer studies were made. It can hardly be necessary to acknowledge in the pages of *Biometrika* the fact that as regards descriptive technique and methods of analysis we have been guided entirely by the procedure which owes its inception to Professor Karl Pearson, but we would wish to record our more direct indebtedness to him for inspiring and aiding us in this work.

Our present purpose is to compare the earlier and later groups of cranial series with one another and thus to provide a *résumé* of the present state of our knowledge of British ethnology as revealed by cranial remains, and, perhaps we ought to add, as envisaged by those who approach such questions from the biometric standpoint. We are probably far more conscious of the incompleteness of the existing evidence than the critics of such methods will be. Of all the available series there is only one—the Farringdon Street—which fulfils the two main requirements of the craniometrician in that it is large enough to furnish reliable statistical criteria and that it has been adequately described by direct measurements and type contours. Macdonell's Whitechapel series nearly fulfils both requirements, but all the other series dealt with are defective in one way or the other or in both. We present the following tentative conclusions, then, with a full consciousness of the limitations of the material and the hope that they will shortly be confirmed or modified by more adequate studies.

The Coefficients of Racial Likeness (C.R.L.'s) between the ♂ and ♀ series of 17th century London skulls measured in the Biometric Laboratory and the Anglo-Saxon and earlier types for which mean measurements could be found are given in Table I. From a comparison of them and of the C.R.L.'s previously given we conclude that:

(*a*) In late Neolithic times, and probably in early Neolithic times also, the population of England and Scotland was racially homogeneous and of a type

TABLE I. *Coefficients of Racial Likeness of 17th Century London and Earlier British Series.*

		British Neolithic	English Bronze Age	Scottish Bronze Age	British Iron Age	Anglo-Saxon
		Male				
		(38·0)*	(32·6)*	(18·7)*	(53·4)*	(41·2)*
Farringdon Street (95·6)	All Characters ...	13·60 ± ·18 (27)	15·11 ± ·18 (27)	7·50 ± ·20 (21)	2·82 ± ·20 (22)	5·27 ± ·17 (30)
	Indices and Angles...	18·84 ± ·30 (9)	19·77 ± ·30 (9)	5·67 ± ·33 (7)	4·32 ± ·36 (6)	7·82 ± ·29 (10)
Whitechapel (93·5)	All Characters ...	7·44 ± ·18 (27)	18·83 ± ·18 (28)	7·12 ± ·20 (22)	0·38 ± ·20 (23)	2·98 ± ·19 (27)
	Indices and Angles...	8·33 ± ·30 (9)	25·35 ± ·29 (10)	5·85 ± ·33 (7)	0·57 ± ·33 (7)	2·14 ± ·29 (10)
Moorfields (28·8)	All Characters ...	8·29 ± ·19 (23)	9·54 ± ·19 (24)	5·87 ± ·22 (18)	1·55 ± ·21 (19)	4·88 ± ·19 (25)
	Indices and Angles...	13·41 ± ·33 (7)	12·60 ± ·32 (8)	6·12 ± ·38 (5)	3·40 ± ·38 (5)	6·96 ± ·32 (8)
Mean Coefficient	All Characters ...	9·78	14·49	6·83	1·58	4·38
		Female				
					(31·9)*	(32·1)*
Farringdon Street (114·2)	All Characters ...	—	—	—	2·81 ± ·21 (19)	6·51 ± ·17 (31)
	Indices and Angles...	—	—	—	3·02 ± ·36 (6)	7·72 ± ·27 (11)
Whitechapel (91·4)	All Characters ...	—	—	—	1·84 ± ·22 (17)	9·61 ± ·20 (22)
	Indices and Angles...	—	—	—	0·55 ± ·38 (5)	7·99 ± ·32 (8)
Moorfields (38·9)	All Characters ...	—	—	—	1·32 ± ·22 (17)	4·66 ± ·19 (25)
	Indices and Angles...	—	—	—	3·22 ± ·38 (5)	6·45 ± ·32 (8)
Mean Coefficient	All Characters ...	—	—	—	1·99	6·93

* These are the mean numbers of skulls on which the Iron Age and Anglo-Saxon mean characters used in computing the female Coefficients are based.

This table indicates that the modern English skull resembles more closely the British Iron Age type than the Anglo-Saxon. In fact the three modern English series are on the average nearer to the Iron Age type than they are to each other (mean C.R.L.'s: ♂'s, 2·67; ♀'s, 4·02), although the three groups were inhabitants of the same city, living practically at the same period, only in different parishes.

which is clearly distinguished from that of the races which predominated in later times. Its salient characters are a most extreme skull length and a low cephalic index. The brain box was of a normal height.

(*b*) An invading race introduced the Bronze Age culture into Britain. The invaders arrived on the east coast, spread westwards and settled in considerable numbers over the whole island. Being brachycephalic they can be clearly distinguished from the people they conquered. There appears to have been some blood admixture between the two races, and some skulls of the pure Neolithic type are found associated with Bronze Age artifacts, but it is highly probable that the remnant of the indigenous population which was not exterminated was absorbed by the invading people without producing any sensible modification of that type. The English and Scottish Bronze Age populations with which we have acquaintance are very similar, but not sufficiently so to warrant the assumption that they belonged to identically the same racial population.

(*c*) It is surprising to find that by the Iron Age the brachycephalic Beaker people had almost entirely disappeared from England. A considerable number of skulls from widely scattered areas and ranging in age from the early Iron Age to Roman times all form, with some doubtful exceptions which are certainly not of the Bronze Age type, a homogeneous population. It is distinguished from the Neolithic type by its shorter length, greater cephalic index—it being only just dolichocephalic—and, most characteristically, by its low vault and retreating frontal bone. The few Scottish Iron Age skulls of which measurements have been given all come from the West Lowlands and they conform to identically the same type. We can only suppose that it represents an invading people which entirely displaced and probably largely exterminated the settled Bronze Age folk over the greater part of Britain. From the fact that a brachycephalic type is found in Scotland in later historic times we may infer that some of the Bronze Age people avoided extinction by seeking shelter in the more inaccessible regions of that country, but more direct evidence in support of such a theory will be needed before it can be considered established. As far as we can tell the miscegenation of the natives with alien elements introduced into England at the time of the Roman occupation had no permanent effect on the predominating Iron Age type.

(*d*) That the Anglo-Saxons came to England and the south of Scotland in large numbers is evidenced by the numerous grave-yards that have been found. But the skulls of which measurements have as yet been published are miserably few in number. They were all interred between the 5th and the end of the 9th century and no distinction can be found between the remains from the earlier and later settlements. Angles, Saxons and Jutes all belonged, as far as we can tell, to a single homogeneous racial type which is in some ways similar to that of the British Iron Age population—the two have almost identical cephalic indices—but it is clearly distinguished from that of the conquered people by its greater skull height and absence of a retreating forehead. The ♀ skulls are of

precisely the same type as the ♂, suggesting that the two peoples lived side by side without mixing until at least as late as the 10th century.

(e) There are no adequately described series of English skulls of mediaeval date. The cranial type of Londoners of the 17th century is now well-known by the three collections—the Whitechapel, Moorfields and Farringdon Street—measured in the Biometric Laboratory. The three are very similar but not sufficiently so to warrant the assumption that they are random samples drawn from a single homogeneous population. All show a decidedly stronger bond of affinity with the British Iron Age type than with any other of the early ones (cf. the C.R.L.'s in Table I), and the C.R.L. with the Whitechapel skulls is so low that the assumption of identity of type is almost justified. The two appear to be more closely related to one another than the Whitechapel series is to the other two London ones contemporary with it. It can be definitely stated that the divergence of the Farringdon Street and Moorfields type from that of the Whitechapel and Iron Age skulls is not towards the Anglo-Saxons but in the opposite direction (see Table I)*. The two former show the distinguishing characters of a low calvarial height and a markedly retreating forehead more accentuated than do the two latter. A population of modern skulls from the Lowlands of Scotland measured by Sir William Turner (the Lowland Type (48·2)) gives means almost indistinguishable from those of the British Iron Age (53·4): Coefficients are found of $0·94 \pm ·19$ for 23 characters and $0·97 \pm ·33$ for 7 indices and angles. The similarity to the Whitechapel English is again very close while the Moorfields and Farringdon Street series are further removed. An almost identical population seems to have been found by Young† in a modern Glasgow cemetery, but we have not used his measurements as the skulls appear to have been inaccurately sexed. Now the undoubtedly significant differences of all these variants of the same type from one another and from the Anglo-Saxon mean measurements are confined to two only of the direct measurements compared—the basio-bregmatic height H', involving the indices $100\,H'/L$ and $100\,B/H'$, and the nasio-basion length (LB). The cephalic index and its component lengths appear to be identical for all. As is shown in Table II, the differentiating characters H' and LB arrange the types in the same order, which is that given by the Coefficient of Racial Likeness.

In the case of the modern Lowland Scottish populations it is reasonable enough to suppose that the type is directly descended from the British Iron Age race and that it was modified slightly by admixture with invaders who settled in the country in the early centuries of the present era and the majority of whom were, in all probability, the founders of the kingdom of Bernicia. The 17th century Londoners had undoubtedly the same origin, but we cannot detect the slightest effect of admixture with Anglo-Saxon elements and, at present, we are unable to

* The more salient differences between the Anglo-Saxon type and that of the 17th century Londoners (Farringdon Street) can be clearly seen by superposing the contours provided (with tracings) in our earlier papers.

† Young, *Trans. Royal Soc. Edinburgh*, Vol. LI. Part II. (No. 9), 1916.

TABLE II. *Mean Male Measurements of some British Cranial Series.*

	100 B/L	L	B	H'	LB	100 H'/L	100 B/H
Moorfields ...	75·5 (42)	189·2 (44)	143·0 (46)	129·8 (34)	98·5 (35)	68·4 (31)	110·2 (34)
Farringdon Street	75·4 (132)	188·8 (139)	142·4 (141)	129·7 (118)	100·1 (118)	68·6 (115)	109·8 (117)
Whitechapel ...	74·3 (131)	189·1 (137)	140·7 (135)	132·0 (122)	101·6 (119)	70·0 (120)	106·6 (122)
British Iron Age...	75·4 (61)	187·4 (61)	141·4 (102)	132·9 (77)	101·6 (67)	70·9 (61)	106·3 (77)
Lowland Scottish (Turner)	75·3 (54)	188·8 (54)	142·1 (54)	133·6 (52)	102·0 (52)	70·9 (52)	106·4 (52)
Glasgow (Young)	74·4 (405)	—	—	—	—	70·8 (405)	105·2 (405)
Anglo-Saxon ...	74·7 (52)	190·6 (58)	141·7 (103)	136·0 (31)	104·1 (31)	71·2 (25)	104·9 (61)

offer any hypothesis which would explain how they acquired their distinctive characters. We are unwilling to free ourselves from this *impasse* by supposing that the modern type was evolved from the ancestral one by gradually acquiring a form which would generally be considered to indicate a more primitive type! The selection of the material from different social strata may possibly account for the seeming anomaly. More ample comparative data will no doubt provide a rational explanation.

(*f*) From data provided by Sir William Turner we have acquaintance with another type of modern Scottish skull which is markedly contrasted to the one predominating in the Lowlands. It is represented by remains found in proximity to the latter (Fifeshire and East Lothian) and in the north-east counties. As it is brachycephalic, comparison can only be made with the Bronze Age types. The following C.R.L.'s are found:

TABLE III. *Coefficients of Racial Likeness between British Brachycephalic Types.*

		English Bronze Age (32·6)	Scottish Bronze Age (21·8)	Turner's Eastern Scottish (21·1)
English Bronze-Age (32·6)	All Characters ...	—	6·58 ± ·20 (22)	3·37 ± ·20 (23)
	Indices and Angles	—	12·69 ± ·33 (7)	3·09 ± ·33 (7)
Scottish Bronze Age (21·8)	All Characters ...	6·58 ± ·20 (22)	—	2·88 ± ·20 (21)
	Indices and Angles	12·69 ± ·33 (7)	—	3·70 ± ·33 (7)
Turner's Eastern Scottish (21·1)	All Characters ...	3·37 ± ·20 (23)	2·88 ± ·20 (21)	—
	Indices and Angles	3·09 ± ·33 (7)	3·70 ± ·33 (7)	—

None of the Coefficients shown in Table III indicate a close similarity—it being noticed that the mean numbers of skulls on which the means are based are small—but they are all of a lower order than the C.R.L.'s found between the three types and the other British ones and we may assume that the populations were inter-related. The modern Scottish brachycephalic skull is apparently characterised by a lower calvarial height than that of the Bronze Age people, but there is no clear evidence to suggest that it acquired that character as the result of miscegenation with a low-browed race like the Lowland Scottish. The material is too meagre to justify us in forming further conclusions.

(*g*) The only other long modern series of British crania for which measurements have been published are the two late mediaeval ones from Hythe (Kent) and Rothwell (Northants). The number of measurements given for the individual skulls is too small to be of much value, nor do the frequency distributions of the characters satisfy the conditions which the biometrician imposes as a test of racially homogeneous populations. The mean measurements of the series neither accord with one another nor with those of any other British series.

To summarise the above conclusions:—We have acquaintance with four distinct cranial types found in England and Scotland during the period extending from late Neolithic times to the present day.

(*a*) An extremely dolichocephalic Neolithic type which was probably extinct before the end of the Bronze Age.

(*b*) A brachycephalic Bronze Age type which had apparently disappeared from England and the greater part of Scotland by the time the Iron Age invaders had settled in the country, but which persisted in the remoter parts of the latter country*.

(*c*) An Iron Age type from which the modern population of the greater part of Britain is directly descended and which has not been essentially modified since its first appearance.

(*d*) The distinct type of the Anglo-Saxon invaders which persisted in a pure form for several centuries and appears to have been finally absorbed by the native population without appreciably modifying the latter.

It is noteworthy that two of these cranial types possess characters which place them on the edge of the distribution of mean characters—the inter-racial distributions—for all races of *homo sapiens*. The extreme values are the great length and low cephalic index of the Neolithic skulls and the extraordinarily low calvarial height in proportion to the calvarial length ($100\,H'/L$) shown by all the variants of the Iron Age type and especially by the Farringdon Street and Moorfields series. It may be noted, too, that the most significant differences between the types are almost entirely confined to calvarial characters. That is, perhaps, partly due to the fact that they are invariably based on larger numbers of skulls than the mean facial measurements. We would wish to emphasize again the fact that the above conclusions can lay no claim to be final. They provide what appears to us to be the most reasonable scheme of the inter-relations between the populations represented by the available scanty material, and if the reader thinks that certain of our statements are not fully substantiated we should to some extent agree with him.

* The fact that the mean cephalic indices of all the longest series of Englishmen that have been measured (see a list given by Pearson and Tippett, *Biometrika*, Vol. XVI. 1924, p. 131) are all close to 78 may indicate descent from either the Iron Age or the Anglo-Saxon populations—the mean index of the living head being supposed as usual about 2·5 points greater than that of the skull—but the value 78 confirms the conclusion suggested by the cranial measurements that the extremely dolichocephalic Neolithic and the brachycephalic Bronze Age types were either exterminated or absorbed by the later races, without leaving any sensible impress upon them.

ON THE COEFFICIENT OF RACIAL LIKENESS.

BY KARL PEARSON, F.R.S.

EVERY craniologist and indeed every physical anthropologist has come up against the difficulty of comparing two races of which it is only possible to secure a limited number of individuals of one or other or both races. Not unnaturally he is driven under the circumstances to seek help by measuring a large number of characters in order to compensate for few individuals*. We have frequently to admit that relatively few individuals are available in many anthropometric inquiries, and that we really must compensate for the smallness of our sample by the largeness of our character series. But how is this to be done? We can compare the means for our two small groups character by character, and if we are trained statisticians we shall compare these mean differences with their standard deviations. But when a considerable number of the characters do not show differences markedly significant with regard to their probable errors, we are left in considerable doubt as to what inference may be safely drawn from the whole series. We need a single numerical measure of the whole system of differences, something which will express by a single coefficient the measure of resemblance (or divergence) of the two races or groups. Such a measure or coefficient I term a *Coefficient of Racial Likeness* (C.R.L.). It should be a measure, not of how far the two races or tribes are alike or divergent, but of *how far on the given data* we can assert significant resemblance or divergence.

Let us suppose m_s to be the mean in the first group of the sth character, σ_s its standard deviation and n the size of the sample; let m_s', σ_s' and n' be the corresponding quantities for the second sample. Then the difference of the means will be $m_s - m_s'$ and, supposing as will be the fact for proper random sampling that m_s is not correlated with m_s', the standard deviation of the difference will be $\sqrt{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}}$. Similarly for a second character t we have to compare $m_t - m_t'$ with

$$\sqrt{\frac{\sigma_t^2}{n} + \frac{\sigma_t'^2}{n'}}.$$

Now if we are really taking samples from the *same* population, the mean of all m_s's and of all m_s''s will be the same, or the mean value of $m_s - m_s'$ or of $m_t - m_t'$ will be zero. Further, the distribution of difference of means will be like the dis-

* I have known cases in which an anthropologist has measured 50 to 100 characters in the 20 to 80 individuals of one tribe and sex, who were accessible to him, and he has pressed me to tell him whether this group was distinguishable from a similar small sample of a second tribe. Thus the problem is a very real one.

tribution of means themselves (if there be no association of the samples), closely normal whatever the original population may be, and will be the more rapidly normal as the size of the sample increases if the original population be, as it actually is in most anthropometric series, approximately normal. Accordingly each series like $m_s - m_s'$ will be given by a normal curve with S.D. equal to $\sqrt{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}}$. All these normal curves will be reduced to one and the same scale if we take as our variate

$$X_s = (m_s - m_s') \Big/ \sqrt{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}},$$

i.e. the variation of the X_s's will be about the origin of a normal curve with standard deviation unity.

If then we could consider X_s, X_t, etc. as independent characters, each would be a random drawing from a normal population of standard deviation unity; and if we took M, such drawings

$$\Sigma = \left\{\frac{1}{M} S \left(\frac{(m_s - m_s')^2}{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}}\right)\right\}^{\frac{1}{2}}$$

should within the error of random sampling approach the value unity.

And further

$$\Sigma^2 = \left\{\frac{1}{M} S \left(\frac{(m_s - m_s')^2}{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}}\right)\right\}$$

should also approach the value unity. We may adopt either of these values we please, but we shall have a difference in the probable error of our result according to our choice of Σ or Σ^2 to be dealt with.

Now what we are really doing here is to sample from a variate X (i.e. values $X_1, X_2, \dots X_s, \dots X_t, \dots$), which is distributed normally, and determining its standard deviation or its standard deviation squared. The distribution of both of these are known and accordingly their probable values, mean values and standard deviations.

The curve for the distribution of Σ is

$$y = y_0 \Sigma^{M-2} e^{-\frac{1}{2}M\Sigma^2},$$

and the most probable value of Σ is

$$\tilde{\Sigma} = 1 - \frac{2}{M},$$

while the standard deviation of Σ is

$$\sqrt{\frac{2}{M}\left(\frac{M-1}{2} - \left\{\frac{\Gamma(\frac{1}{2}M)}{\Gamma(\frac{1}{2}(M-1))}\right.\right.}$$

$$\frac{1}{\sqrt{2M}}\left(1 - \frac{1}{8M}\right),$$

approximately.

If M be large it will be adequate to suppose the distribution of Σ to be normal; for practical purposes this may be supposed reached by $M = 20$*, in which case we may take $\sigma_{\Sigma} = \frac{1}{\sqrt{2M}}$ and represent our result as

$$(\text{C.R.L.})_1 = \left\{ S\left(\frac{1}{M} \frac{(m_s - m_s')^2}{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}} \right) \right\}^{\frac{1}{2}} - 1 \left[+ \frac{2}{M} \right] \pm \frac{\cdot 67449}{\sqrt{2M}} \quad \text{.........(i)},$$

the value of C.R.L. varying round zero, if the two groups are from the same race with the probable error $\cdot 67449/\sqrt{2M}$. As a rule the races are sufficiently divergent to make the term $\left[+ \frac{2}{M} \right]$ of small importance; for twenty characters it only contributes $\cdot 1$, while it is the whole digits which have really to be considered.

If we prefer to deal with Σ^2 instead of Σ, the curve for its distribution is

$$y = y_0 (\Sigma^2)^{\frac{M-3}{2}} e^{-\frac{1}{2} M \Sigma^2}.$$

Here the most probable value of Σ^2 is

$$(\tilde{\Sigma}^2) = 1 - \frac{1}{M},$$

and the standard error σ_{Σ^2} is given by

$$\sigma_{\Sigma^2} = \sqrt{\frac{2}{M}} \left(1 - \frac{1}{2M} \right),$$

approximately.

Accordingly:

$$(\text{C.R.L.})_2 = S \left\{ \frac{1}{M} \left(\frac{(m_s - m_s')^2}{\frac{\sigma_s^2}{n} + \frac{\sigma_s'^2}{n'}} \right) \right\} - 1 \left[+ \frac{1}{M} \right] \pm \cdot 67449 \sqrt{\frac{2}{M}} \quad \text{......(ii)},$$

where the value of C.R.L. will vary round zero with a probable error of

$$\cdot 67449 \sqrt{\frac{2}{M}}.$$

Either value of C.R.L. might be taken as our standard measure of racial resemblance, and I considered both in 1919, and preferred the second, because the term in square brackets could be more frequently neglected. But unfortunately it appeared in *Biometrika*, Vol. XIII. p. 248, with the probable error of (i), i.e. $\frac{\cdot 67449}{\sqrt{2M}}$, instead of (ii), i.e. $\cdot 67449 \sqrt{\frac{2}{M}}$, and this slip has been perpetuated in craniometric papers since. I did not notice the slip till reading through the proofs of Miss Hooke's paper in the current number. The slip is corrected in that paper and also in those of Mr Morant, and of Mr Morant and Miss Hooke in this issue.

* See Table, *Biometrika*, Vol. X. p. 529.

In the papers on the Tibetan*, Nepalese† and Egyptian‡ crania by Mr Morant and on the Burmese crania by Miss Tildesley§, the probable errors of the C.R.L. require *doubling*. This error, however, makes no difference in the conclusions drawn, for the probable error is merely intended to enable an appreciation to be made of how far the intensity of the coefficient is influenced by random sampling and how far by the fact that the two groups compared belong to markedly different races. In nearly all cases dealt with, even in comparing English with English, it is seen at once that the coefficient is influenced in the first place by the differential characterization and not by the sampling, which is of a quite secondary order.

We must halt, however, here to remark on another important point. σ_s and σ_s' are supposed to be the standard deviations of the sth character in the two populations of which our groups are respectively samples. They do not therefore vary with the sample or contribute to the probable error of C.R.L. But unfortunately we do not know σ_s and σ_s' and if we determine them from our samples, which *à priori* are supposed somewhat small, they will have large probable errors. Indeed the determination of variability from small samples constantly leads to larger divergences between the variability in the two samples than exists between the variabilities of two different races based on adequate numbers. For this reason I concluded that it would be unwise to use the values of σ_s and σ_s' derived from the small samples themselves, but that it would, having regard to the fact that the different races of men are not widely divergent in variability, be best to use the system of standard deviations obtained from large numbers, rather than those from the small samples under immediate consideration. For this purpose I selected the 1700 Dynastic Egyptian skulls which had been measured in the Laboratory and gave reasonable values of the standard deviations based on 700 to 800 crania of each sex. The formula then simplifies to

$$\text{C.R.L.} = S\left(\frac{1}{M}\frac{nn'}{n+n'}\frac{(m_s - m_s')^2}{\sigma_s^2}\right) - 1\left[+\frac{1}{M}\right] \pm \cdot 67449\sqrt{\frac{2}{M}},$$

where σ_s^2 is the variance of the sth character in the standard population.

If the series of either group be sufficiently long, as in the case of the Farringdon Street English, then we may use the σ_s's as found from it, but for the special case for which the C.R.L. was devised, where n and n' are relatively small, and this smallness is to be compensated by measuring many characters M, this is unwise. It is better, I think, to use the idea of a general human variability, slightly modified from one race to a second, than to increase the random errors of our coëfficient by making crude approximations to the variabilities of the unknown populations from our small samples themselves.

Another and most important aspect of the matter now arises for consideration. Let us suppose the means of the sth character in the two sampled populations to

* *Biometrika*, Vol. XIV. pp. 207 *et seq.*
† *Biometrika*, Vol. XVI. pp. 54—78.
‡ *Biometrika*, Vol. XVII. pp. 1—52.
§ *Biometrika*, Vol. XIII. pp. 248—351.

be $\bar{m}_s$ and $\bar{m}_s'$ respectively. We can write $m_s = \bar{m}_s + \delta\bar{m}_s$ and $m_s' = \bar{m}_s' + \delta\bar{m}_s'$, where $\delta\bar{m}_s$ and $\delta\bar{m}_s'$ are statistical differences and not infinitely small mathematical differentials. Accordingly we have

$$\text{C.R.L.} = S\left\{\frac{1}{M}\,\frac{nn'}{n+n'}\cdot\frac{(\bar{m}_s-\bar{m}_s')^2}{\sigma_s^2}\right\}$$

$$+\,2S\,\frac{nn'}{M}\left\{\frac{(\bar{m}_s-\bar{m}_s')}{(n+n')\,\sigma_s^2}\right\}\delta\bar{m}_s$$

$$-\,2S\left\{\frac{nn'}{M}\;\frac{\bar{m}_s-\bar{m}_s'}{(n+n')\,\sigma_s^2}\right\}\delta\bar{m}_s'$$

$$+\;S\left\{\frac{1}{M}\,\frac{nn'}{n+n'}\,\frac{(\delta\bar{m}_s)^2}{\sigma_s^2}\right\}+S\left\{\frac{1}{M}\,\frac{nn'}{n+n'}\,\frac{(\delta\bar{m}_s')^2}{\sigma_s^2}\right\}-2S\left\{\frac{1}{M}\,\frac{nn'}{n+n'}\right\}\delta\bar{m}_s\,\delta\bar{m}_s'-1.$$

Now when we take the mean value of all these summations for a large number of samples, the first is a constant and does not vary, the second and third vanish because the mean value of $\delta\bar{m}_s$ and $\delta\bar{m}_s'$ is zero, the mean value of $(\delta\bar{m}_s)^2$ is σ_s^2/n and of $(\delta\bar{m}_s')^2$ is σ_s^2/n', while lastly the mean value of $\delta\bar{m}_s\,\delta\bar{m}_s'$ is zero, for they are uncorrelated. Thus the mean value of the terms in $(\delta\bar{m}_s)^2$ and $(\delta\bar{m}_s')^2$ is $S\left(\frac{1}{M}\right)=1$, which cancels with the -1, and we see that the main component of the C.R.L. is the term $S\left\{\frac{1}{M}\,\frac{nn'}{n+n'}\,\frac{(\bar{m}_s-\bar{m}_s')^2}{\sigma_s^2}\right\}$, that is it is determined by the difference in the means of the sampled populations. It is really these $\delta\bar{m}_s$, $\delta\bar{m}_s'$ which give rise to the variability of the C.R.L. when we take continuous pairs of samples, and which is expressed by the probable error. It is thus not likely, unless the means are the same for all corresponding characters in the two sampled populations, that the C.R.L. will be zero. It is easily shown from a slight experience in comparing data for various races that the variations expressed in $\delta\bar{m}_s$, $\delta\bar{m}_s'$ (σ_s^2 being considered constant) contribute little to the value of the C.R.L.; that depends for its intensity on the fundamental term

$$S\left\{\frac{1}{M}\,\frac{nn'}{n+n'}\,\frac{(\bar{m}_s-\bar{m}_s')^2}{\sigma_s^2}\right\}.$$

The question may of course be raised, whether another and better expression might not be found as a coefficient of racial likeness. It may be quite well argued that product terms ought to be introduced into its expression. We know that while m_s is not correlated with m_s', yet m_s is correlated with m_t, in fact if we are taking a series of samples from the same two populations the correlation between m_s and m_t is r_{st} and between m_s' and m_t' will be r_{st}', where r_{st} and r_{st}' are the correlations of the sth and tth characters in the two populations themselves*. Now it would not be unreasonable to suppose that just as we have taken $\sigma_s = \sigma_s'$ we may also take approximately $r_{st} = r_{st}'$. In this case it is easily shown that

$$\frac{nn'}{n+n'}\,\frac{(m_s-m_s')(m_t-m_t')}{\sigma_s\sigma_t}$$

* The correlation coefficient of the mean of two variates is the same as the correlation coefficient of the two variates themselves.

takes for its mean value

$$\frac{nn'}{n+n'}\frac{(\bar{m}_s-\bar{m}_s')(\bar{m}_t-\bar{m}_t')}{\sigma_s\sigma_t}+r_{st}.$$

As in the case of $\frac{nn'}{n+n'}\frac{(m_s-m_s')^2}{\sigma_s^2}$ it will probably be the first term in this expression which provides the chief contribution. Although some characters of the skull are fairly highly correlated, others have hardly any correlation at all, and some may have positive and some negative signs.

If, however, we start with our variates

$$X_s=\sqrt{\frac{nn'}{n+n'}}\,\frac{m_s-m_s'}{\sigma_s},\quad X_t=\sqrt{\frac{nn'}{n+n'}}\,\frac{m_t-m_t'}{\sigma_t}$$

and *suppose our two samples to belong to the same population*, then we have $\sigma_{X_s}=1$, $\sigma_{X_t}=1$ and $r_{X_sX_t}=r_{st}$. Hence if we form the determinant

$$R=\begin{vmatrix}1, & r_{12}, & \dots\dots\dots & r_{1M}\\ r_{21}, & 1, & \dots\dots\dots & r_{2M}\\ \dots & \dots & \dots\dots\dots & \dots\\ \dots & \dots & \dots\dots\dots & \dots\\ r_{M1}, & r_{M2} & \dots\dots\dots & 1\end{vmatrix}$$

and its minors, we should have

$$\chi^2=\frac{1}{R}\left\{\overset{M}{\underset{1}{S}}(R_{ss}X_s^2)+2S'(R_{st}X_sX_t)\right\}$$

and our distribution surface with an element $z_0e^{-\frac{1}{2}\chi^2}\,dX_1\,dX_2\dots dX_M$ of frequency. This would lead us to a χ^2, P test of the improbability of the material of the two groups being drawn from the same population, which might well be better theoretically than the C.R.L. discussed previously. Now it will be seen that in order to get the value of R and its minors we must know the correlations of M characters in a standard population. The ideal value of M would be 40 to 50, this would involve the calculation of 780 or 1225 correlation coefficients. The largest number of correlation coefficients yet computed are those for the Egyptian 26th to 30th Dynastic series. But in that case after very strenuous labour only 312 correlations were found. Thus, if the particular characters there chosen had been so chosen as to give a closed series, i.e. if 24 or 25 characters had been selected and correlated only among themselves, we could only take $M=24$ or 25 and must *always* take the same 24 or 25 characters, whatever pairs of races were being compared. This is an impossibility unless all craniologists agree to consider the same standard characters. But suppose this were done, and that there were 30 to 40 standard characters and the 435 to 780 correlations were all known, then indeed the real difficulty of the task would begin, we should have to compute a series of determinants, 780 to 820 in number, each consisting of 30 to 40 rows and columns. The task would be gigantic and if completed would be of no service should a series of crania be measured in which even one of the standard characters had been omitted. For the statistician, as for the statesman, the ideally

best is not always the wisest course. Even if these coefficients could be computed, we should have to deal with the determining and adding together of 465 (30 + 435) to 820 (40 + 780) terms instead of the 30 to 40 terms of the present cruder coefficient. Further, while some of these terms are positive others are negative, and the great bulk, although by no means all, are very small*. Hence the 435 to 780 terms may not contribute as much as we might possibly anticipate to the 465 to 820 terms suggested for our χ^2. The mean correlation of the characters of the skull—without regard to sign—is only about ·3, and would be considerably less paying attention to sign. Still the fundamental weakness of the Coefficient of Racial Likeness lies in the fact that it neglects the correlations between the characters dealt with. If we examine such determinantal relations as R_{ss}/R and R_{st}/R, we see that

$$R_{ss}/R = 1 - \text{squares and higher powers and products of correlations,}$$
$$R_{st}/R = r_{st} - \text{products and higher powers and products of correlations,}$$
$$= 1 - \Delta_{ss} \text{ and } r_{st} - \Delta_{st}, \text{ say, respectively.}$$

I have found it possible to express Δ_{ss} and Δ_{st} as approximate functions of *mean* correlation values, but I have not been able to determine how close this approximation is in the case of 15 to 20 rowed determinantal ratios without numerical experiment, the labour of which would be very great. But were these approximate expressions adequate in the case of cranial correlations, I cannot conceive that any craniologist could at present be induced to calculate the hundreds of correlations requisite (i.e. the r_{st}'s), or having found them to compute the hundreds of terms requisite to determine χ^2.

However, I do see an entirely different method of approaching the subject when once we have 50 to 100 cranial series, each containing 50 to 100 individuals of one sex measured in a standardised manner. But that method is for co-operative work in the future.

Meanwhile the C.R.L. as now in use seems to me the best test available, if used with discretion, i.e. tested in male and female series, and for indices and angles, as well as for all characters, and compared in its results with conclusions drawn from the correspondence or divergence of the mean racial cranial contours†. If any one has a sounder coefficient to propose, I shall not be the last to welcome and use it.

Assuming, however, that the theoretical difficulties of the C.R.L. can be disregarded, and that it can be looked upon as *practically* an approximate measure of racial association, if not an ideally adequate one, we may ask: how has it fulfilled this purpose? Does it give on the whole a rough measure of racial likeness, when we classify races by the general impressions which anthropologists have hitherto adopted rather than by accurate numerical relations? On the whole, while it contradicts some current anthropological beliefs, and suggests some

* See, on all these points, the paper by Pearson and Davin, "Biometric Constants of the Human Skull," *Biometrika*, Vol. XVI. pp. 347—368.

† A single number to represent the degree of resemblance between two mean racial cranial contours is badly needed.

hitherto unsuspected relationships, it does not give results wildly discordant with the beliefs and impressions of anthropology. It seems rather to confirm, to extend and in special cases to correct them.

Up to the present nearly 760 C.R.L.'s have been determined by the Biometric School of craniometry. It was considered originally possible that there might be a resemblance in shape between two races, when there failed to be a resemblance in absolute size. For this reason C.R.L.'s were worked out for indices and angles only; of these we have 340, while for absolute measurements, angles and indices, i.e. for characters of all types combined, we have 417 cases. It was actually found that in certain cases there was a greater resemblance in shape than size, but it may be doubted whether it is worth while to separate size and shape characters, as this of course lessens the total number of characters available for computing the coefficient. We prefer to use two series only, one for shape characters and one for *all* characters.

The following table gives the distribution of coefficients found:

Values of C.R.L.'s.

	Less than *1* ...	*1—4*	*4—7*	*7—10*	*10—13*	*13—16*	*16—19*	*19—22*	*22—25*	*25—28*	*28—31*	Above *31*	Totals
All Characters	54	131	73	39	26	23	15	12	8	11	2	23	417
Indices and Angles	56	106	59	35	19	16	9	12	4	6	2	16	340
Totals	110	237	132	74	45	39	24	24	12	17	4	39	757

Of course the main object of the biometric inquirers was to find resemblances, not to search for widely divergent races. Hence no special stress is to be laid on the frequency distributions, but knowing the races involved in each group of values it seemed possible to arrange a classification giving five grades of association and seven grades of divergence, and after stating these—at any rate as provisional terms—we will then consider what pairs of races fall into some of these categories.

Degrees of Association			Degrees of Divergence		
Grade	Range	Class	Grade	Range	Class
I	Less than *1*	Very intimate Association	I	*13—16*	Slight Divergence
II	*1— 4*	Close Association ...	II	*16—19*	Moderate Divergence
III	*4— 7*	Moderate Association ...	III	*19—22*	
IV	*7—10*	Slight Association ...	IV	*22—25*	Marked Divergence
V	*10—13*	Doubtful Association ...	V	*25—28*	
—	—	—	VI	*28—31*	
—	—	—	VII	Over *31*	Very wide Divergence

Of course, assuming the origin of man to be monogenetic, "Association" and "Divergence" are only relative terms of a continuous grade of relationship, indicating only the greater length of differentiated ancestry. But they are convenient terms. It is proper to look upon Anglo-Saxons and modern English as associated races, but on Chinese and English as divergent races, if we only mean by this that the forerunners of Chinese and English diverged much earlier from a common ancestry than English and Anglo-Saxons. To fix the limit of association and divergency at a C.R.L. = 13 is of course arbitrary, but it is convenient for practical purposes. It signifies that it would be difficult to place two races with this coefficient in the same family of races.

I will examine individually first some of the pairs which fall into my "Very wide Divergence" category. They are, considering only all characters:

The Dravidian Race as represented by the Maravar with Malays (31), with Burmese (38), with English (Whitechapel) (43), with Moriori (49), and with Aino (55). We should *à priori* probably have asserted all these races to be widely divergent, but in this marked divergence that the Malays and Burmese should be less marked is satisfactory, if again it be what some might anticipate*.

The Aino race with Tibetans (31), with Hindus (32), with Malayans (33), with Moriori (33), with Burmese (34), with English (35), with Nepalese (38), with Dravidians (55), and with Altai Telengites (71)†.

English 17th Century with Nepalese (35), with Burmans (46), and with Malays (77); the Prehistoric Egyptians (Naqada) with Burmans (55), and with Malayans (65); the Moriori with the Hindus (32) and with the Nepalese (33) are all self-explanatory. We are dealing with races which every one admits to be widely divergent. In practically all these cases our results are confirmed, if we limit our characters to indices and angles only. There are indeed some cases in which for shape only the divergence is more conspicuous than when we deal with both size and shape. Thus:

Eskimos with Maori (32), with Fuegians (35) with Moriori (51), with Aino (64); while for both size and shape the corresponding values are: (13), (20), (25) and (26) respectively. These connote rather considerable divergence but not so great as for shape only. We get measures of wide divergence if we deal with shape only in the case of the Malayans, e.g. with Aino (37), with Moriori (37), with Prehistoric Egyptians (110), and with 17th Century English (126). In all these cases we are dealing with what are admittedly widely divergent races. A very limited craniological experience would enable anyone to distinguish without computing a C.R.L. between the skulls of the last races. But we have cited the values here to show that the C.R.L. is a real criterion of cranial divergence.

It would not be fair, however, to pass by two remarkable values wherein the C.R.L. appears at first to fail. The skulls at Hythe have been measured to the

* It is easy to find races closely allied to this Dravidian stock, thus with Bengal Hindus (2), with Veddahs (2), with Andamanese (5), with Nepalese (6·5), with Burmese Hybrids (7), and with Karens (10).

† Again it is easy to find out associated races: with Fuegians (7), with Japanese (8), with Maori (8), with Koreans (9), and with Northern Chinese (16).

number of 315 by Professor Parsons, unfortunately for very few characters. Only eight are available for a C.R.L. They have been compared with the 17th Century English (Farringdon Street) (83) and with the Anglo-Saxons (73). These numbers indicate either (i) that the Hythe crania are very widely divergent from English and Anglo-Saxon, or (ii) that Professor Parsons' methods of measurement are very widely divergent from those laid down by the international concordat. Until the Hythe crania are measured for a large variety of characters in the standardised manner it will be impossible to say what is the real significance of the above C.R.L.'s.

If we now turn to the other extreme of our scale "very intimate association," we find ourselves dealing with: (*a*) the same race sampled by two craniologists, (*b*) local varieties of the same race, or (*c*) the same race at different epochs of its existence. Of course all cases of (*a*), (*b*) and (*c*) do not fall into Grade I of Association, but it is difficult to find any pair in Grade I which we are certain embraces two craniologically distinct races. A few examples of each class will suffice: (*a*) Eskimos measured by Fürst and Hansen, and those measured by Hrdlička (– 0·90), two series of Chinese (0·59), Moriori—Scott's series and Thomson's series—(0·41), Trans-Himalayan and Cis-Himalayan Bhotias (– 0·34) Nepalese Central and Nepalese Eastern (– 0·45), etc. The last two cases probably might be classified under (*b*). (*b*) Tibetans and Trans-Himalayan Bhotias (0·48), Burmans and Burmese Hybrids (0·78), Annamese and Southern Chinese (0·01), Siamese and Annamese (0·14), Torgods and Kalmucks (0·89), etc. (*c*) 17th Century English (Whitechapel) and British Iron Age (0·38), 1st and 2nd Dynasty (Royal Tombs) and 18th—20th Dynasty (0·9), Ptolemaic Period and Roman Period Egyptians (0·72), 1st and 2nd Dynasty Royal Tombs and Roman Period Egyptians (0·65), Prehistoric Egyptians (Naqada) and Ptolemaic Period Egyptians (0·04), etc.

The above illustrations of the two extremes of "marked divergence" and "intimate association" prepare us for having confidence in the C.R.L. in the intermediate grades.

Once this confidence is won—and those who have examined the 750 coefficients already computed will find in them confirmation of many conclusions reached in other ways—we cannot reject straight off results which are novel or against impressions often based on no well-defined quantitative research. I may mention one or two of these results, which I believe should not be straight off rejected, but deserve full consideration.

(i) The Moriori are more closely related to the Fuegians (4·6) than to the Maori (8·5). This may indicate an early transfer from Antarctic lands to South America.

(ii) The Aino are more closely related to the Fuegians (6·7) than to the Japanese (8·1), or to the Koreans (8·9), or to the Northern Chinese (16·1). A study of "fringe" peoples by aid of the C.R.L. may lead us to new ideas on the passage of human racial waves over the whole earth's surface. The presence of Lemuroids

in Borneo and Madagascar led to strange hypotheses, until the fossils of Lemuroids were found in Europe, Asia, and America.

(iii) The crania of modern Abyssinians of the Tigré District are as closely related to Dynastic Egyptians (1·2 to 3·7) as the Dynastic are to the Predynastic Egyptians. The ancient Egyptian type has therefore been preserved in more than one form to modern days.

(iv) The English skull is nearer to that of the men of the British Iron Age than to that of Anglo-Saxons. Thus

17th Century English		Men of Iron Age	Anglo-Saxons
Males	Whitechapel Crania ...	(0·38)	(2·98)
	Moorfields Crania ...	(1·55)	(4·88)
	Farringdon Street Crania	(2·82)	(5·27)
Females	Whitechapel Crania ...	(1·84)	(9·61)
	Moorfields Crania ...	(1·32)	(4·66)
	Farringdon Street Crania	(2·81)	(6·51)

It will be evident from these figures that though the Anglo-Saxon is associated with the English skull, it stands in nothing like so close a relationship as the skull of the Iron Age men. Nor indeed is the Anglo-Saxon closer to the English than the Long Barrow cranium as measured by Schuster, which gave with Whitechapel (3·7) and with Farringdon Street (5·3).

If the coefficient of racial likeness is to be trusted, the belief that the English are in the main Anglo-Saxons must be discarded. This does not mean that there is not association of a "moderate" kind with the Anglo-Saxons—it is closer than the "doubtful association" of Bavarians with Würtemberger (12·1) or than the "marked divergence" of English and French (24·5)—but it is not of the "close association" type which the English have with men of the Iron Age (Grade II of Association).

I have said enough to indicate that not only the C.R.L. can confirm current impressions, but that it can raise new and suggestive problems. With the work done in the Biometric Laboratory by Miss M. Tildesley, Mr Morant and Miss B. Hooke we are now in a position to state that the coefficient can be a serviceable tool in craniometric research. Such a statement was not possible when the coefficient only stood on a not wholly adequate theoretical basis, but the present practical basis of 750 computed coefficients, capable of being set against many accepted racial relations, has given it a sounder position, and this we owe entirely to their assiduous labour.

The change in the C.R.L. can be illustrated in reference to the number of characters dealt with by comparing a few cases in which it has been calculated for two series of characters on the same series of crania.

Thus we have:

Series compared	Number of Characters	
Naqada *A*, *Q* Graves with Naqada *B*, *T*, *R* Graves	14 (1·62)	31 (0·65)
Modern Abyssinian with El Kubanieh South Graves	14 (1·16)	23 (1·87)
Naqada *B*, *T*, *R* Graves with 1st Dynasty Royal Tombs	14 (2·59)	31 (2·37)
El Kubanieh South Graves with 18th—21st Dynasty Tombs ...	14 (3·51)	27 (2·65)
Naqada *A* and *Q* Graves with 1st Dynasty Royal Tombs ...	14 (9·68)	31 (6·36)

While the experienced craniometrician would lay no real stress on the differences in the C.R.L. occurring between those with long and those with short series of characters—especially as the numbers of crania involved were not large—it might appear as if the longer series of characters involved in general a smaller value of the C.R.L. To test this point I took out of the 417 coefficients for all types the 313 for which the number of characters used was directly stated, and formed a correlation table for size of coefficient and number of characters used. The resulting table was somewhat "lumpy" owing to necessity or preference leading the biometric workers to adopt certain groups of characters, but I think the final result may be relied on. It is:

Correlation of the C.R.L. and Number of Characters used

$$r = \cdot 0527 \pm \cdot 0380.$$

In other words there is no evidence that the coefficient of racial likeness is influenced in one direction by the number of characters adopted. I think this must really *à priori* be obvious, if no selection has been made of those characters which in the two races differ most or differ least from one another.

I next proceeded to consider what influence the number of individual crania dealt with had on the coefficient. Considering only coefficients based on all characters (i.e. absolute size and indices and angles), I obtained a correlation table of 580 entries, entering each coefficient twice, once for each race dealt with. The table was still more "lumpy" than the previous one, as the skull frequencies run from 6 to 885. The answer I found for the Correlation of the C.R.L. and Number of Crania used was

$$r = + \cdot 1270 \pm \cdot 0276,$$

i.e. there was a not very important correlation between the number of skulls used and the coefficient, the *greater* the number of skulls the *larger* the coefficient. As a matter of fact the influence of the size of the two samples is largely obscured by the variation of the ratio $(\bar{m}_s - \bar{m}_s')^2/\sigma_s^2$ (see our p. 109). Thus we see that the actual values of n and n', the numbers of individuals in the two series compared, is not so influential as might have been anticipated. It would undoubtedly be well if n and n' as well as the cranial characters selected could be standardised. We should then make far more rapid progress in placing the various races of man into a classified scheme and seeing more clearly the nature of human evolution.

What we need are, say 50 to 100 crania of each sex of each race, and then 40 to 50 characters measured in a standardised manner. I think the C.R.L., as it has been already used, forms a very good rough guide to racial association—or in some few cases, perhaps, to the extraordinary personal equations of certain craniologists—but if we had such series as I have suggested its value would be markedly emphasised. Owing to the steady measuring and tabling work of German and English investigators, such long series for an adequate number of characters are becoming greater in number and they will one day form a sound basis for a theory of racial evolution in man. Any argument from series of 6 to 10 crania—even using the C.R.L.—is, I think, to be deprecated*. It may be all that is feasible at present, but conclusions based on such series cannot be treated as final.

* I noted that out of the 580 coefficients tabled by me nearly a sixth, 94, were for series with less than 16 crania measured.

REMAINS OF SAINT MAGNUS AND SAINT ROGNVALD, ENTOMBED IN SAINT MAGNUS CATHEDRAL, KIRKWALL, ORKNEY.

By R. W. REID, M.D. (Aberdeen), LL.D. (Aberdeen), F.R.C.S. (England), Emeritus Professor of Anatomy, University of Aberdeen.

Owing to the courtesy of the Magistrates of Kirkwall, Orkney, trustees of the bequest of the late Sheriff Thoms for the restoration of Saint Magnus Cathedral in Kirkwall, Orkney, the privilege was granted to me of examining, in August 1925, the remains of two human skeletons entombed in two of the pillars of that cathedral.

These skeletal remains were supposed to be those of Saint Magnus and Saint Rognvald, two famous Norse Saints, whose life histories formed the main theme of the Orkneyinga Saga.

The object of the examination was to decide, if possible, whether this supposition might be accepted as verified, and if so, to make an anthropological record of the skeletal remains of two celebrated Norsemen who had both been canonized in the twelfth century.

The only means of attaining this object was by an investigation of the history of these individuals, in so far as it could be obtained from the Orkneyinga Saga and other relative documents, and by associating the facts so ascertained with the physical appearances presented by the skeletal remains and the locations in the cathedral in which they were found.

History.

Magnus was the son of Erlend, Joint-Earl of Orkney. Erlend held his Earldom from the King of Norway and died in Norway about the close of the 11th century. Magnus was described in a somewhat imaginative strain as being "Of large stature, a man of noble presence and intellectual countenance" (1). At first he ruled Orkney conjointly with his cousin Hakon, but after some time dissensions arose between them, with the result that Magnus was slain by order of Hakon, on the island of Egilshay in 1115. His day in the Roman Calendar is 16th April.

As the manner of his death had an important bearing upon the identification of the skeletons, the following extract from the Orkneyinga Saga may be quoted: "Then when God's friend (Magnus) was led to execution he said to Lifolf (executioner), 'Stand before me and hew me a mighty stroke on the head, for it is not fitting that high-born lords should be put to death like thieves. Be firm, poor man, for I have prayed God for you, that he may have mercy upon you'" (2).

"After that Earl Magnus signed himself with the sign of the Cross and bowed him to the stroke, but Lifolf hewed him on the head a great blow with an axe. Then Earl Hakon said, 'Hew him a second time.' Then Lifolf hewed into the same wound. Then Saint Magnus the Earl fell on his knees and fared with this martyrdom from the wretchedness of this world to the everlasting bliss of the kingdom of heaven" (3).

Another passage from the Saga is also of interest: "The place where Earl Magnus was slain was previously covered with moss and stones, but shortly afterwards his merits before God became manifest in this wise that it became greensward where he was beheaded" (4).

As a consequence of the entreaties of his mother, his body was buried in a coffin in the ground in Christ's Kirk, Birsay, and after lying twenty years there it was placed in a shrine which was "set over the altar" of that church. The shrine with its contents was thereafter removed to Saint Olaf's in Kirkwall and it was described as "set over" the altar of that church (5). It was then transferred to the Cathedral in Kirkwall when that edifice was ready to receive it (6).

Hakon after repenting of his evil deeds went upon a pilgrimage to Rome and Jerusalem and returned to Orkney where he promoted all the arts of peace and died beloved and respected by his people.

He was succeeded by his two sons, Paul and Harald. The latter was mysteriously done to death; the former was kidnapped by Swein, Asleif's son, and taken to Scotland, his ultimate fate being uncertain.

Rognvald, a nephew of Magnus, now became Earl of Orkney and in fulfilment of a vow he founded a cathedral in Kirkwall in memory of his uncle, Magnus (A.D. 1137). He was described as being "a middle man in growth, well set up, with light brown hair" (7). He was assassinated in Caithness by a stab from a spear, having been previously wounded in the chin by a sword.

His body was brought to Orkney and ultimately buried in the choir of the Cathedral which he had founded (8).

There is thus conclusive evidence that the relics of Saint Magnus and Saint Rognvald were placed in the Cathedral.

It is now necessary to discuss the evidence in favour of the supposition that the bones discovered in the *two pillars* of the Cathedral are those of Saint Magnus and Saint Rognvald. Such evidence has been given with considerable fulness by Mr John Mooney, F.S.A. (Scotland), Kirkwall (9).

It is probable that prior to the extension of the choir eastwards, which was made more than a century after the founding of the Cathedral in 1137, the relics occupied places of honour near the high altar. Before the extension, the high altar stood in close proximity to the pillars in which the relics have been found, and it has been surmised that the relics were subsequently transferred to these pillars when the altar was removed during the extension of the choir. There is, however, no documentary evidence from the early history of the Cathedral as to the date of the transference.

These pillars are represented by the darkly shaded parts of the pillars marked A and B in Fig. 1, the lightly shaded parts corresponding to the additions which had been made during the extension of the choir.

It has been known for several generations locally that there were bones in the third pier from the East on the north side of the choir (A). These were surmised to be those of Saint Magnus. The bones were viewed by the late Marquis of Bute, who rejected the idea that they were those of Saint Magnus, and regarded them as being probably those of Saint Rognvald.

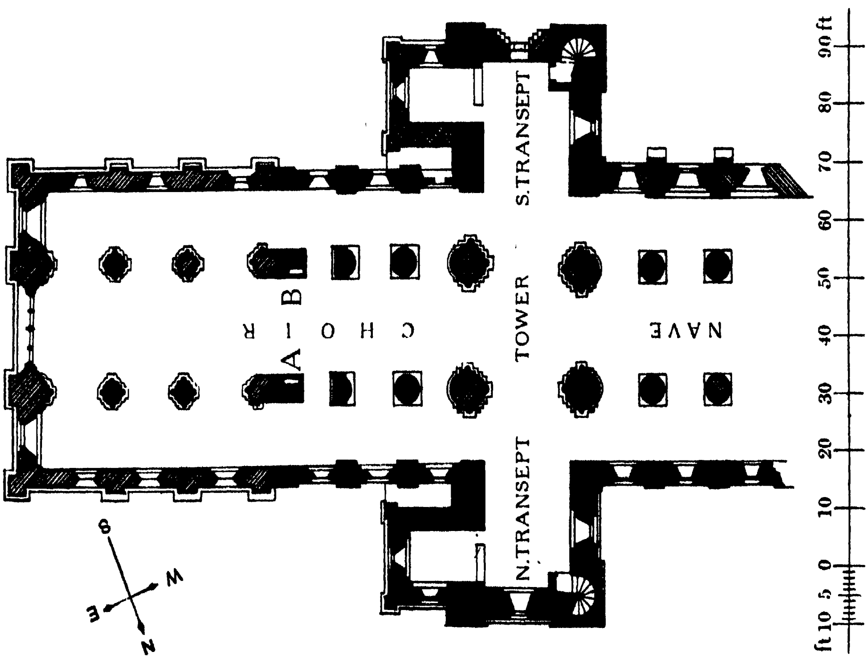
In March 1919 a chest containing bones was discovered in the corresponding pier on the south side of the choir.

It is submitted that the researches embodied in the following paper leave little doubt that they are those of Saint Magnus. Their position in the south pillar corresponds with that of the bones in the north pillar.

On the assumption that the bones in the south pillar (B) are those of Saint Magnus, it seems certain that the bones in the corresponding position on the north side were those of one comparable to him in dignity, and in the veneration of those who placed them there. No other person than Saint Rognvald fulfils these conditions. Saint Magnus and Saint Rognvald were the only saints having a special connection with Orkney. There was no other local canonization. Other early dedications in the country are to Scottish or Irish saints whose disciples as christian missionaries founded churches there, or to Saint Olaf of Norway, who claimed the sovereignty of the Islands. Whether the bones received sepulture in the pillars (A and B) as a token of special honour, or with a view to special security (10), there is no one other than Rognvald to share with Magnus in such honour or solicitude for his relics.

One of the two skeletons now under examination was found along with some fragments of wood in a rudely walled cavity situated in the older part of the pillar marked A, which will be subsequently named, from its position, the *north pillar*. This cavity, which was 9 ft. 6½ in. above the floor of the Cathedral, had evidently been excavated *after* the pillar had been built. It was placed towards the south-west corner of the pillar which flanked the original position of the high altar, Fig. 2.

The other skeleton under examination was found in a more carefully excavated cavity in the older part of the pillar marked B in Fig. 1. This pillar will subsequently be designated the *south pillar*. It was at a distance of 9 ft. 2 in. from the floor of the Cathedral and was placed towards the north-west corner of the pillar which flanked the original position of the high altar. The skeleton was contained in a case made of Scots pine (*Pinus sylvestris*), its sides, top, bottom and ends being held together by wooden pegs. The case practically filled the cavity, and lay with its long axis directed from east to west. The wood was somewhat decomposed at the west end of the case. Its outside dimensions were: length, 74·8 cm. (29½ in.), breadth, 26 cm. (10¼ in.), depth on the average 21 cm. (8¼ in.), and thickness of wood 2·5 cm. (1 in.). The skull lay at the west end of the case with its face directed towards the east.



A indicates position of remains of Saint Rognvald.
B " " " Saint Magnus.
Walls shown solid black, 12th century.
" crosshatched, late 12th century.
" hatched, late 13th century.

Fig. 1. Ground plan of part of Saint Magnus Cathedral, Kirkwall, Orkney. Adapted from Dryden's "Illustrations of some parts of the Cathedral-Church, dedicated to Saint Magnus at Kirkwall in Orkney," *Transactions of the Architectural Institute of Scotland*, 1868—1871.

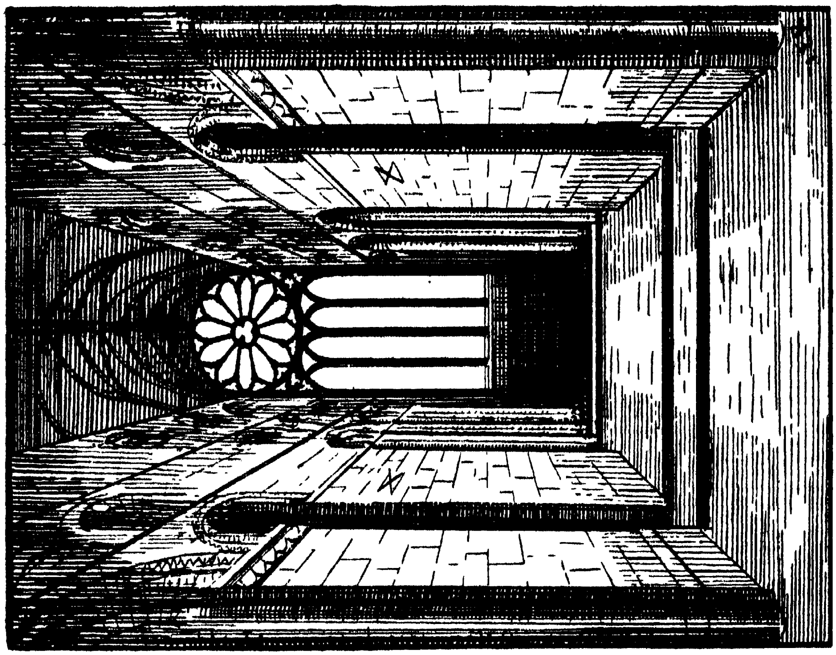


Fig. 2. Sketch of interior of choir of Saint Magnus Cathedral, Kirkwall, Orkney: looking east. Crosses on north and south pillars indicate sites of burial of remains of Saint Rognvald and Saint Magnus.

Remains found in the South Pillar.

The remains found in the case in the south pillar consisted of portions of the skeleton of a man whose age might be roughly estimated as from twenty-five to thirty-five years and of about 171·7 ± 2·03 cm. (5 ft. 7½ in.) in stature, and of six fragments of the limb bones of birds of about the size of geese.

The human remains were: skull (imperfect), lumbar vertebra (imperfect), six ribs (imperfect), left clavicle (imperfect), right and left scapulae (imperfect), right and left humeri (imperfect), right radius (imperfect), left radius, left ulna, three metacarpal bones (imperfect), innominate bone (imperfect), right and left femora (imperfect), left astragalus, right os calcis, right scaphoid, right internal cuneiform, left external cuneiform, left cuboid, third and fourth right metatarsal bones, first, third and fifth left metatarsal bones.

*Skull**. The skull was in a fairly good state of preservation, but the lower jaw was wanting.

In order to facilitate description a comparison was made of the appearances which it presented with those of a modern Scottish skull of a native of an inland parish in Aberdeenshire in the north-east of Scotland, whose measurements represented a blend of the Alpine and Nordic types.

As regards *capacity* the cranium was small. It fell into the mesocephalic group, having a capacity of 1380 c.c. as estimated by the use of shot. It was thus 117 c.c. less than the average 1497 c.c. given by Flower for mixed Europeans (13).

As regards shape of head the cranium was mesaticephalic having a breadth index of 79·3.

Norma verticalis. (Figs. 4, 5, 6, Plate II, Fig. 3.) The superciliary ridges and glabella were indefinitely marked.

The zygomatic arches projected freely from the outline of the cranium. This appearance was in marked contrast to that which was shown in the Aberdeenshire skull where the zygomatic arches did not appear free of the contour of the skull. It was due partly to a narrowing of the frontal bone between the two temporal ridges, and partly to a pronounced sweeping outwards of the arches themselves. The zygomatic breadth was 14·2 cm. as compared with 14·0 cm. in the Aberdeenshire skull, 13·2 cm. as given by Turner for the Scottish, and 14·5 cm. by Hrdlička for Eskimo skulls, the last of which are noted as having a very great distance between their zygomatic arches.

The contour in the occipital regions formed a uniform curve and was slightly more acute than that of the Aberdeenshire skull. The sutures were open with the

* The abbreviations employed in the figures and in Table I were those used by Benington (11) and Morant (12). For list of abbreviations used in the figures see p. 147.

It is important to note that the contours were made by means of the dioptrograph, and that therefore they are not comparable with sectional type contours, since the sectional and the dioptrographic contours taken from the same skull may differ widely, especially in the case of the contour of the norma verticalis, where the contours of the parietal eminences bulge beyond the vertical section through the auricular axis.

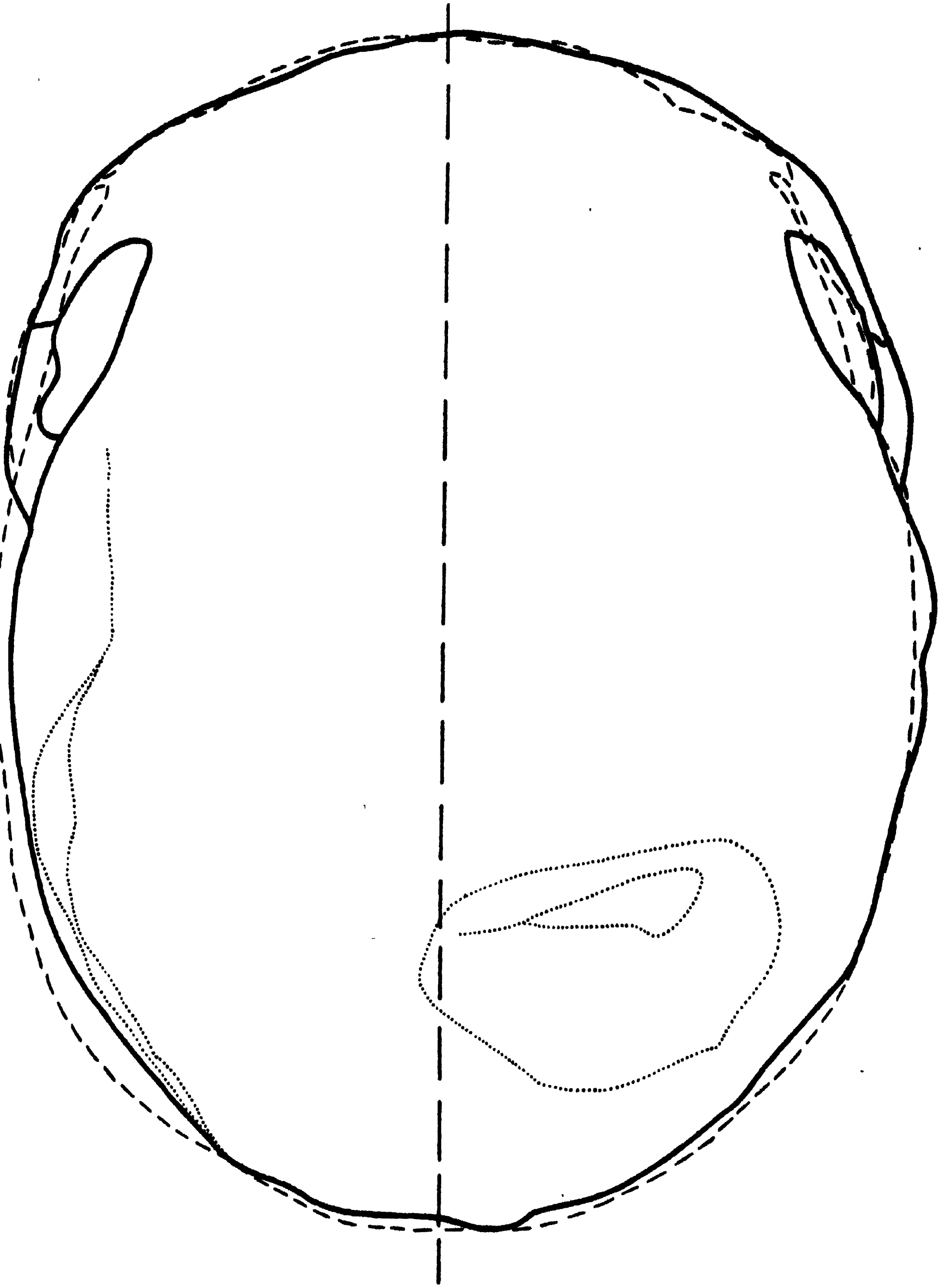


Fig. 4. Dioptrographic projection of skull of Saint Magnus (continuous line) superimposed upon dioptrographic projection of skull of native of Aberdeenshire, Scotland (broken line). Dotted line indicates injuries to skull of Saint Magnus. Orientated in Frankfort horizontal plane.

Norma verticalis.

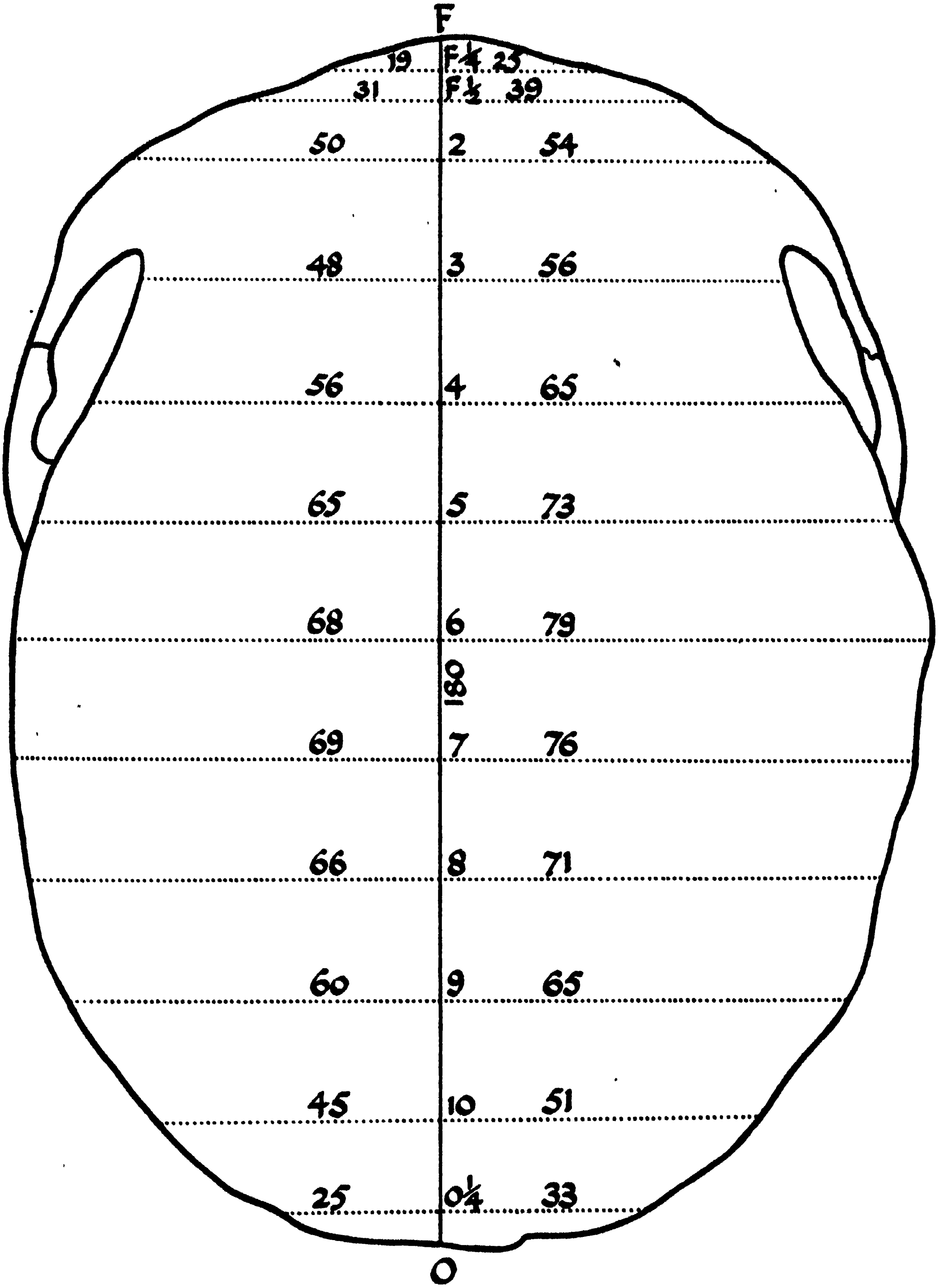


Fig. 5. Horizontal contour of the skull of Saint Magnus.

Dimensions in millimetres.

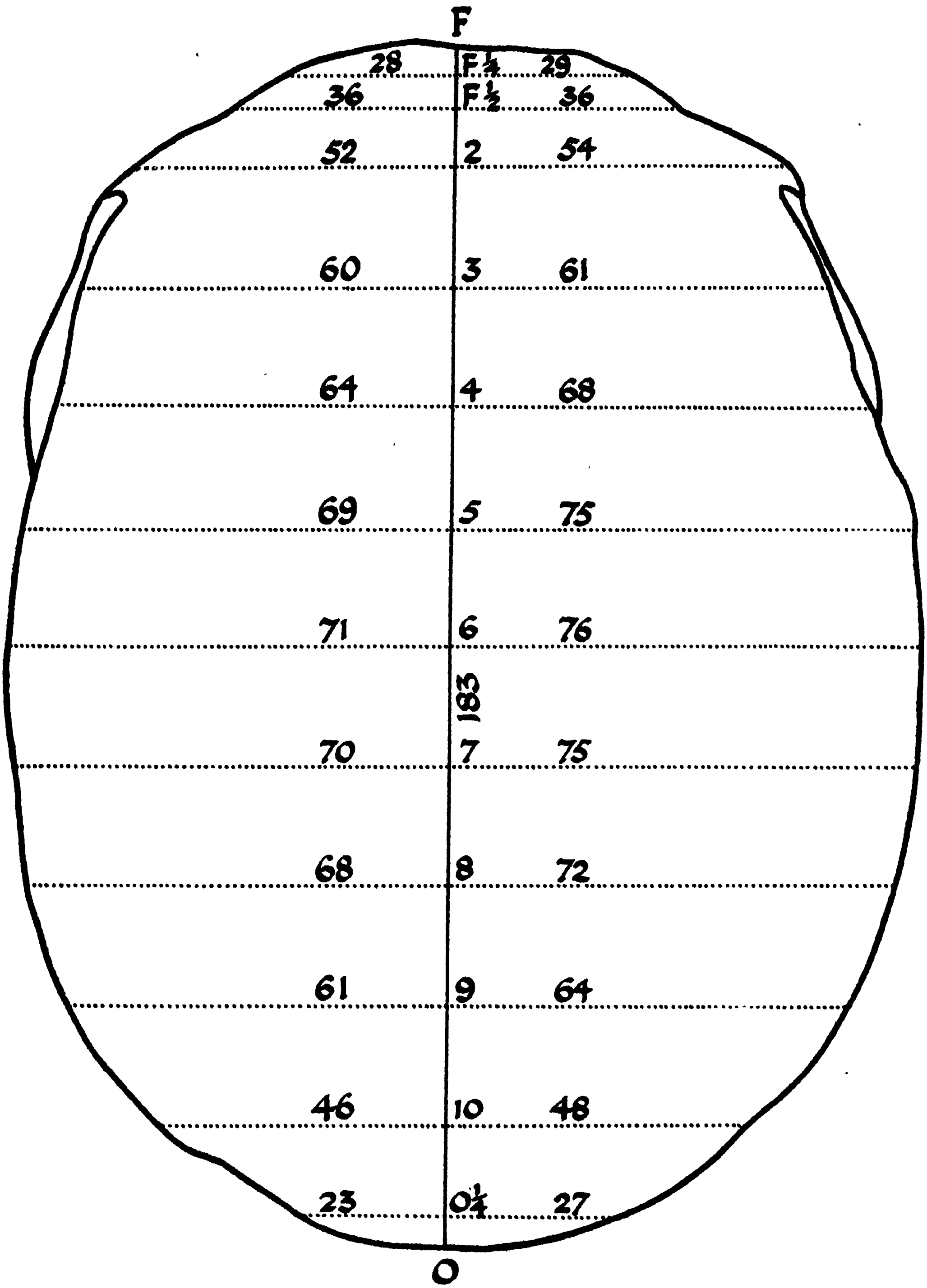


Fig. 6. Horizontal contour of the skull of a native of Aberdeenshire, Scotland. Dimensions in millimetres.

exception of the sagittal which was closed in the neighbourhood of the obelion. Patches of post-mortem erosion were scattered over the surface of the skull.

On the right parietal bone above and behind the parietal eminence there was a smooth, broadly tongue-shaped area having its base directed anteriorly and its apex in the opposite direction. It measured 5·0 cm. from side to side and 3·5 cm. from before backwards. It exposed the diploë, and led at its base into the cavity of the cranium by an elongated aperture which measured 3·0 cm. from side to side and 1·0 cm. in its greatest measurement from before backwards.

This area had evidently been produced by a swift blow from a heavy sharp cutting instrument, which had first struck the cranium at right angles to its surface, and then passing downwards and backwards had removed a flake which involved both tables and the diploë anteriorly and thinned off so as to involve only the external table of the bone posteriorly.

Towards the back part of the left side of the norma was seen an elongated aperture which led into the cavity of the cranium and communicated anteriorly with a fissure which extended forwards as far as the lower part of the coronal suture. This aperture and fissure will be more fully described in connection with the norma lateralis.

Norma lateralis (Figs. 9, 10, 11, 12, Plate I, Figs. 7, 8). Looking at this aspect of the cranium it was noticeable that, as a whole, it was flattened from above downwards, this general impression being confirmed by the length-height index which on calculation was found to be 69·6. This index was lower than the corresponding index of 70·4 for the Aberdeenshire skull. It corresponded closely to the average index given by Turner for modern Scottish crania (16) and according to Martin's list of height indices of crania indicated a marked degree of general depression of the vault (14).

The alveolar process of the upper jaw had been removed along the line of the bases of the fangs of the teeth. The position of the alveolar point was thus left somewhat indefinite. The anterior nasal spine was prominent. The lower parts of the nasal bones were absent, but the profile of the nose was seen to be somewhat flattened.

Nasion and glabella were practically in the same vertical plane. The angle which this plane formed with the profile of the nose was wider than the corresponding angle in the Aberdeenshire skull.

The face was orthognathous. The alveolar index was approximately 90·2 as compared with the alveolar index of the Aberdeenshire skull which was 93·5. The frontal region of the skull was flattened towards the nasio-bregmatic line. The inion was very prominent. The interparietal part of the occipital bone was flattened, and the outline of the cerebellar fossa was but slightly curved. On both sides the temporal ridges were particularly apparent especially anteriorly.

Many patches of post-mortem erosion were visible on both sides of the skull. The parietal margins of the squamous portion of the temporal bones were particularly

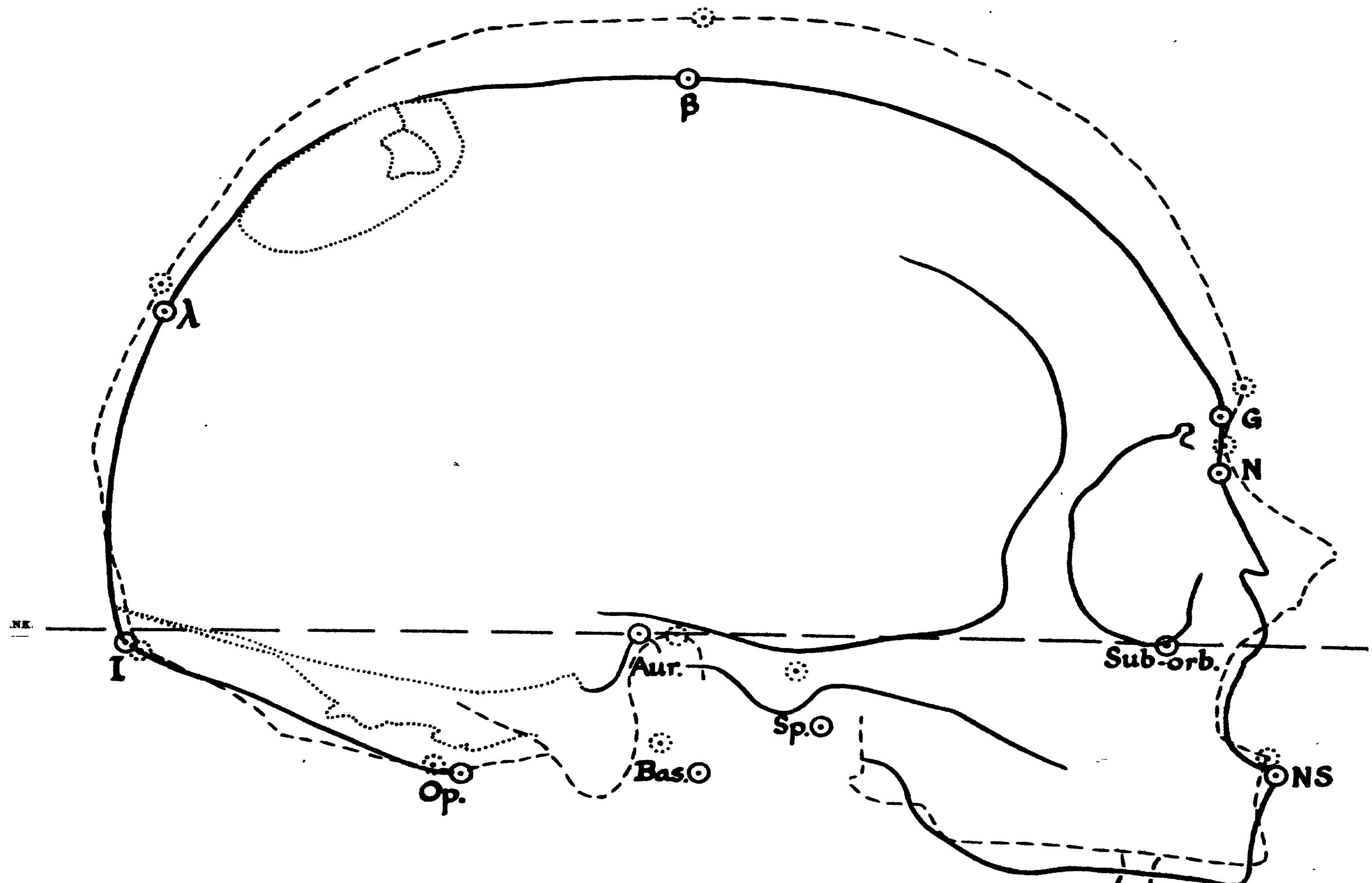


Fig. 9. Dioptrographic projection of skull of Saint Magnus (continuous line) superimposed upon dioptrographic projection of skull of native of Aberdeenshire, Scotland (broken line). Dotted line indicates injuries to skull of Saint Magnus. Orientated in Frankfort horizontal plane. *Norma lateralis* (right profile).

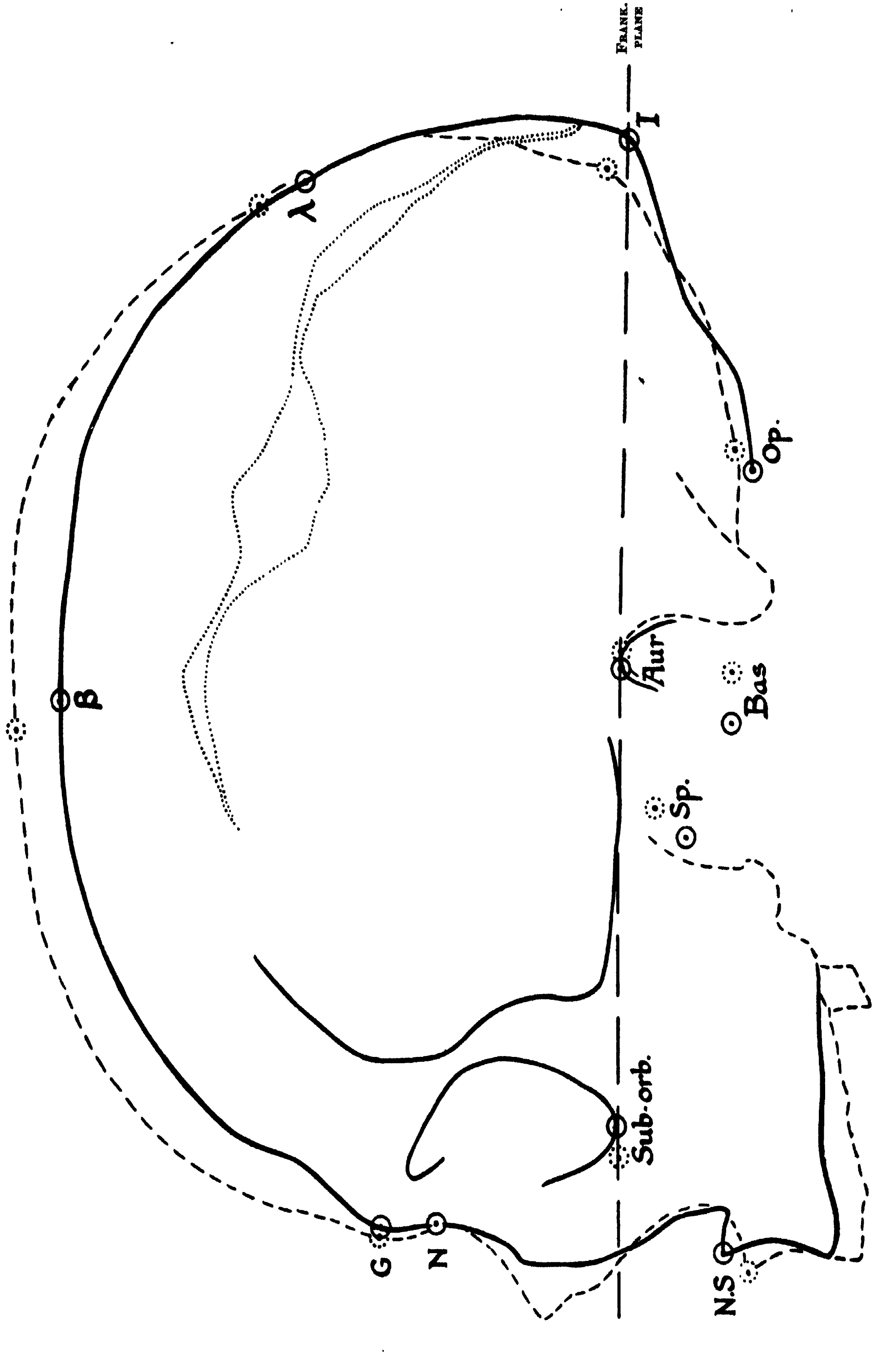


Fig. 10. Dioptrographic projection of skull of Saint Magnus (continuous line) superimposed upon dioptrographic projection of skull of native of Aberdeenshire, Scotland (broken line). Dotted line indicates injuries to skull of Saint Magnus. Orientated in Frankfort horizontal plane.

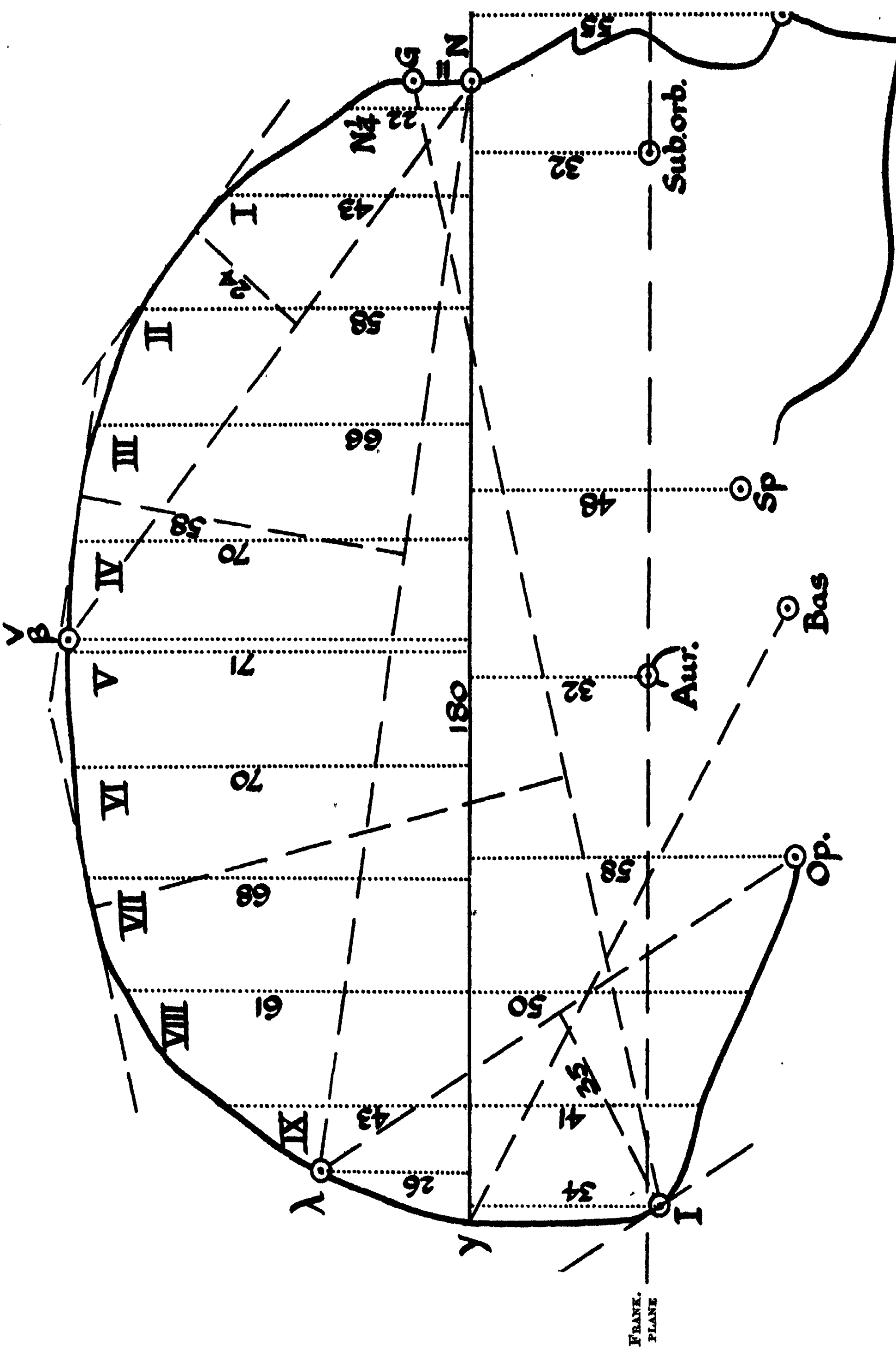


Fig. 11. Sagittal contour of skull of Saint Magnus. Dimensions in millimetres.

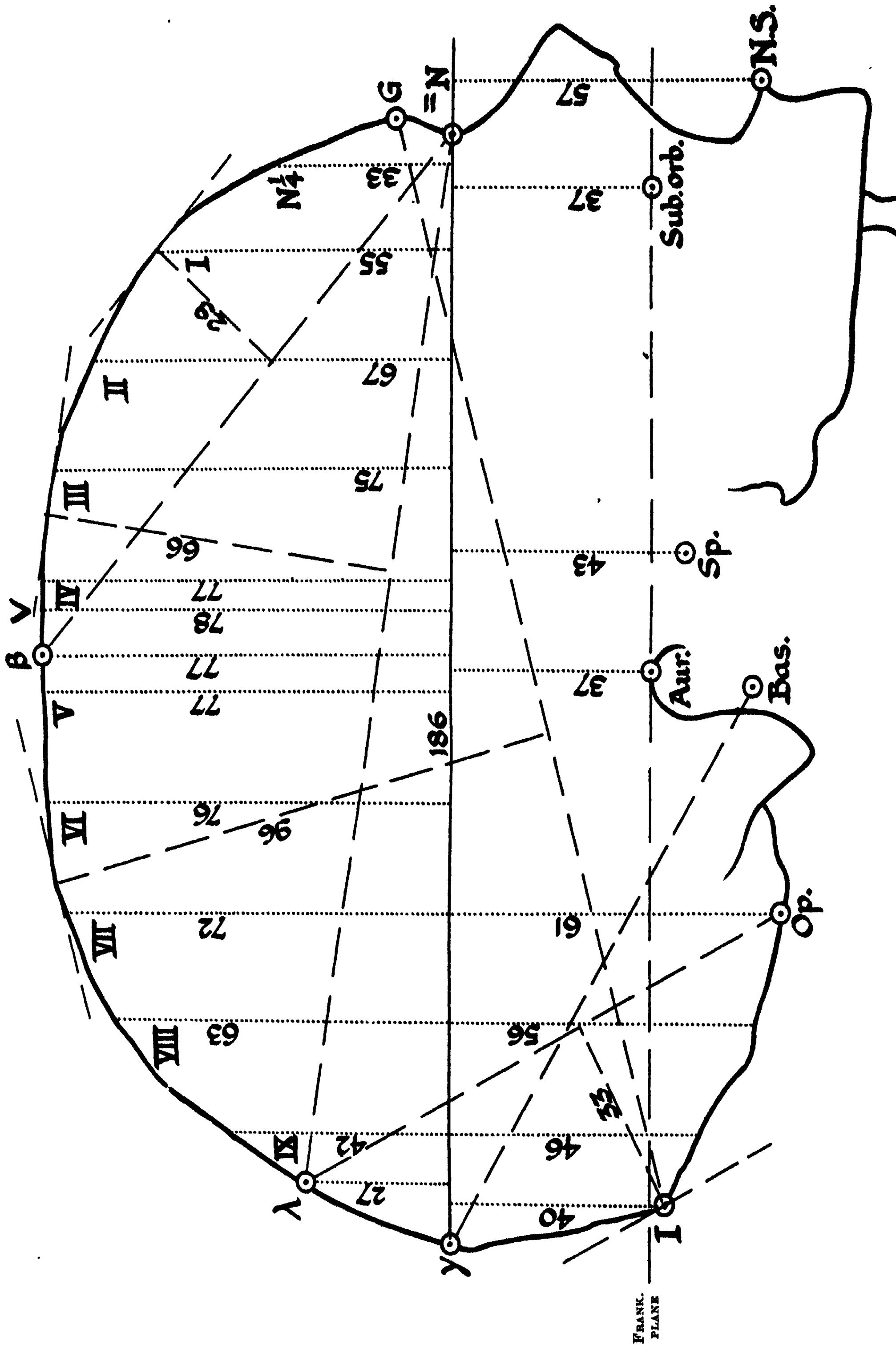


Fig. 12. Sagittal contour of skull of native of Aberdeenshire, Scotland.

eroded in the same way. All the sutures visible on both normae laterales were open. Those in the region of the pterion were well marked and the spheno-parietal articulation measured 1·5 cm. in length.

The lateral aspect of the injury already described as seen on the vertex was visible on the right norma lateralis although very much fore-shortened in this aspect.

On viewing the norma lateralis on the left side a sinuous fissure was seen in the parietal and occipital bones extending backwards and downwards from a point in the coronal suture 4·0 cm. above the pterion, across the lambdoidal suture 3·0 cm. from the lambda, and passing through the right side of the occipital bone 1·5 cm. above the inion. This fissure became continuous with the cut surface to be described on the norma basalis. The fissure throughout its whole length communicated directly with the interior of the cranium and was widely open as it passed across the parietal eminence where its edges showed much post-mortem decay.

Norma occipitalis (Figs. 14, 15, 16, Plate III, Fig. 13). A conspicuous feature presented by this aspect was the relation of the height to the width of the cranium, for, as was shown by the breadth auricular-height index of 70·6—several points less than the average index 75·3 of Swiss skulls as given by Martin (14)—the skull was very much flattened from above downwards; this feature was well demonstrated by comparison with the Aberdeenshire skull whose breadth auricular-height index was 80·1 (Fig. 14).

The sutures were open with the exception of the sagittal which was closed in the posterior third of its length, and post-mortem erosions similar to those already mentioned were also visible in this view of the cranium.

The injuries already described in association with the vertical and lateral aspects of the skull were also seen in this view. The one descending from the left parietal region became continuous with a deficiency of the floor of the cerebellar fossa on the right side of the skull.

Norma basalis (Plate II, Fig. 17). All sutures were open with the exception of the occipito-sphenoid which was closed.

The widely spreading zygomatic arches were particularly well shown on this aspect, which also presented a series of three smoothly cut surfaces lying in different planes.

One had been produced by the removal of the inner part of the right eminentia articularis, the floor of the right middle cerebral fossa, the right styloid process, the right mastoid process, and the right half of the occipital bone as far as a point half an inch above the external occipital protuberance where it became continuous with the fissure already described as existing on the left side of the cranium.

Another had been produced by the removal of a portion of the floor of the left cerebellar fossa, the left mastoid process through its antrum, the lower part of the petrous portion of the temporal bone, and both condyles of the occipital bone.

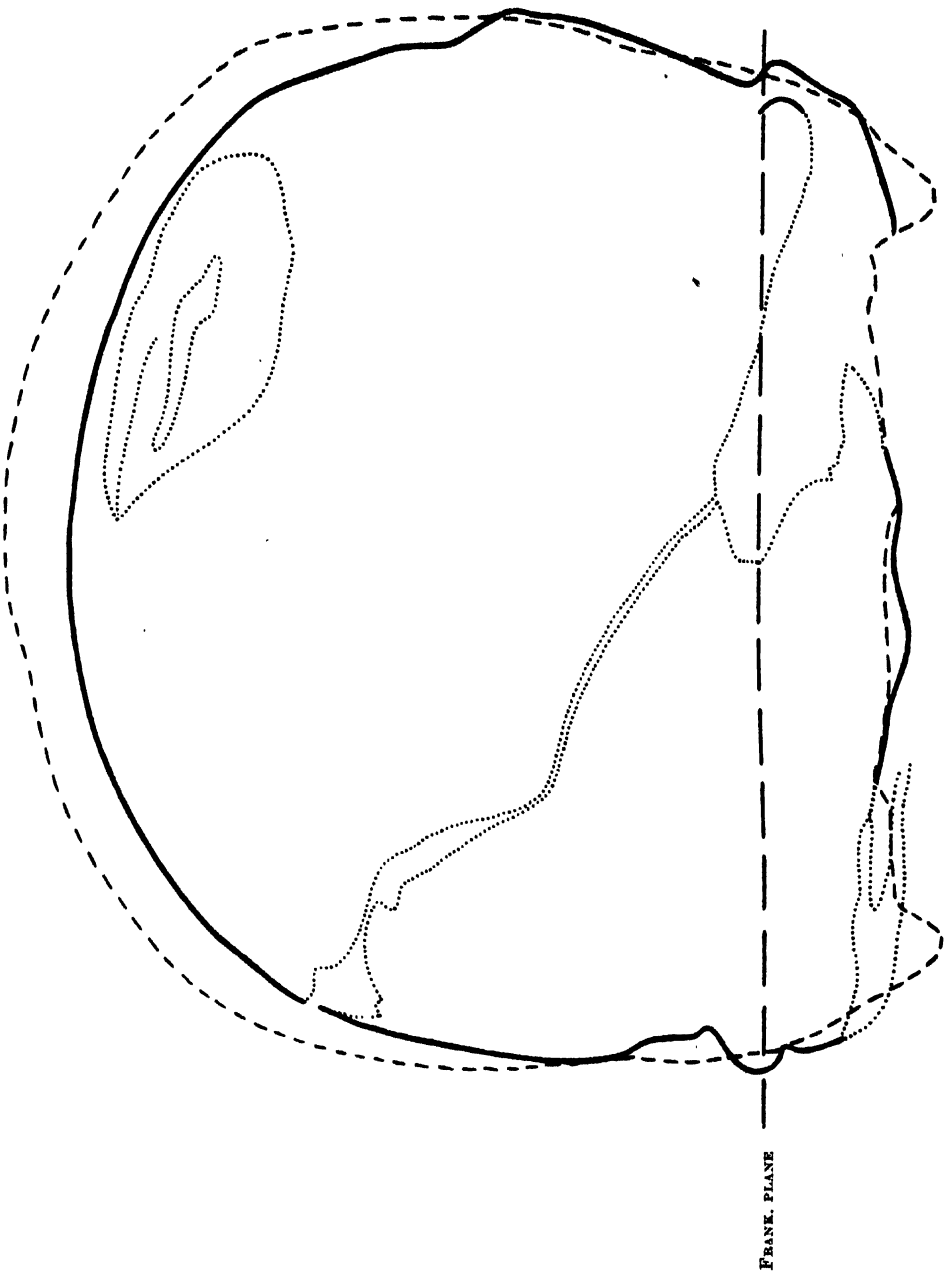


Fig. 14. Dioptrographic projection of skull of Saint Magnus (continuous line) superimposed upon dioptrographic projection of skull of native of Aberdeenshire, Scotland (broken line). Dotted line indicates injuries to skull of Saint Magnus. Orientated in Frankfort horizontal plane.

Norma occipitalis.

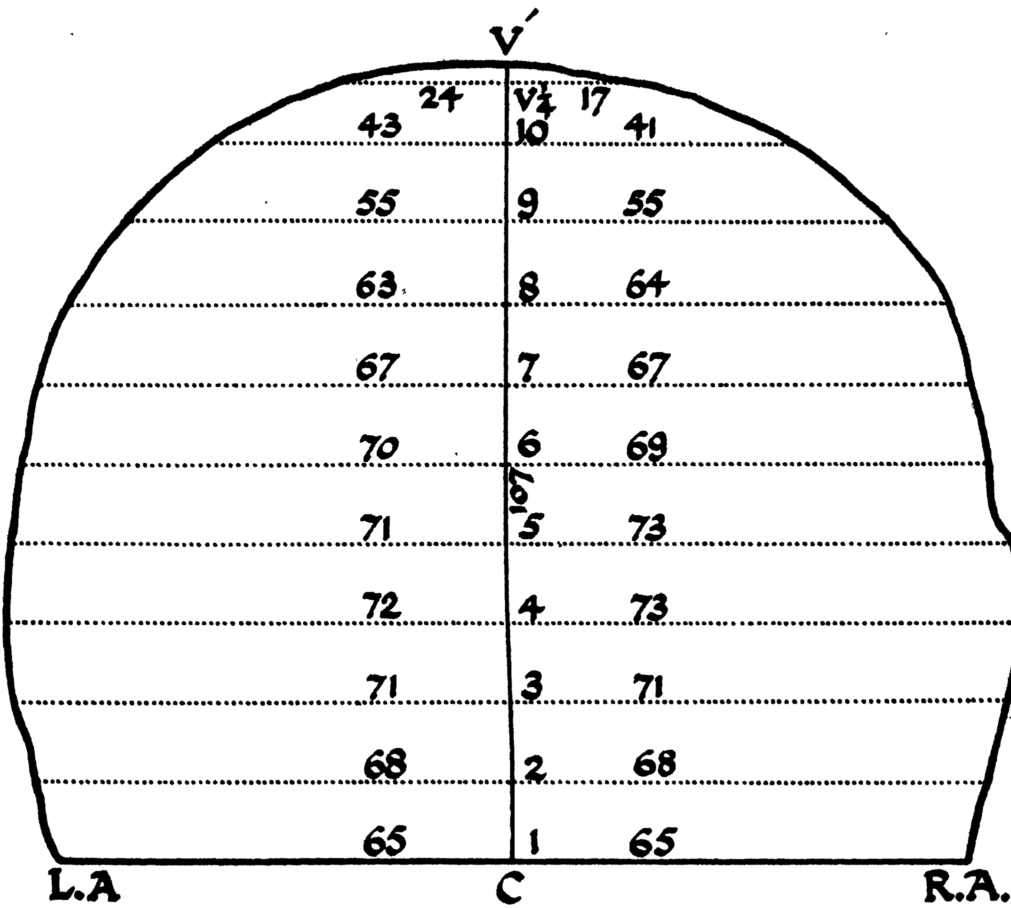


Fig. 15. Transverse vertical contour of the skull of St Magnus.
Dimensions in millimetres.

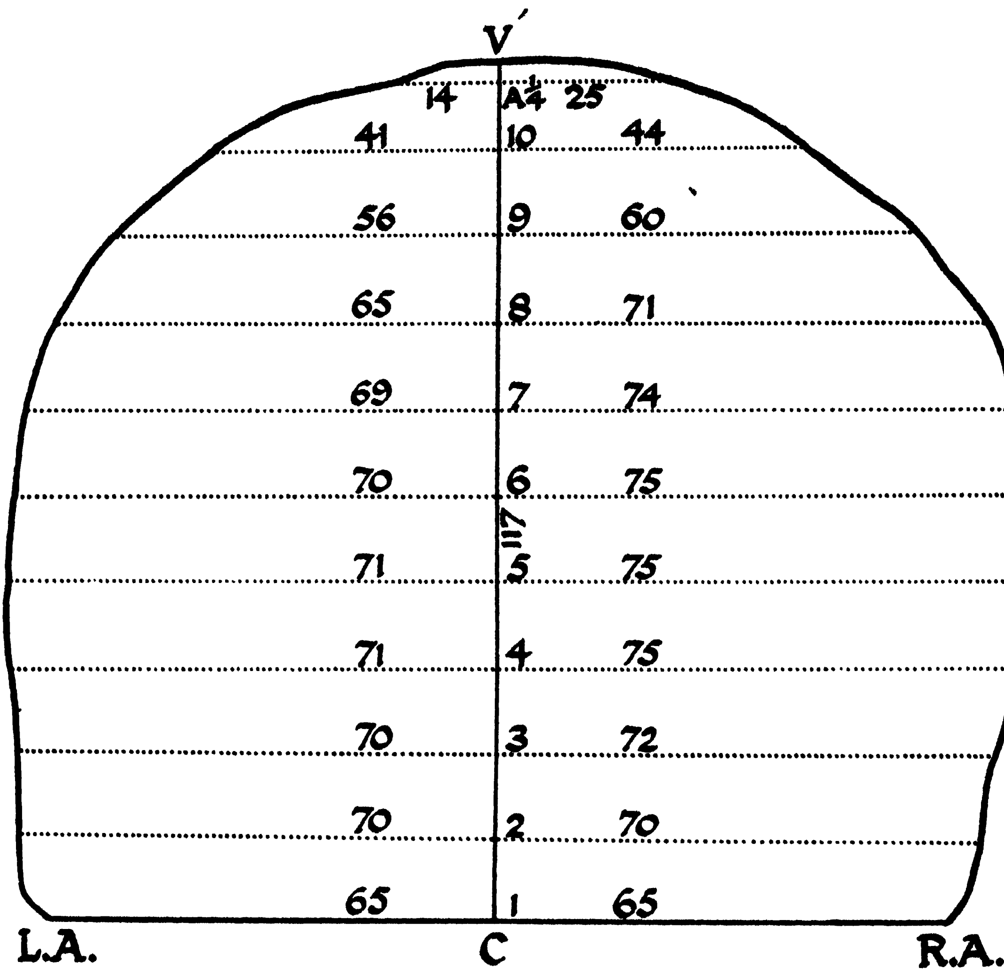


Fig. 16. Transverse vertical contour of the skull of a native of Aberdeenshire, Scotland. Dimensions in millimetres.

A third had been made by the removal of the alveolar processes of both superior maxillary bones at the level of the junction of the necks of the teeth with their fangs. The plane of the last was not exactly flat, but presented a very slight uniform, antero-posterior hollow looking downwards.

The portions of all the teeth above the line of incision were present in their sockets in the alveolar processes with the exception of those of the second left bicuspid and the third right molar which had evidently fallen out after death. The cut surfaces of the teeth were exactly level with the cut surface of the surrounding bone in which they lay.

These three smoothly cut surfaces had evidently been produced by swift blows from some heavy sharp cutting instrument similar to that which had been used to produce the wound in the vertex. There was no appearance of a saw having been used to occasion these injuries.

The blow which caused the cut surfaces on the right side of the base had evidently given rise to the fracture described as seen on the left norma lateralis and on the norma basalis.

Norma facialis (Fig. 19, Plate III, Fig. 18). The flattening of the head from above downwards was again visible. The right frontal sinus was open as the result of post-mortem decay and showed that, although there was no very pronounced superciliary ridge, the frontal sinus was well developed. The zygomatic arches were prominent. The cut edge of the alveolar process was apparent.

The bridge of the nose was flattened and had a remarkably low simotic index of 34 as compared with the average 51 given by Ryley and Bell for the Whitechapel English skulls (15). The right nasal bone was broader than the left, so that the inter-nasal suture joined the fronto-nasal suture to the left of the middle line. The canine ridges upon the upper jaw, especially the right, were well marked.

As regards breadth the face occupied a position midway between the long and short varieties. Its upper face index was 52·8 as compared with 57·8 for the Aberdeenshire skull.

The orbits were mesoseme with an index of 86·4 and the nose was mesorhine with an index of 48·2 and was broad as compared with that of the narrow nosed Aberdeenshire skull with a nasal index of 44·2.

Vertebral column. The only part of the column which was present was the spine of a lumbar vertebra from the middle of the lumbar region.

Ribs. Six fragments of these bones were present—those of the left first rib, a last rib, and four shafts of ribs whose numbers could not be ascertained. The fragments of the first rib consisted of a part of the body, tubercle, and most of the neck, and although it was somewhat decayed its appearance suggested that its vascular and muscular markings had never been very distinct.

Clavicle. The acromial and greater part of the prismatic portion of the left clavicle existed but was slightly decayed. It was a slender bone with a very slight curvature. The greatest circumference of the prismatic part was 4·2 cm.

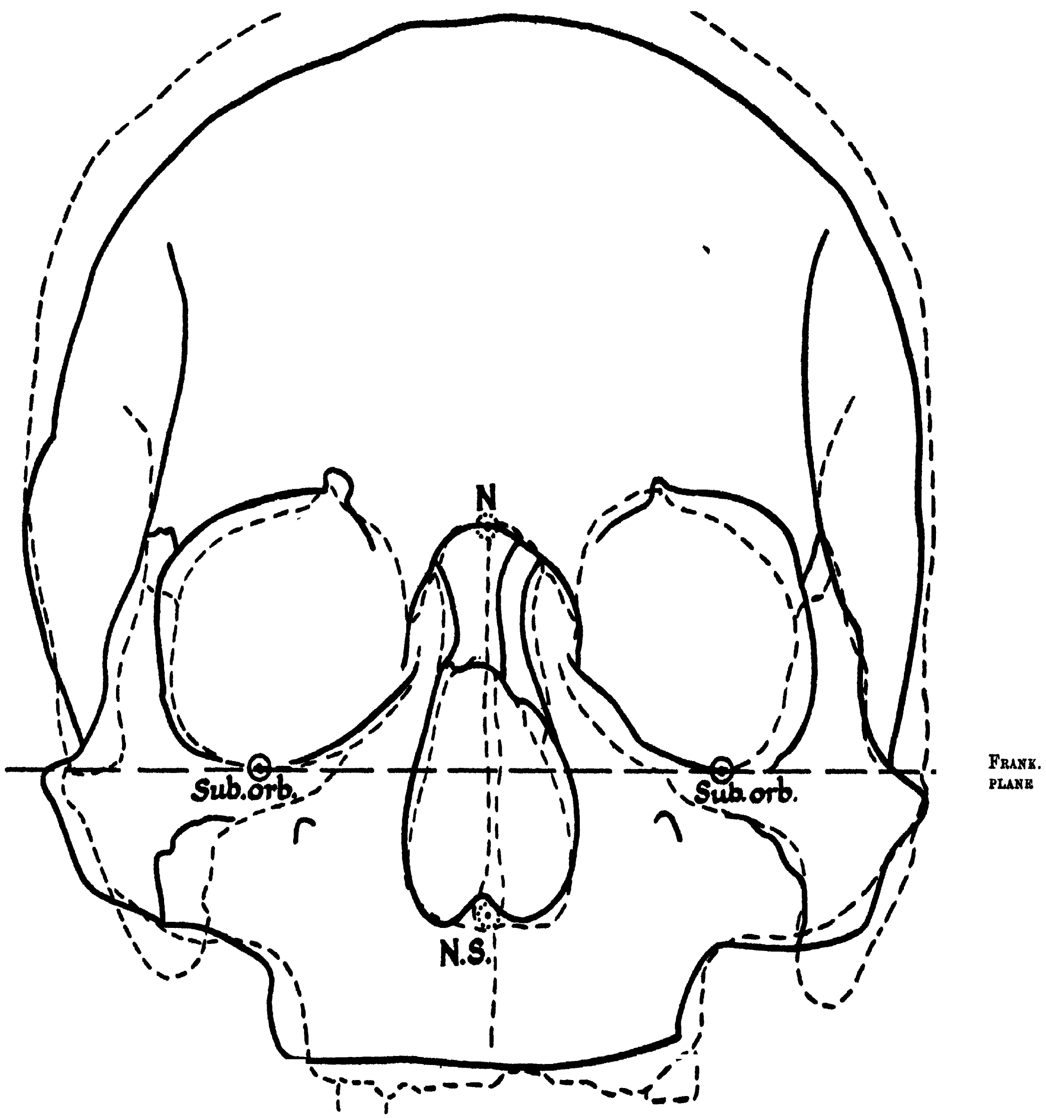


Fig. 19. Dioptrographic projection of skull of Saint Magnus (continuous line) superimposed on dioptrographic projection of skull of native of Aberdeenshire, Scotland (broken line). Orientated in Frankfort horizontal plane.

Norma facialis.

Scapula. The right was represented by a fragment consisting of the acromion process, coracoid process, glenoid cavity and upper third of the axillary border. The length-breadth index of the glenoid fossa was 70·3.

The fragment of the left scapula consisted of the acromion, posterior part of the spine and part of the neck.

In both scapulae the angle of the acromion was very evident and the areas for attachment of muscles and ligaments were well seen.

Humerus. In the right humerus the lower epiphyseal portion was wanting and the smallest circumference of the shaft below the deltoid eminence was 6·2 cm.

The lower extremity of the left humerus was absent, and the lower third of the shaft was much attenuated from decay.

The muscular markings, especially that for the deltoid muscle, were of normal size in each bone and in neither was a condyloid process or a coronoid foramen present.

Radius. Of the two radii the left one was entire. The curvatures of its shaft were conspicuous, especially in the region of insertion of the pronator radii teres muscle.

Its greatest length was 26·1 cm. Its physiological length, i.e., the distance between the deepest parts of the humeral and scaphoid articular surfaces, was 24·9 cm.

The smallest circumference below the middle of the shaft was 4·3 cm. The index of robusticity or length-girth index was 17·3 approximately.

The upper extremity with neck and tuberosity and immediately adjacent part of shaft of the right radius had disappeared from decay. The smallest circumference below the middle of the shaft, which was much eroded, was 3·7 cm. The curvatures in this bone were similar to those which were seen in the right.

Ulna. The left bone was present in its entirety. Its greatest length was 27·9 cm. Its physiological length, i.e., the distance between the deepest point of the greater sigmoid cavity and the deepest point of the articular surface on the head, was 24·9 cm. The smallest circumference of the shaft was 4·0 cm.

The index of robusticity or length-girth index was 16·1, and the curvatures of the shaft and muscular markings showed average appearances.

Hand. The only bones of the hand found among the remains were the base and part of the shaft of the third right metacarpal and portions of the shafts of two other metacarpals whose identity could not be ascertained.

Os innominatum. The right innominate bone was represented by a fragment composed of part of ilium, part of ischium, whole of pubis. The cotyloid cavity and margin of obturator foramen were intact. The greatest breadth of the cotyloid cavity was 5·4 cm. The edges of the depression at the bottom of this cavity and the cotyloid notch were very definitely defined.

The greatest breadth and greatest length of the obturator foramen were 3·7 cm. and 5·5 cm. respectively giving an index of 62·3 which practically corresponded with the average index—61·4—given by Martin for male Europeans (14).

A fragment consisting of part of the ilium and upper three-fourths of the cotyloid cavity was the only part of the left innominate bone preserved.

Femur. Both femora were complete with the exception of the great trochanters and the upper borders of the necks which had been destroyed by decay.

The oblique length, i.e., the distance between the highest point of the head and a plane tangential to the under surfaces of the condyles, was 47·2 cm. in the case of the right and 46·7 cm. in the case of the left bone.

The greatest length, i.e., the distance between the highest point of the head and the lowest point of the internal condyle, was 47·4 cm. in the case of the right and 47·1 cm. in the case of the left bone.

The platymeric index was 94·0 for the right and 75·8 for the left bone, which indicated that there was no undue flattening in the upper part of the shaft in the right bone, but that flattening showed itself very slightly in the corresponding region of the left bone.

Both the right and left bones gave a pilasteric index of 100.

The circumference in the middle of the shaft was 8·7 cm. in both cases which, when taken in conjunction with the total length of the bones, gave an index of robusticity or a length-girth index of 18·5 for the right and 18·6 for the left bone, indicating that the bones conformed in this respect with those of the average European.

Patella. The left patella was present but was much eroded along its edges. The articular surfaces, however, were intact. Its greatest length was 3·9 cm., its greatest breadth 3·0 cm. and its greatest thickness 1·9 cm.

Tibia. Both tibiae were complete. They were slender bones with muscular markings indistinct.

The greatest length, i.e., the distance between the tip of the spine and the tip of the internal malleolus, measured 39·6 cm. for the right and 39·4 cm. for the left bone.

The total length, i.e., the distance between the articular surface of the external tuberosity and the tip of the internal malleolus, measured 39·3 cm. for the right and 39·0 cm. for the left bone.

The smallest circumference of the shaft was 7·1 cm. in the case of both bones. The index of robusticity or length-girth index, i.e., the circumference of the shaft expressed as a percentage of its total length, was 18·1 for the right and 18·2 for the left bone.

As regards the degree of flattening of the shaft the platycnemic index in the case of the right bone was 72·5 and in the case of the left 66·7.

The retroversion angle of the right bone was 9° and of the left bone 8°. This indicated that neither tibial head was unduly turned backwards. The right tibia had a torsion angle of 12°·5, and the left of 12°.

The ratio of the total tibial to the total femoral length as expressed by the tibio-femoral index was 83·3 for the right and 83·6 for the left limb, indicating

that, in each case, the index was very slightly in excess of that associated with Europeans, which Turner gives as being below 83 (16).

Fibula. The head and upper inch of the shaft of the right fibula were absent. The remainder of the upper third of the shaft showed much post-mortem decay. The greatest thickness and smallest thickness of the mid-shaft were 1·4 cm. and 1·1 cm. respectively. The index of the transverse section of the mid-shaft, i.e., the smallest transverse thickness divided by the greatest thickness, was 78·6. The maximum circumference of the shaft was 4·3 cm.

The left fibula was complete but showed superficial erosion in the head and upper part of the shaft. The greatest length was 38·2 cm. The greatest thickness and smallest thickness of the mid-shaft were 1·4 cm. and 1·0 cm. respectively, which gave an index of the transverse section of the mid-shaft of 71·5. The maximum circumference of the shaft was 4·3 cm. and the smallest circumference of the bone below the head was 3·0 cm. The index of robusticity or length-girth index was 7·85.

Astragalus. Of the astragali the left alone was found among the remains, but it was well preserved.

Its length, i.e., the distance between the deepest part of the groove for musculus flexor hallucis longus and the most anterior part of the navicular articular surface, was 5·4 cm.

Its greatest length, i.e., the distance from the tip of the lateral tubercle of the groove for the musculus flexor hallucis longus to the most anterior part of the navicular articular surface, was 5·5 cm.

Its height, i.e., the perpendicular distance between the highest point of the inner lip of the trochlea and the horizontal plane upon which the bone rested in its normal position in the foot, was 3·4 cm.

Its breadth, i.e., the distance between the tip of the lateral process and medial side of the astragalus in the transverse plane and the superior trochlear surface, was 4·3 cm.

The antero-posterior length of the deepest part of the trochlea was 3·6 cm. and the length of the head and neck taken together was 1·9 cm.

The deviation angle of the neck, i.e., the angle between the sagittal axis of the trochlea and the longitudinal axis of the neck, was 15°, which lay between the average 11° and 17°·8, given respectively by Sewell and Volkov for Europeans (12).

The length-breadth index, i.e., the breadth of the astragalus expressed as a percentage of its total length, was 79·6, which appeared to be a normal figure for Europeans.

The neck-talus index, i.e., the length of the neck plus that of the head of the astragalus expressed as a percentage of the length, was 35·2, but statistics were lacking for the accurate determination of the significance of this index.

Os Calcis. The right os calcis was present and intact.

Of this bone, the greatest length, i.e., the distance between the perpendiculars dropped from the most projecting point of the great tuberosity and the highest point of the articular surface for the cuboid to the horizontal plane on which the bone rested when present in the living body in the erect posture, was 7·5 cm.

The middle breadth, i.e., the perpendicular distance between lines which passed through the most projecting points of the sustentaculum tali and the anterior extremity of the posterior facet for articulation with the talus and which were parallel to the axis of the bone, was 4·2 cm.

The length-breadth index was 56·0.

The height, i.e., the height of the deepest part of the groove upon the upper surface of the bone behind the posterior facet for articulation with the astragalus above the horizontal plane described in connection with the definition of the greatest length, was 4·2 cm.

The deviation angle of the posterior articular surface, i.e., the angle which the long axis of the posterior articular surface made with the long axis of the bone, was 50°, which corresponded closely with the angle given by Martin as the average for the Tyrolese (14).

Navicular. The right navicular was present and well preserved. It showed no abnormal features.

Cuneiforms. The right internal and the left external cuneiforms were also found and were normal.

Cuboid. The left cuboid was discovered and showed a facet for articulation with the scaphoid.

Metatarsals. There were five metatarsal bones present, viz., the right third and fourth and the left first, third and fifth. All were normal. The length of the first was 6·0 cm. and the greatest breadth of its shaft was 1·3 cm. The length-breadth index was 21·7.

Stature. The stature of the individual during life as calculated from the greatest length of the right femur and the total length of the right tibia according to Professor Pearson's formula (*f*) for dry bones (18) was 171·7 cm. (5 ft. 7½ in.), with a probable error of 2·03 cm. The stature, therefore, practically corresponded to the average of modern Norwegian males, i.e., 171·98 ± ·11 cm., or that of the average modern Swede, viz., 171·37 ± ·02 cm. (17).

Summary of the Description of the Human Skeletal Remains found in the South Pillar.

(1) The remains were those of a man whose age, roughly estimated, was from twenty-five to thirty-five years, whose stature was about 171·7 cm. (5 ft. 7½ in.), and who was rather poorly developed physically.

(2) His head as regards cephalic index was mesocephalic but tended towards the brachycephalic type.

(3) His cranial capacity (1380 c.c.) was much below (117 c.c.) that of the average (1497 c.c.) of the modern male European.

(4) He had a broad nose, prominent cheek bones, orbits like those of the average modern European, somewhat receding forehead and markedly flattened cranial vault.

(5) With the exception of incised wounds on the vault and base of the skull probably produced by swift blows from a heavy sharp cutting instrument which, if inflicted during life, must have resulted in his death, there were no other indications of injury or disease in his skeleton.

(6) He presented characters which were not purely of the type which we now associate with the taller element in the Scandinavian countries.

REMAINS FOUND IN THE NORTH PILLAR.

The remains found in the north pillar consisted of portions of the skeleton of a man whose age was roughly estimated as from forty to fifty years and who was about $168{\cdot}9 \pm 2{\cdot}03$ cm. (5 ft. 6½ in.) in stature, of the lower jaw of an aged person, and of the metacarpal bone of a young pig. With the exception of a small splinter of pine wood none of the small fragments of wood previously mentioned as having been found were present.

The human skeletal remains were: skull (imperfect), mandible, fourth and fifth lumbar vertebrae (imperfect), sacrum (imperfect), shaft of rib (imperfect), right and left femora (imperfect), right and left tibiae, left fibula, astragalus, os calcis, second right (imperfect), fifth right, first left, second left, and fifth left metatarsal bones.

Skull (Fig. 21, Plate IV, Figs. 20, 22). The portions of the skull which were present were in a fairly good state of preservation.

The capacity of the cranium as estimated by the use of shot was 1530 c.c., which was above the average, 1497 c.c., of mixed Europeans as given by Flower (13), and as regards its shape it was mesaticephalic with a breadth index of 78·1.

Owing to absence of bones of the face it was impossible to orient the skull in the Frankfort plane and it was thus valueless to make tracings of any other norma than the lateralis.

As regards the face the only portions present were a detached fragment consisting of the right malar bone and the malar process of the right superior maxilla, also the upper parts of the nasal bones and nasal processes of the superior maxillae which were in articulation with the frontal bone. The latter fragments formed a well marked high nasal bridge with a simotic index of 60 approximately.

The superciliary ridges and glabella were moderately developed. The subtense to the nasio-bregmatic arc was 2·7 cm. This was 0·2 cm. less than that of the Aberdeenshire skull.

The length-height index was 69·4, being slightly less than 70·9 given by Turner as the average of Scottish skulls (16).

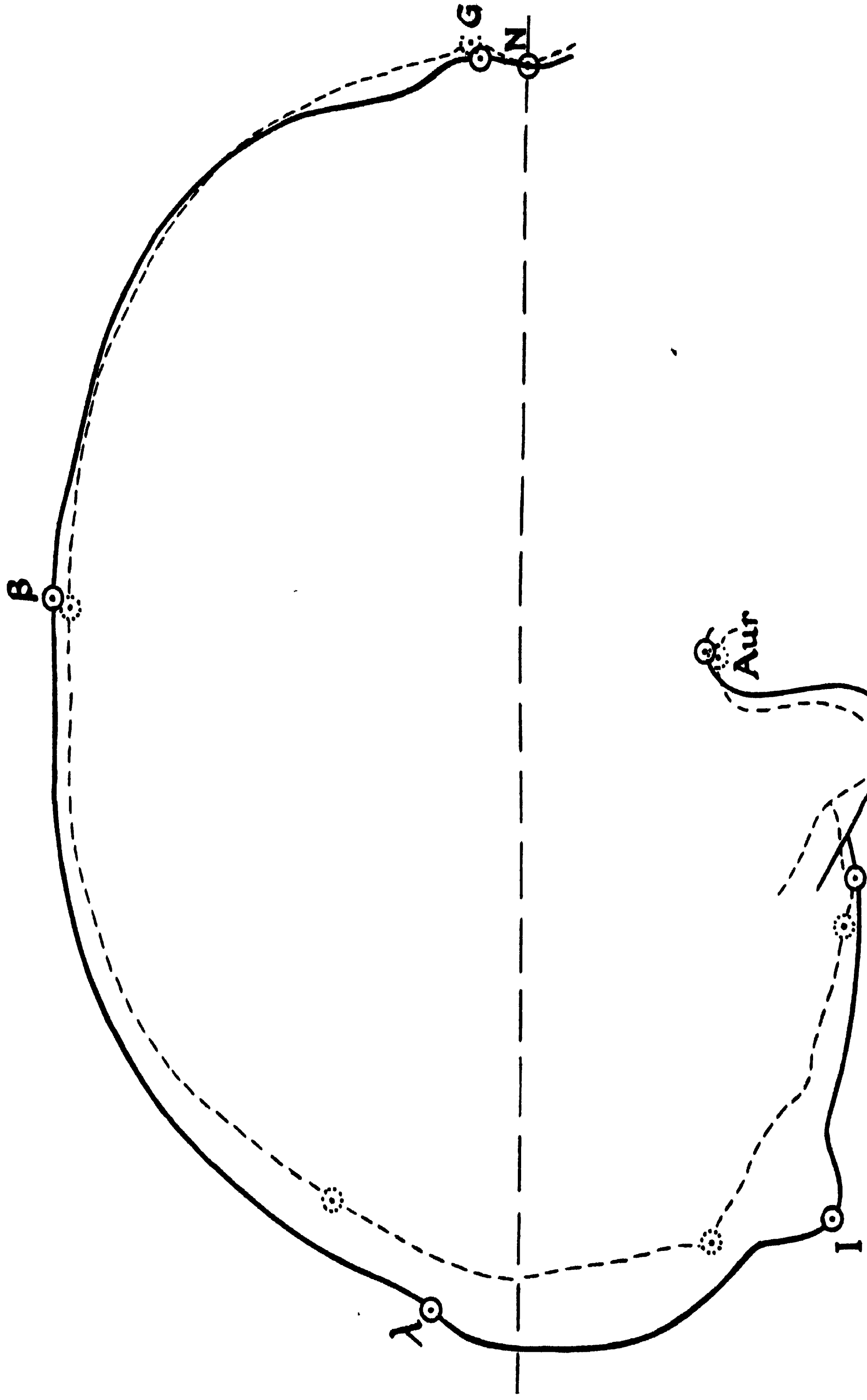


Fig. 21. **Dioptrographic projection of skull of Saint Rognvald (continuous line) superimposed upon dioptrographic projection of skull of native of Aberdeenshire, Scotland (broken line). Orientated on line drawn from nasion to point most distant from nasion in median sagittal plane.**

Norma lateralis (right profile).

In general outline the back of the skull was flattened, and showed a particular bulge in the occipital region especially at the inion.

The temporal ridges were pronounced and in their sweep upwards and backwards caused a marked narrowing of the forehead at the level of the ophryon.

All the cranial sutures were open with the exception of the sagittal in its whole length, the spheno-frontal, and the spheno-parietal. There was a well marked but smooth shallow depression behind the bregma in the position occupied by the hinder part of the anterior fontanelle in early childhood. It showed no appearance of having been produced by injury. There were many patches of erosion over the surface of the cranium, one of which was immediately above and parallel to the spheno-frontal suture.

A lower jaw was present with the remains in the north pillar, but it did not belong to this or to the skull found in the south pillar. Its intercondyloid exceeded the interglenoid measurement of the skull by 1 cm. It was the jaw of an old individual and was much attenuated and decayed along its angles and the posterior borders of its rami. All the teeth were absent, the alveolar process was almost completely absorbed, and the mental foramina were distant 1·5 cm. and 1·2 cm. from the upper and lower borders of the body of the bone respectively.

Vertebral column. The only parts of this column present were the complete fifth and part of the body of the fourth lumbar vertebra, and two fragments of sacrum one consisting of the bodies of the second, third, and fourth sacral vertebrae and the other of a fragment of the left sacral ala.

On the edges of the imperfect fourth lumbar vertebra there was marked appearances of the effects of rheumatism.

Ribs. A portion only of the body of one of the true ribs was present but it was impossible to identify its number.

Humerus. Both humeri were present, but their lower epiphyseal portions were absent. The muscular markings, especially the deltoidal eminences, were pronounced. No supracondyloid process or coronoid foramen was present in either bone. The smallest circumference of the shaft below the deltoid eminence was 7·4 cm. in the case of the right and 6·7 cm. in the case of the left bone. It was impossible to make the other usual measurements on account of the absence of the lower ends of the bones.

Femur. Both femora were present. The head, neck and great trochanter of the right femur were absent owing to decay. The great and small trochanters and upper border of the neck of the left bone were absent from the same cause. Connected with the small trochanter of the right femur, and the bone immediately surrounding it, was a spongy bony mass which had evidently been associated with the insertion of the ilio-psoas muscle. It was continuous below with the spiral line and the inner lip of the linea aspera, both of which were prominently marked. The area of insertion of the gluteus maximus muscle was also very evident. Signs of the effects of rheumatism were visible on the margins of the articular surfaces of the lower extremities of both bones.

The oblique length of the left femur as measured from the highest point of the head to a plane which was placed so as to be a tangent to the under surfaces of the condyles was 46·2 cm.

The greatest length of the bone, i.e., the distance between the highest point of the head and the lowest point of the internal condyle, was 47·0 cm.

The circumference in the middle of the diaphysis was 10·3 cm. in the case of the right and 10·2 cm. in the case of the left bone. The index of robusticity, i.e., the circumference of the left diaphysis expressed as a percentage of its total length of the bone, was therefore 22·1.

The degree of flattening of the upper part of the left femur as measured by the platymeric index was 102·8. The pilasteric index was 109·7 for each bone.

These measurements and indices indicated that both femora corresponded in shape and size to that of the average European male of the present day.

Tibia. Both tibiae were complete. Origins and insertions of muscles were fairly well marked.

The greatest length, i.e., the distance between the tip of spine and tip of internal malleolus, measured 37·8 cm. in the case of the right and 38·1 cm. in that of the left bone.

The total length, i.e., the distance between the articular surface of the external tuberosity and the tip of the internal malleolus, of the right bone measured 37·2 cm. and 37·8 cm. in the case of the left bone.

The smallest circumference of the shaft was 8·3 cm. for the right and 8·2 cm. for the left.

The index of robusticity or the length-girth index, i.e., the circumference of the shaft expressed as a percentage of its total length, was 22·3 for the right and 21·7 for the left bone. The platycnemic index was 78·9 for both bones. The torsion angle was 36° for the right bone and 38°·5 for the left. The retroversion angle was 10°·5 in both cases. The tibio-femoral index for the left extremity was 81·8.

The only remark to be made about these measurements was that all corresponded to those found in the average European with the exception of the markedly forward and outward twisting of the inner part of lower ends of tibiae. No facet was seen on the anterior margin of the lower extremity of either tibia for articulating with the astragalus in extreme dorsiflexion of the joint.

Fibula. The left bone was present and its grooves, ridges and longitudinal curves were particularly worthy of notice as they indicated that powerful muscle had been attached to the bone.

Its greatest length was 36·2 cm. The greatest thickness and the smallest thickness of the mid-shaft were 1·7 cm. and 1·5 cm. respectively, giving an index of the transverse section of the mid-shaft of 88·3. The greatest circumference of the mid-shaft was 5·5 cm. and the smallest circumference of the shaft just below the head was

4·0 cm. The circumference of the bone in relation to its length as indicated by the index of robusticity or the length-girth index was 11·0.

Astragalus. Both astragali were present and in good condition.

The length, i.e., the distance between the deepest part of the groove for musculus flexor hallucis longus and the most anterior part of the navicular articular surface, was 5·7 cm. for both bones.

The greatest length, i.e., the distance from the tip of the lateral lip of the groove for the musculus flexor hallucis longus to the most anterior part of the navicular articular surface, was 6·2 cm. for each bone.

The breadth, i.e., the distance between the tip of the lateral process of the body and the medial side of the astragalus in a plane tangential to the trochlear surface, was 4·5 cm. for the right and 4·7 cm. for the left bone.

The height, i.e., the perpendicular distance between the highest point of the inner lip of the trochlea and the horizontal plane upon which the bone lay in its normal position in the foot, was 3·8 cm. for the right and 3·7 cm. for the left bone.

The antero-posterior length of the deepest part of the trochlea was 3·8 cm. for the right and 3·7 cm. for the left bone. The length of neck and head taken together was 2·2 cm. for each astragalus.

The deviation angle of the neck, i.e., the angle between the sagittal axis of the trochlea and the longitudinal axis of the neck, was 18° for the right and 19° for the left bone, corresponding closely with the average 17·8° given by Volkov for Europeans (14). No facets existed upon the upper or the lateral surface of the neck.

The length-breadth index for the right bone was 79·0 and for the left 82·5. The neck-talus index for the right bone was 38·6 and for the left 40·0.

Os calcis. Both bones were present and in good condition.

The greatest length, i.e., the distance between perpendiculars dropped from the most projecting point of the great tuberosity and highest point of articular surface for the cuboid to the horizontal plane on which the bone rested in the erect posture, was 8·4 cm. for each bone.

The middle breadth, i.e., the distance between perpendiculars directed backwards from the most projecting point of the sustentaculum tali and the most anterior extremity of the posterior facet for articulation with the talus to a plane drawn at right angles to the line of its total length, was 4·4 cm. for both the right and left bones.

The height, i.e., the distance between the deepest part of the groove upon the upper surface of the bone behind the posterior facet for articulation with the astragalus and the point where the bone most closely approached the ground in the standing position, was 4·5 cm. for the right and 4·9 cm. for the left bone.

The deviation angle of the long axis of the posterior articular surface to the long axis of the bone was 52° in the case of the right and 53° in the case of the left bone.

Metatarsals. As already noted the only metatarsal bones present were the first left, the second right and left, and the fifth right and left. All were normal.

The length of the first left was 6·2 cm. and the greatest breadth of its shaft was 1·3 cm. The length-breadth index was 21·0.

Stature. The stature of the individual during life as calculated from the greatest length of the left femur and the total length of the right tibia according to Professor Karl Pearson's formula (*f*) for dry bones (18) was 168·9 cm. (5 ft. 6½ in.), with a probable error of 2·03 cm. This stature therefore was not significantly different from that of the average modern Norwegian, viz. 171·98 ± ·11 cm., or that of the average modern Swede, viz. 171·37 ± ·01 cm. (17).

Summary of the Description of the Human Skeletal Remains found in the North Pillar.

(1) The remains were those of a well developed man whose age, roughly estimated, was from forty to fifty years and whose stature was about 168·9 cm. (5 ft. 6½ in.).

(2) The skull was, as regards cephalic index, between the long and short varieties.

(3) The cranial capacity was rather above that of the average modern male European.

(4) With the exception of slight effects of rheumatism in the spine and lower limbs the bones were otherwise healthy and showed no indication of their having met with any injury during life.

(5) The measurements indicated that the skeleton possessed physical characters which were not purely of the type which we now associate with the taller element in the Scandinavian countries.

Conclusions.

(1) The results obtained from an investigation of the human remains found in the south and north pillars of the Cathedral of Saint Magnus, Kirkwall, Orkney, conformed with the descriptions and histories of Saint Magnus and Saint Rognvald as obtained from the Orkneyinga Saga and other relative documents.

(2) The investigation on the whole confirmed the conclusion that the human remains which were contained in a wooden case in the south pillar belonged to Saint Magnus and that those which were found in a cavity in the north pillar belonged in all probability to Saint Rognvald.

(3) It was interesting to note that as regards racial characters neither of the remains of the two Saints possessed characters which were purely of the type we now associate with the tall Nordic type.

Fig. 7. The skull of Saint Magnus.
Norma lateralis (right profile) (circa ½ linear).

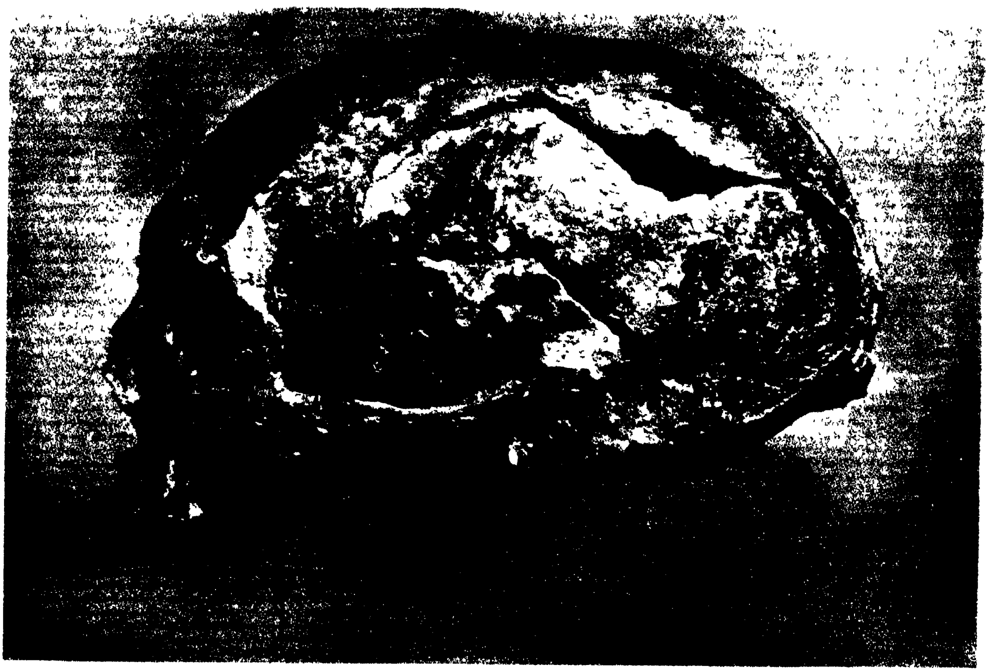


Fig. 8. The skull of Saint Magnus.
Norma lateralis (left profile) (circa ½ linear).

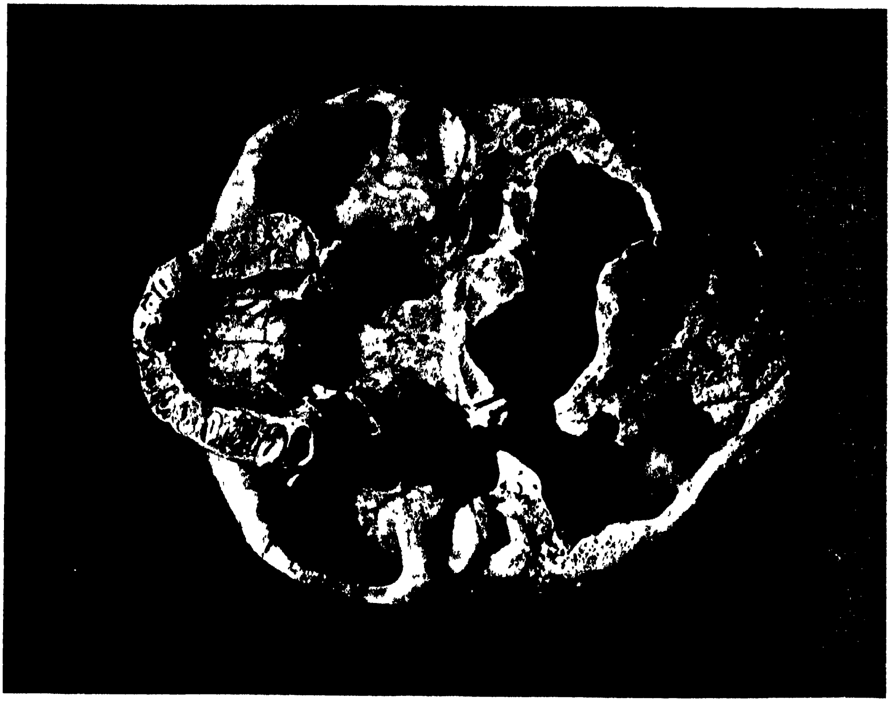


Fig. 17. The skull of Saint Magnus.

Norma basalis (circa ½ linear).

Fig. 3. The skull of Saint Magnus.

Norma verticalis (circa ½ linear).

Fig. 13. The skull of Saint Magnus.
Norma occipitalis (circa $\frac{1}{2}$ linear).

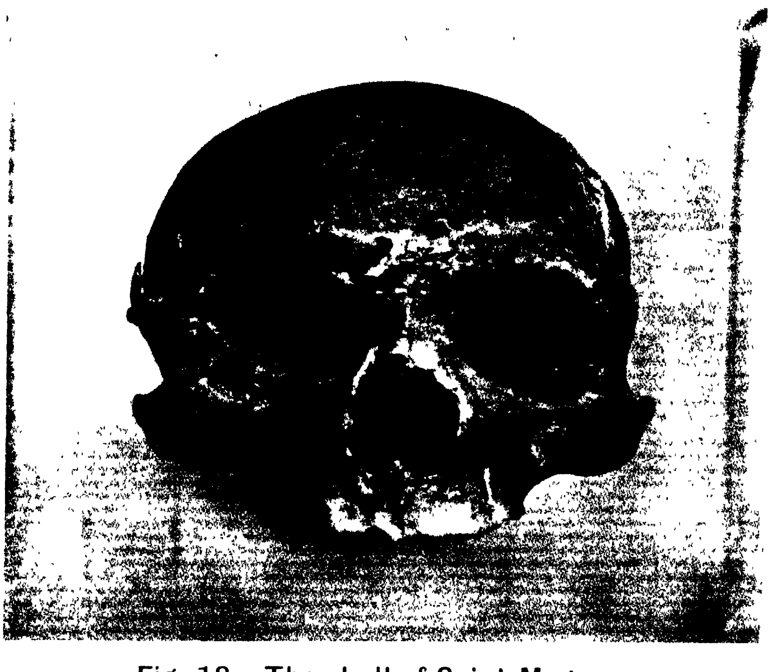


Fig. 18. The skull of Saint Magnus.
Norma facialis (circa $\frac{1}{2}$ linear).

Fig. 22. The skull of Saint Rognvald.
Norma verticalis (circa ½ linear).

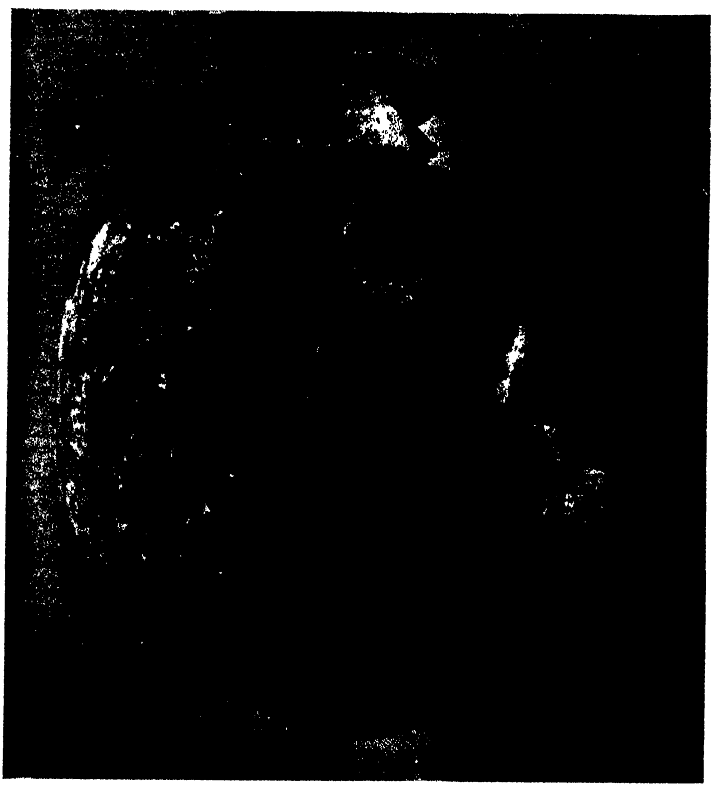


Fig. 20. The skull of Saint Rognvald.
Norma lateralis (right profile) (circa ½ linear).

It was largely due to the kindness of the Rev. George Walker, D.D., Minister of the East Church of Saint Nicholas of Aberdeen, who entertains a hereditary veneration for the Cathedral, and who possesses a profound knowledge of the structure and history of that noble pile, that sanction was given to me by the Magistrates of Kirkwall, and the Trustees of the Bequest of the late Sheriff Thoms for the restoration of the Cathedral, to prosecute this research. My cordial thanks are due to Dr Walker for this and for the assistance which he gave me in the elucidation of the historical part of the research.

It is impossible to overestimate the help which I received from Dr Mulligan, my former Assistant and Carnegie Teaching Fellow in the Anatomy Department of the University of Aberdeen, in carrying out this work and I wish to acknowledge it most gratefully.

I am indebted to Miss I. E. Reid, M.B.E., for the sketch of the interior of the choir of the Cathedral showing the position in which the relics were found.

I wish to record my appreciation of the manner in which the Thoms' Trustees, the Magistrates and Town Council of Kirkwall, enabled me to obtain access to the relics. My thanks are due to the Rev. William Barclay, M.A., for allowing me to use the Vestry of the Cathedral in which to conduct the investigation.

I desire to record my warmest thanks to Professor Karl Pearson, F.R.S., for many valuable suggestions and much kindness in facilitating the publication of this paper.

Explanation of Abbreviations used in Figures.

Alv. = Alveolar point. Aur. = Auricular point. Bas. = Basion. C. = Mid-point of inter-auricular line. F. = Point immediately above glabella on horizontal contour. Frank. plane = Frankfort horizontal plane. G. = Glabella. I. = Inion. LA. = Left auricular point. N. = Nasion. NS. = Tip of anterior nasal spine. O. = Point immediately below lambda on horizontal contour. Op. = Opisthion. RA. = Right auricular point. Sp. = Mid-point of spheno-occipital junction. Sub. Orb. = Sub-orbital point. V. = Vertex of skull. V'. = Point on transverse vertical contour of skull vertically above C. β. = Bregma. λ. = Lambda. γ. = Gamma, i.e., the point on the surface of the occipital region of the skull in the median plane on the same horizontal level as the nasion with the skull in the Frankfort horizontal plane.

TABLE I.

Measurements in centimetres and indices of the skulls of Saint Magnus and Saint Rognvald.

Symbols of Characters	Characters	Saint Magnus	Saint Rognvald
F	Ophryo-occipital length	18·2	19·3
L	Glabello-occipital length	18·4	19·6
B	Maximum horizontal breadth on parietal bones ...	14·6	15·3
	Breadth index ($100\ B \div L$)	79·3	78·1
B'	Least forehead breadth	10·1	9·3
H'	Basio-bregmatic height	12·8	13·6
	Height index ($100\ H' \div L$)	69·6	69·4
H	Height from basion to point vertically above it (with skull adjusted to Frankfort horizontal plane) ...	12·7	—
	Height index derived from H ($100\ H \div L$)	69·0	—
	Transverse vertical index ($100\ H' \div B$)	87·7	88·9
OH	Auricular height	10·3	—
	Breadth auricular height index	70·6	—
	Auriculo-bregmatic height	10·3	11·5
LB	Basion to nasion	10·2	9·8
Q'	Transverse arc between auricular points through vertex of skull	29·0	33·2
S	Sagittal arc from nasion to opisthion	35·4	40·3
S_1	Sagittal arc from nasion to bregma	12·6	13·2
S_2	Sagittal arc from bregma to lambda	10·5	14·0
S_3	Sagittal arc from lambda to opisthion	12·3	13·1
S_3'	Chord from lambda to opisthion	9·7	10·0
U	Greatest circumference of cranium	51·7	55·4
PH	(?) Alveolar point to tip of anterior nasal spine ...	2·1	—
$G'H$	Nasion to (?) alveolar point	7·5	—
GB	Distance between lower ends of malo-maxillary sutures	10·0	—
	Upper face index derived from GB ($100\ G'H \div GB$) ...	75·0	—
J	Zygomatic breadth	14·2	—
	Upper face index ($100\ G'H \div J$)...	52·8	—
NH, R	Nasion to lowest part of right nasal notch	5·7	—
NH, L	Nasion to lowest part of left nasal notch	5·5	—
NB	Greatest breadth of anterior nares	2·7	—
	Nasal index right ($100\ NB \div NH, R$)	47·4	—
	Nasal index left ($100\ NB \div NH, L$)	49·1	—
DS	Dacryal subtense	1·55	—
DC	Dacryal chord	2·60	—
	Mesodacryal index ($100\ DS \div DC$)	60·0	—
DA	Dacryal arc	4·30	—
SC	Simotic chord, i.e., the minimum between the two naso-maxillary sutures	1·09	0·60
SS	Simotic subtense	0·37	0·36
	Simotic index ($100\ SS \div SC$)	34·0	60·0
SA	Simotic arc	1·35	1·20
$O_1 R$	Greatest width of right orbit	4·4	—
$O_1 L$	Greatest width of left orbit	4·4	—
$O_2 R$	Greatest height of right orbit	3·8	—
$O_2 L$	Greatest height of left orbit	3·8	—
	Orbital index right ($100\ O_2 R \div O_1 R$)	86·4	—
	Orbital index left ($100\ O_2 L \div O_1 L$)	86·4	—
GL	Basion to (?) alveolar point	9·2	—
	Alveolar index (approximately)	90·2	—
fml	Antero-posterior length of foramen magnum	3·9	3·6
fmb	Greatest breadth of foramen magnum	—	3·2
C	Capacity taken with shot	1380 c.c.	1530 c.c.
	Capacity taken with mustard seed	1400 c.c.	1540 c.c.

TABLE II.

Measurements in centimetres, angles and indices of bones of extremities of Saint Magnus and Saint Rognvald.

Characters	Saint Magnus		Saint Rognvald	
	Right	Left	Right	Left
Radius:				
Greatest length	—	26·1	—	—
Physiological length	—	24·9	—	—
Smallest circumference of shaft	3·7	4·3	—	—
Index of robusticity	—	17·3	—	—
Ulna:				
Greatest length	—	27·9	—	—
Physiological length	—	24·9	—	—
Smallest circumference of shaft	—	4·0	—	—
Index of robusticity	—	16·1	—	—
Femur:				
Greatest length	47·4	47·1	—	47·0
Oblique length	47·2	46·7	—	46·2
Circumference of mid-shaft	8·7	8·7	10·3	10·2
Index of robusticity	18·5	18·6	—	22·1
Platymeric index	94·0	75·8	—	102·8
Pilasteric index	100·0	100·0	109·7	109·7
Tibia:				
Greatest length	39·6	39·4	37·8	38·1
Oblique length	39·3	39·0	37·2	37·8
Smallest circumference of shaft	7·1	7·1	8·3	8·2
Index of robusticity	18·1	18·2	22·3	21·7
Platycnemic index	72·5	66·7	78·9	78·9
Torsion angle	12°·5	12°·0	36°·0	38°·5
Retroversion angle	9°·0	8°·0	10°·5	10°·5
Tibio-femoral index	83·3	83·6	—	81·8
Fibula:				
Greatest length	—	38·2	—	36·2
Smallest circumference of shaft	—	3·0	—	4·0
Index of transverse section of mid-shaft ...	78·6	71·5	—	88·3
Index of robusticity	—	7·85	—	11·0
Astragalus:				
Length	—	5·4	5·7	5·7
Greatest length	—	5·5	6·2	6·2
Breadth	—	4·3	4·5	4·7
Length-breadth index	—	79·6	79·0	82·5
Height	—	3·4	3·8	3·7
Length of deepest part of trochlea	—	3·6	3·8	3·7
Length of head plus neck	—	1·9	2·2	2·2
Neck-talus index	—	35·2	38·6	40·0
Deviation angle of the neck	—	15°·0	18°·0	19°·0
Os calcis:				
Greatest length	7·5	—	8·4	8·4
Middle breadth	4·2	—	4·4	4·4
Length-breadth index	56·0	—	52·4	52·4
Height	4·2	—	4·5	4·9
Deviation angle of posterior articular surface	50°·0	—	52°·0	53°·0

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THE CORPUSCLE PROBLEM.

SECOND MEMOIR. CASE OF ELLIPSOIDAL CORPUSCLES.

By S. D. WICKSELL, *Lund*.

TABLE OF CONTENTS.

1. In my previous paper (*Biometrika*, Vol. XVII. pp. 84—99) I treated the case of spherical corpuscles and formulated the problem as follows: In an opaque body there are suspended a large number of corpuscles of different sizes, the density of the corpuscles and their size-distribution being the same in all parts of the body. To express the distribution of the sizes of the corpuscles in terms of the distribution of the sizes of the contours of corpuscles found in a plane section by which the body is split in two.

Denoting by $F(r)$ the frequency function of the diameter r of the corpuscles in general (the function required); by $f(r)$ the frequency function of the diameters of those special corpuscles which are cut by the plane of intersection; by $\phi(x)$ the frequency function of the diameters x of the circular contours in the section plane (the observed function) and by r_0 the mean diameter of the corpuscles in general, I found the following fundamental relations:

$$rF(r) = r_0 f(r) \qquad (1),$$

$$\phi(x) = x \int_x^R f(r) \frac{dr}{r\sqrt{r^2 - x^2}} \qquad (2).$$

Inserting $F(r)$ in equation (2) by the aid of (1), I obtained the integral equation for $F(r)$, from which the numerical solutions of the problem were subsequently worked out.

As pointed out already in the previous paper the relation (2) is valid also when the corpuscles have ellipsoidal forms, if x and r are defined as the geometric mean diameters of the elliptical contours cut out of the corpuscle by the section plane and by a plane through the centre of the corpuscle and parallel to the section plane. This was shown to be true whatever be the eccentricities or distribution of the eccentricities and whatever be the manner in which the axes are directed. Incidentally, it may be pointed out that as the two elliptic contours in question have the same form the relation (2) will be valid also if x and r denote for instance the major diameters or the minor diameters or, indeed, their arithmetic instead of their geometric mean.

The further reduction corresponding to the simple "spherical" relation (1) will, in the ellipsoidal case, however, be different and not so simple (compare equation II, p. 156), and the matter is still more complicated by a third reduction necessary in the ellipsoidal case, i.e. to find the distribution of the true sizes—in any way defined—from the distribution of the measure r.

As already stated in my previous paper it was above all the anatomical applications, and therein especially the spherical case, which first caused me to take up the corpuscle problem for discussion. I now take up the ellipsoidal case for a more complete discussion, partly on account of its general theoretical interest and its possible practical value for anatomists, but principally because the problem, and especially the ellipsoidal aspect of it, has been found to be of importance also in other fields, namely in geology and mineralogy. Although it was not published until 1925 the principal contents of my first paper in *Biometrika* were read before the

Actuarial Society in Copenhagen, in November 1923, in a lecture of which a short review appeared in the Danish journal, *Assurandören*, in December 1923. In the meantime, however, and unknown to me, a paper discussing the corpuscle problem was published by the Swedish geologist, Hagerman, in *Geologiska Föreningens i Stockholm Förhandlingar*, 1924. Hagerman had run across the problem quite independently, in connection with some technical questions in the investigation of granulary minerals, and without noticing that the same problem had already been receiving for more than ten years the attentions of the Swedish anatomists. He points out that in the materials investigated by the petrographists the corpuscles or grains are generally not spherical but rather of ellipsoidal form, and he gives an interesting and so far quite correct, but from a mathematical point of view rather primitive, discussion of the problem. For the mathematical aspects involved he refers to an appendix to his paper written by a young mathematician and treating the special case when the corpuscles are ellipsoids of revolution of constant form. As I have had occasion to show in a short notice published in the same geological journal, in December 1925, this mathematical treatment of the problem is, however, fundamentally incorrect. Its results are quite misleading even in the case of spherical corpuscles. As the papers referred to are written in Swedish I do not think it necessary here to enter into details of their contents and my criticism, but I wish to acknowledge that it is the paper of Hagerman and the interesting applications there suggested which have made it clear to me that a full mathematical discussion of the ellipsoidal corpuscle problem is urgently wanted. This is, indeed, the primary cause of my now taking the matter up again. As referred to in the last section of this paper it has appeared subsequently that in mineralogical researches the problem is not only to determine the number and size distribution of the corpuscles, but also to find their total surface. Thus I have devoted the last section of this paper to a discussion of this more special problem.

2. In treating of the ellipsoidal corpuscle problem, the first important question is to define what is to be meant by the *size* or *diameter* of an ellipsoidal corpuscle, and of the elliptic contour cut out of it by the section plane. As will be seen later on, several different definitions are possible, and in special cases the one or the other will be preferable from the point of view of mathematical simplicity. Thus, in cases where the corpuscles are prolate or oblate ellipsoids of revolution it will be most convenient to use the minor or major diameter, respectively, as size measures.

In the general case of tri-axial ellipsoids I have, however, found the geometric mean of the principal diameters the most suitable measure. Besides the relatively greater simplification of the mathematical expressions following from this choice of size measure, considerations of a more general nature also favour it. The geometrical mean of the principal diameters has the important property of giving a measure which is proportional to the cube root of the volume and the square root of the area, respectively, of the corpuscles or corpuscle contours in question. Generally speaking it seems most rational to define the size of a corpuscle by its

volume and the size of a section contour by its area, and when linear measures are wanted the diameter of the sphere or circle of equal volume or area will probably give the best measure.

3. I here put together some of the notations, definitions and relations used in the following discussion.

Section: an arbitrary plane section through the material containing the corpuscles.

Section Ellipse: the ellipse cut out of a corpuscle by the section. The principal diameters of the section ellipse are denoted by ξ_1 and ξ_2.

Central Ellipse: the ellipse cut out of a corpuscle by a plane through the centre and parallel to the section. The principal diameters of the central ellipse are denoted by σ_1 and σ_2.

The principal diameters of a corpuscle are denoted by s_1, s_2 and s_3, and by α_1, α_2 and α_3 we denote the direction cosines of the normal to the section with reference to the axes of the corpuscle. Denoting by h the "height" of a corpuscle, i.e. the distance between the two tangent planes, parallel to the section, it is easily found that we have

$$h^2 = s_1^2\alpha_1^2 + s_2^2\alpha_2^2 + s_3^2\alpha_3^2.$$

Hence, taking s_1 to be the major diameter, and denoting the two principal eccentricities by e_2 and e_3, we also have, since

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

the relation

$$h = s_1\sqrt{1 - \alpha_2^2e_2^2 - \alpha_3^2e_3^2} \quad\text{..............................(3).}$$

As "diameter" of a corpuscle we take the geometric mean of its principal diameters. Thus the "diameter" of a given corpuscle is

$$\rho = \sqrt[3]{s_1s_2s_3}.$$

As "diameter" of the central ellipse, we take

$$r = \sqrt{\sigma_1\sigma_2},$$

and as "diameter" of the section ellipse

$$x = \sqrt{\xi_1\xi_2}.$$

Between the parameters h, ρ, r and x the following relations hold. By y is here denoted twice the distance of the centre of a corpuscle from the section plane. We then have, as can be easily verified,

$$\rho^3 = hr^2 \quad\text{.. (4),}$$

$$\frac{y}{h} = \frac{\sqrt{r^2 - x^2}}{r} \quad\text{..................................(5).}$$

Further, since $s_2 = s_1\sqrt{1 - e_2^2}$; $s_3 = s_1\sqrt{1 - e_3^2}$, we have

$$\rho = s_1\left((1 - e_2^2)(1 - e_3^2)\right)^{\frac{1}{6}}.$$

We thus get

$$h = \rho \frac{(1 - e_2^2\alpha_2^2 - e_3^2\alpha_3^2)^{\frac{1}{2}}}{((1 - e_2^2)(1 - e_3^2))^{\frac{1}{6}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and, according to (4),

$$r = \rho \frac{((1 - e_2^2)(1 - e_3^2))^{\frac{1}{12}}}{(1 - e_2^2\alpha_2^2 - e_3^2\alpha_3^2)^{\frac{1}{4}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Finally we denote by

n the number of corpuscles to the unit area in the section,

$\phi(x)$ the relative frequency function of x for the corpuscles contained in the section,

$f(r)$ the relative frequency function of r for the same corpuscles,

N the number of corpuscles in general per unit volume of tissue,

$F(r)$ the relative frequency function of r for the corpuscles in general,

$P(\rho)$ the relative frequency function of ρ for the corpuscles in general,

$\Pi(\alpha_2, \alpha_3)$ the correlation function of α_2 and α_3, defining the way in which the principal axes of the corpuscles are directed,

h_r the regression of h on r for the corpuscles in general,

$M'(a)$ the arithmetic mean of any measure a in the corpuscles, contained in the section,

$M(a)$ the arithmetic mean of a for the corpuscles in general.

Here $\phi(x)$ is the observed, apparent distribution of the "diameters" and $P(\rho)$ is the distribution of the actual "diameters." The problem is to express $P(\rho)$ in terms of $\phi(x)$. This can be done in several ways, the most practical one, however, seems to be to express the moments of the function $P(\rho)$ in terms of the moments of the function $\phi(x)$, i.e. to express $M(\rho^t)$ in terms of $M'(x^s)$.

4. According to (5), and as already shown in § 7 of my previous paper, we must have

$$\phi(x) = x \int_x^R f(r) \frac{dr}{r\sqrt{r^2 - x^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I.}$$

This integral equation depends only on the assumption that the corpuscles have ellipsoidal form and are uniformly distributed in space. The equation is, in particular, independent of the way in which the corpuscles are orientated with regard to the direction of their axes (i.e. the form of the function $\Pi(\alpha_2, \alpha_3)$) and also of the way in which the eccentricities may vary.

5. In deducing the relations between $f(r)$ and $F(r)$ or $P(\rho)$ it is, however, necessary to make assumptions concerning the way in which the eccentricities and directions of the axes vary.

With respect to the former variation it is, indeed, very difficult to find any sufficiently general and at the same time simple law of distribution. That the variation of the eccentricities must be assumed to be independent of the direction parameters α_2 and α_3 is evident. But it seems indispensable also to assume it

independent of the size, as expressed by the "diameter" ρ. Otherwise there seems to be no way to solve the problem.

However, it will be possible to postpone the discussion of this question, which will not be prejudiced by our regarding for the present the eccentricities as constant, i.e. the corpuscles as having the same form.

Regarding the variation of the directions, I assume that it takes place in quite haphazard fashion, and quite independently of the sizes of the corpuscles. This implies the following: imagine a sphere of unit radius to be described around the centre of every corpuscle, and that the point in which the normal from the centre to the section plane intersects this sphere is marked off; imagine further that all those spheres are made to coincide, by referring them to the same centre, and rotating them, until the lines corresponding to the axes of the corpuscles coincide: then the points marked off will be uniformly distributed on the surface of this sphere. In other words, any direction of the normal, relative to the principal axes of the corpuscle, will be equally likely. Now, α_1, α_2 and α_3 are the rectangular coordinates of the point of intersection of the normal and the unit sphere. Hence the probability that α_2, α_3 falls within a given "element" of area" $d\alpha_2 d\alpha_3$ will be proportional to the element of surface on the sphere of which it is the projection. It is thus immediately seen that we must have

$$\Pi(\alpha_2, \alpha_3) = \frac{2}{\pi\sqrt{1-\alpha_2^2-\alpha_3^2}} \quad \text{..........................(8).}$$

6. As the section will cut any corpuscle which has its centre within the distance $h/2$ on either side of the section plane, it is evident that we have

$$n = h_0 N,$$

h_0 being the mean value of the "heights" of the corpuscles. Furthermore, taking into consideration only corpuscles with a given central "diameter" r, it is as easily seen that we have

$$nf(r) = NF(r)\,h_r,$$

where h_r is the mean value of h for those corpuscles in which the central "diameter" is equal to r. Hence we have

$$f(r) = F(r)\frac{h_r}{h_0} \quad \text{..................................II.}$$

This equation evidently corresponds to the relation (1), deduced for the case of spherical forms.

Now, as

$$P(\rho)\,.\,\Pi(\alpha_2, \alpha_3)\,d\rho\,d\alpha_2\,d\alpha_3$$

is the probability that a given corpuscle has the "diameter" ρ and the direction parameters α_2 and α_3, it is evident that we have

$$h_0 = \int_0^R d\rho \int_0^1 d\alpha_2 \int_0^{\sqrt{1-\alpha_2^2}} d\alpha_3 h\,P(\rho)\,\Pi(\alpha_2, \alpha_3) \quad \text{..................(9),}$$

where we may insert for h the expression (6). But if the integration is not, as above, extended to all possible combinations of the variables ρ, α_2 and α_3, but is

restricted to such combinations as give, when inserted in (7), an assigned value to r, we get, instead of h_0, the value of $F(r)\,h_r$ corresponding to that particular r. On account of II we have therefore

$$f(r)\,h_0 = \iiint h\;P(\rho)\,\Pi\,(\alpha_1,\,\alpha_2)\,d\rho\,d\alpha_2 d\alpha_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{III},$$

the integration being extended over all values of the variables, which make

$$\rho\,((1-e_2{}^2)\,(1-e_3{}^2))^{\frac{1}{12}}\,.\,(1-e_2{}^2\alpha_2{}^2-e_3{}^2\alpha_3{}^2)^{-\frac{1}{4}}$$

equal to a given quantity r.

Equations I and III may be regarded as the fundamental equations of the ellipsoidal corpuscle problem. Solving I (in any one of the ways indicated in the previous paper) $f(r)$ is obtained; $f(r)$ being known $P(\rho)$ is determined by the integral equation III.

7. For solving III, I have found no better method than the method of moments. Multiplying both sides by r^s and integrating over all values of r we get, inserting on the right side the expressions (6) and (7) for h and r,

$$h_0\int_0^R r^s f(r)\,dr = \int_0^R d\rho\int_0^1 d\alpha_2\int_0^{\sqrt{1-\alpha_2{}^2}} d\alpha_3\,\frac{((1-e_2{}^2)\,(1-e_3{}^2))^{\frac{s-2}{12}}\,\Pi\,(\alpha_2,\,\alpha_3)\,.\,\rho^{s+1}P(\rho)}{(1-e_2{}^2\alpha_2{}^2-e_3{}^2\alpha_3{}^2)^{\frac{s-2}{4}}}$$
$$\ldots\ldots\ldots(10),$$

or, using the notation explained in § 3,

$$h_0\,.\,M'(r^s) = M(\rho^{s+1})\,.\,M\left(\frac{((1-e_2{}^2)\,(1-e_3{}^2))^{\frac{s-2}{12}}}{(1-e_2{}^2\alpha_2{}^2-e_3{}^2\alpha_3{}^2)^{\frac{s-2}{4}}}\right)\quad\ldots\ldots\ldots\ldots\ldots(11).$$

This relation, in the form in which it stands, is independent of any other assumption than that α_2, α_3, e_2 and e_3 vary independently of ρ.

If now for the sake of convenience we write (11) in the form

$$h_0\,.\,M'(r^s) = M(\rho^{s+1})\,.\,\lambda_s \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11\text{ bis}),$$

we have, using the expression (8) for $\Pi\,(\alpha_2,\,\alpha_3)$,

$$\lambda_s = [(1-e_2{}^2)\,(1-e_3{}^2)]^{\frac{s-2}{12}}\,\frac{2}{\pi}\int_0^1 d\alpha_2\int_0^{\sqrt{1-\alpha_2{}^2}} d\alpha_3\,\frac{(1-\alpha_2{}^2e_2{}^2-\alpha_3{}^2e_3{}^2)^{\frac{2-s}{4}}}{\sqrt{1-\alpha_2{}^2-\alpha_3{}^2}}\quad\ldots(12).$$

This integral is not in general an elementary function of e_2 and e_3. But as

$$\frac{2}{\pi}\int_0^1 d\alpha_2\int_0^{\sqrt{1-\alpha_2{}^2}} d\alpha_3\,\frac{\alpha_2{}^{2n}\,\alpha_3{}^{2m}}{\sqrt{1-\alpha_2{}^2-\alpha_3{}^2}} = \frac{(2m)\,!\,(2n)\,!\,(m+n)\,!}{(2m+2n+1)\,!\;m\,!\;n\,!},$$

λ_s may easily be obtained in the form of an expansion in powers of e_2 and e_3.

Performing the expansion in the usual manner, and putting

$$p_2 = e_2{}^2 + e_3{}^2,$$
$$p_4 = e_2{}^4 + \tfrac{2}{3}\,e_2{}^2\,e_3{}^2 + e_3{}^4,$$
$$p_6 = e_2{}^6 + \tfrac{3}{5}\,e_2{}^4\,e_3{}^2 + \tfrac{3}{5}\,e_2{}^2\,e_3{}^4 + e_3{}^6,$$
$$p_8 = e_2{}^8 + \tfrac{4}{7}\,e_2{}^6\,e_3{}^2 + \tfrac{18}{35}\,e_2{}^4\,e_3{}^4 + \tfrac{4}{7}\,e_2{}^2\,e_3{}^6 + e_3{}^8,$$

we obtain the following series:

$$\left.\begin{aligned}
\lambda_{-1} &= [(1-e_2^2)(1-e_3^2)]^{-\frac{1}{4}}\ \ [1 - \tfrac{1}{4}\,p_2 - \tfrac{3}{160}\,p_4 - \tfrac{5}{896}\,p_6 - \tfrac{5}{2048}\,p_8 \cdots]\\
\lambda_0 &= [(1-e_2^2)(1-e_3^2)]^{-\frac{1}{6}}\ \ [1 - \tfrac{1}{6}\,p_2 - \tfrac{1}{40}\,p_4 - \tfrac{1}{112}\,p_6 - \tfrac{5}{1152}\,p_8 \cdots]\\
\lambda_1 &= [(1-e_2^2)(1-e_3^2)]^{-\frac{1}{12}}\, [1 - \tfrac{1}{12}\,p_2 - \tfrac{3}{160}\,p_4 - \tfrac{1}{128}\,p_6 - \tfrac{77}{18432}\,p_8 \cdots]\\
\lambda_2 &= 1\\
\lambda_3 &= [(1-e_2^2)(1-e_3^2)]^{\frac{1}{12}}\ \ [1 + \tfrac{1}{12}\,p_2 + \tfrac{1}{32}\,p_4 + \tfrac{15}{896}\,p_6 + \tfrac{65}{6144}\,p_8 \cdots]\\
\lambda_4 &= [(1-e_2^2)(1-e_3^2)]^{\frac{1}{6}}\ \ \ [1 + \tfrac{1}{6}\,p_2 + \tfrac{3}{40}\,p_4 + \tfrac{5}{112}\,p_6 + \tfrac{35}{1152}\,p_8 \cdots]
\end{aligned}\right\}\ \ldots(13).$$

Since the exponent to the first factor and the coefficient of p_2 in the expansions are equal, the important fact comes out, that the deviation of λ_s from unity depends only on the fourth and higher powers of the eccentricities. Thus, expanding the first factor and multiplying, we find, putting

$$P_4 = e_2^4 - e_2^2\, e_3^2 + e_3^4,$$

the following expressions for the functions λ_s:

$$\left.\begin{aligned}
\lambda_{-1} &= 1 + \tfrac{27}{360}\,P_4 + \text{terms of higher order}\\
\lambda_0 &= 1 + \tfrac{16}{360}\,P_4 + \qquad ,, \qquad\quad ,,\\
\lambda_1 &= 1 + \tfrac{7}{360}\,P_4 + \qquad ,, \qquad\quad ,,\\
\lambda_2 &= 1\\
\lambda_3 &= 1 - \tfrac{5}{360}\,P_4 - \qquad ,, \qquad\quad ,,\\
\lambda_4 &= 1 - \tfrac{8}{360}\,P_4 - \qquad ,, \qquad\quad ,,
\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots(14).$$

It must be observed, however, that the series in (14) generally do not converge quite so rapidly as the series in (13).

8. In order to get an idea of the order of magnitude of the coefficients λ_s I have computed their values for the two special cases where $e_2 = e_3 = e$, and $e_2 = e$, $e_3 = 0$.

We have clearly, making the substitutions

$$\alpha_2 = \sin v \sin w,\ \ \alpha_3 = \cos v \sin w,$$

and

$$\alpha_2 = \cos w,\ \ \alpha_3 = \cos v \sin w,$$

respectively, the following two expressions for (12):

$$\lambda_s = [(1-e_2^2)(1-e_3^2)]^{\frac{s-2}{12}}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} dv\,dw \sin w\,[1 - e_2^2 \sin^2 v \sin^2 w - e_3^2 \cos^2 v \sin^2 w]^{\frac{2-s}{4}} \ \ldots\ldots\ldots(15),$$

$$\lambda_s = [(1-e_2^2)(1-e_3^2)]^{\frac{s-2}{12}}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} dv\,dw \sin w\,[1 - e_2^2 \cos^2 w - e_3^2 \cos^2 v \sin^2 w]^{\frac{2-s}{4}} \ \ldots\ldots\ldots(15^*).$$

Hence we have, for the case $e_2 = e_3 = e$,

$$\lambda_s = (1-e^2)^{\frac{s-2}{6}} \int_0^{\frac{\pi}{2}} dw \sin w\,[1 - e^2 \sin^2 w]^{\frac{2-s}{4}} \ \ldots\ldots\ldots\ldots\ldots(16),$$

and, for the case $e_2 = e$, $e_3 = 0$,

$$\lambda_s = (1 - e^2)^{\frac{s-2}{12}} \int_0^{\frac{\pi}{2}} dw \sin w \left[1 - e^2 \cos^2 w\right]^{\frac{2-s}{4}} \quad \text{............(16*)}.$$

Case I. Prolate ellipsoids of revolution.

From (16) we get, putting

$$1 - e^2 \sin^2 w = \frac{1 - e^2}{\cos^4 \phi},$$

$$\lambda_s = (1 - e^2)^{\frac{8-s}{12}} \cdot \frac{\sqrt{2}}{e} \int_0^{\arccos \sqrt[4]{1-e^2}} \frac{d\phi}{\cos^{5-s} \phi \sqrt{1 - \frac{1}{2} \sin^2 \phi}} = (1 - e^2)^{\frac{8-s}{12}} \cdot \frac{\sqrt{2}}{e} y_{\frac{5-s}{2}}$$
$$\text{............(17)}.$$

When s is an even number, the integral reduces to an elementary form. We get after some reduction,

$$\left.\begin{aligned} \lambda_0 &= (1 - e^2)^{-\frac{1}{3}} \frac{1}{2} \left[1 + \frac{1 - e^2}{2e} \ln \frac{1 + e}{1 - e}\right] \\ \lambda_2 &= 1 \\ \lambda_4 &= (1 - e^2)^{\frac{1}{3}} \frac{1}{2e} \ln \frac{1 + e}{1 - e} \end{aligned}\right\} \quad \text{.... .(17 bis)}.$$

When s is an odd number the integral may be computed from a table of elliptic integrals. In Jahnke und Emde: *Funktionentafeln*, Teubner, 1909, the normal elliptic integrals

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad E(k, \phi) = \int_0^{\phi} d\phi \sqrt{1 - k^2 \sin^2 \phi}$$

are tabulated to four decimal places. Putting

$$F = F(\sqrt{\tfrac{1}{2}}, \arccos \sqrt[4]{1 - e^2}), \quad E = E(\sqrt{\tfrac{1}{2}}, \arccos \sqrt[4]{1 - e^2}),$$

and reducing the integral y_r in the usual way (compare Serret-Scheffer: *Lehrbuch der Differential- und Integralrechnung*, Vol. II. Teubner, 1911), I find the following relations:

$$y_0 = F, \quad y_{-1} = 2E - F,$$

$$y_1 = \frac{2e}{\sqrt{2}} (1 - e^2)^{-\frac{1}{4}} - y_{-1},$$

$$y_2 = \frac{1}{3} \left(\frac{2e}{\sqrt{2}} (1 - e^2)^{-\frac{3}{4}} + y_0\right),$$

$$y_3 = \frac{1}{5} \left(\frac{2e}{\sqrt{2}} (1 - e^2)^{-\frac{5}{4}} + 3y_1\right).$$

From these relations y_1, y_2 and y_3 are readily computed, and thus also λ_{-1}, λ_1 and λ_3.

The following table gives the values of λ_{-1}, λ_0, λ_1, λ_2, λ_3 and λ_4 to three decimal places.

TABLE I.

Values of the Coefficients λ_s to Three Decimal Places for $e_2 = e_3 = e$.

e	λ_{-1}	λ_0	λ_1	λ_2	λ_3	λ_4
0	1	1	1	1	1	1
0·1	1·000	1·000	1·000	1	1·000	1·000
0·2	1·000	1·000	1·000	1	1·000	1·000
0·3	1·001	1·000	1·000	1	1·000	1·000
0·4	1·002	1·001	1·001	1	1·000	0·999
0·5	1·007	1·004	1·002	1	0·999	0·998
0·6	1·016	1·009	1·004	1	0·997	0·996
0·7	1·036	1·021	1·009	1	0·994	0·990
0·8	1·067	1·050	1·021	1	0·985	0·977
0·9	1·249	1·140	1·058	1	0·962	0·941
0·95	1·537	1·291	1·118	1	0·926	0·888
0·99	2·967	1·943	1·345	1	0·811	0·720

Case II. Oblate ellipsoids of revolution.

From (16*) we get, putting

$$1 - e^2 \cos^2 w = \cos^4 w,$$

$$\lambda_s = (1 - e^2)^{\frac{s-2}{12}} \frac{\sqrt{2}}{e} \int_0^{\text{arc cos} \sqrt[4]{1-e^2}} \frac{\cos^{5-s}\phi \, d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} = (1 - e^2)^{\frac{s-2}{12}} \frac{\sqrt{2}}{e} y'_{\frac{5-s}{2}} \quad \ldots(18).$$

When s is an even number the integral becomes, after some reduction, expressible in terms of the *arc sin* functions. I find

$$\left.\begin{aligned} \lambda_0 &= (1 - e^2)^{\frac{1}{3}} \frac{1}{2}\left[1 + \frac{1}{e\sqrt{1-e^2}} \text{arc sin}\, e\right] \\ \lambda_2 &= 1 \\ \lambda_4 &= (1 - e^2)^{\frac{1}{6}} \frac{1}{e} \text{arc sin}\, e \end{aligned}\right\} \quad .(18 \text{ bis}).$$

For odd values of s the integral may, in a manner similar to that used in *Case I*, be reduced to the two normal elliptic integrals F and E. I have found,

$$y_0' = F,$$

$$y_1' = 2E - F,$$

$$y_2' = \frac{1}{3}\left(\frac{2e}{\sqrt{2}}(1 - e^2)^{\frac{1}{4}} + y_0'\right),$$

$$y_3' = \frac{1}{5}\left(\frac{2e}{\sqrt{2}}(1 - e^2)^{\frac{3}{4}} + 3y_1'\right).$$

Taking E and F from the tables of Jahnke und Emde we compute y_1', y_2' and y_3', and thus also λ_{-1}, λ_1 and λ_3. The following table gives the values of λ_{-1}, λ_0, λ_1, λ_2, λ_3 and λ_4 to three decimal places.

TABLE II.

Values of the Coefficients λ_s to Three Decimal Places for $e_2 = e$, $e_3 = 0$.

e	λ_{-1}	λ_0	λ_1	λ_2	λ_3	λ_4
0	1	1	1	1	1	1
0·1	1·000	1·000	1·000	1	1·000	1·000
0·2	1·000	1·000	1·000	1	1·000	1·000
0·3	1·001	1·000	1·000	1	1·000	1·000
0·4	1·002	1·001	1·001	1	1·000	0·999
0·5	1·006	1·004	1·002	1	0·998	0·998
0·6	1·014	1·009	1·004	1	0·997	0·996
0·7	1·032	1·019	1·009	1	0·994	0·990
0·8	1·072	1·043	1·019	1	0·986	0·978
0·9	1·182	1·108	1·047	1	0·965	0·943
0·95	1·345	1·203	1·088	1	0·936	0·895
0·99	1·932	1·523	1·219	1	0·847	0·752

9. From equation I it follows, by multiplying both sides by x^s and integrating over all values of x from 0 to R (compare § 5 of the previous paper), that we have

$$M'(x^s) = k_s M'(r^s) \qquad \text{(19)},$$

where k_s is a numerical factor given by the integral

$$k_s = \int_0^1 \frac{\xi^{s+1}\, d\xi}{\sqrt{1-\xi^2}} = \begin{cases} \dfrac{(s+1)!}{\left(\dfrac{s+1}{2}\right)!^2} \dfrac{\pi}{2^{s+2}} & \text{when } s \text{ is odd} \\[2ex] \dfrac{\left(\dfrac{s}{2}\right)!^2\, 2^s}{(s+1)!} & \text{when } s \text{ is even} \end{cases} \qquad \text{(20)}.$$

Hence, according to (11 bis), we have the relation

$$h_0 M'(x^s) = k_s \lambda_s M(\rho^{s+1}) \qquad \text{(21)}.$$

Now $M'(x^s)$ is the so-called sth moment (about the origin) of the "diameters" observed in the section. In my previous paper I denoted it by the symbol v_s'. Similarly $M(\rho^t)$ is the tth moment of the true "diameters" ρ. In my previous paper it was denoted by the symbol N_t'. And it was shown in § 5 of this previous paper that, when the corpuscles are spherical, we must have

$$N_1' v_s' = k_s N_{s+1}' \qquad \text{(22)}.$$

It is now seen, using the same notation here, that when the corpuscles are ellipsoids of constant form we have, instead of (22),

$$h_0 v_s' = k_s \lambda_s N_{s+1}' \qquad \text{(23)}.$$

Putting here $s = 0$ we get

$$h_0 = \lambda_0 N_1',$$

and we have, finally,

$$N_1' v_s' = k_s N_{s+1}' \frac{\lambda_s}{\lambda_0} \qquad \text{(24)}.$$

As we have $N_0' = 1$ we have, in particular,

$$N_1' = \frac{k_{-1}\lambda_{-1}}{v_{-1}'\lambda_0} \quad\text{.................................(25)},$$

that is, N_1' is obtained from the harmonic mean of x. N_1' being found, any moment of the true distribution may be determined from the moments of the observed distribution by the aid of (20), (24) and (13), provided, of course, that the eccentricities are known.

We may now write $$N_{s+1}' = \frac{k_{-1}}{k_s}\,\frac{v_s'}{v_{-1}'}\,\frac{\lambda_{-1}}{\lambda_s}. \quad\text{............................(26).}$$

Denoting by N'' the moments of the distribution obtained, when any one of the methods of reduction, discussed in the previous paper (i.e. what may be called a "spherical reduction"), is applied to the observed section "diameters," we have from (22) the relation

$$N_{s+1}'' = \frac{k_{-1}}{k_s}\,\frac{v_s'}{v_{-1}} \quad\text{...................... ....(27),}$$

and hence from (26), $$N_s' = N_s''.\mu_s, \text{ where } \mu_s = \frac{\lambda_{-1}}{\lambda_{s-1}} \quad\text{......................(28).}$$

From the tables in § 8 we can now obtain the following tables for the coefficients μ_s for cases of ellipsoids of revolution.

These tables illustrate very well the fact that the eccentric form has no appreciable influence on the first five moments about the origin, when the eccentricity does not exceed 0·7, say. Although it is not, of course, mathematically stringent to draw the conclusion that if the first five moments of the true distribution and the "spherically reduced" observed distribution are practically identical, the two distributions are also identical, yet I think that—considering the nature of the problem under discussion—this conclusion may be drawn here without serious risk of error.

TABLE III.

Values of the Coefficients $\mu_s = \dfrac{\lambda_{-1}}{\lambda_{s-1}}$ *to Three Decimal Places for* $e_2 = e_3 = e$.

e	μ_1	μ_2	μ_3	μ_4	μ_5
0	1	1	1	1	1
0·1	1·000	1·000	1·000	1·000	1·000
0·2	1·000	1·000	1·000	1·000	1·000
0·3	1·001	1·001	1·001	1·001	1·001
0·4	1·001	1·001	1·002	1·002	1·003
0·5	1·003	1·005	1·007	1·008	1·009
0·6	1·007	1·012	1·016	1·019	1·020
0·7	1 015	1·027	1·036	1·042	1·046
0·8	1·035	1·045	1·067	1·084	1·092
0·9	1·096	1·180	1·249	1·298	1·327
0·95	1·191	1·375	1·537	1·660	1·731
0·99	1·532	2·213	2·967	3·669	4·133

TABLE IV.

Values of the Coefficients $\mu_s = \dfrac{\lambda_{-1}}{\lambda_{s-1}}$ *to Three Decimal Places for* $e_2 = e$, $e_3 = 0$.

e	μ_1	μ_2	μ_3	μ_4	μ_5
0	1	1	1	1	1
0·1	1·000	1·000	1·000	1·000	1·000
0·2	1·000	1·000	1·000	1·000	1·000
0·3	1·001	1·001	1·001	1·001	1.001
0·4	1·001	1·001	1·002	1·002	1·003
0·5	1·002	1·004	1·006	1·007	1 008
0·6	1·006	1·010	1·014	1·017	1·019
0·7	1·013	1·024	1·032	1·038	1·042
0·8	1·028	1·052	1·072	1·087	1·097
0·9	1·067	1·128	1·182	1·225	1·253
0·95	1·118	1·234	1·345	1·437	1·503
0·99	1·269	1·585	1·932	2·281	2·571

In order to get a standard of comparison for judging the effect of the factors μ_s, that is the factors by which we must multiply the "spherical moments" in order to obtain the actual moments of the true "diameters" ρ, I proceed in the following way.

It is well known that the standard deviations of the moments about the origin are obtained from the following formula:

$$\Sigma^2 (N_s') = \frac{N_{2s}' - N_s'^2}{n}$$

where $\Sigma (N_s')$ is the standard deviation of N_s' in samples of n, and the moments are supposed to be determined in the usual way, as the means of the corresponding powers of the variates. Assuming now that the distribution is of the special type xe^{-x^2}, considered in the previous paper, § 6, the ratio $N_{2s}'/N_s'^2$ will be a given numerical constant, and we find

$$\frac{\Sigma (N_1')}{N_1'} = \sqrt{\frac{0{\cdot}27}{n}}; \quad \frac{\Sigma (N_2')}{N_2'} = \sqrt{\frac{1}{n}}; \quad \frac{\Sigma (N_3')}{N_3'} = \sqrt{\frac{2{\cdot}40}{n}};$$

$$\frac{\Sigma (N_4')}{N_4'} = \sqrt{\frac{5}{n}}; \quad \frac{\Sigma (N_5')}{N_5'} = \sqrt{\frac{9{\cdot}83}{n}}.$$

Table V, p. 154, shows the rates by which the moments are affected by the standard deviations in samples of $n = 4000$, 1000 and 250.

Comparing Table V with Tables III and IV we see that, when reducing the observations by the spherical method, we obtain a distribution with moments which, when the corpuscles are ellipsoidal with the eccentricities as great as 0·7, or so, although differing systematically, will not differ from the moments of the actual distribution by amounts greater than the standard deviation to be expected in samples of 4000.

TABLE V.

n	$1+\frac{\Sigma(N_1')}{N_1'}$	$1+\frac{\Sigma(N_2')}{N_2'}$	$1+\frac{\Sigma(N_3')}{N_3'}$	$1+\frac{\Sigma(N_4')}{N_4'}$	$1+\frac{\Sigma(N_5')}{N_5'}$
4000	1·01	1·02	1·03	1·05	1·05
1000	1·02	1·03	1·05	1·07	1·10
250	1·03	1·06	1·10	1·14	1·20

But even when the eccentricities are greater, if not extremely so, the spherical reduction of the section observations will give a fair approximation to the actual distribution of the "diameters" ρ. For eccentricities of the magnitude 0·8 the approximation may be compared to the errors of sampling in a frequency curve, determined by the method of moments from 1000 observations. When the eccentricities are of the order of magnitude of 0·9 the accuracy may be compared to that of a curve determined from 250 observations.

The above comparison with the errors of sampling only purposes to give a general idea of the order of magnitude of the perturbations caused by the eccentricities. Contrary to the errors of sampling the eccentricities cause systematic deviations and there is, of course, only a small chance of the errors of sampling affecting all the moments in the same direction and by amounts equal to the standard deviations.

10. It has so far been assumed that the corpuscles are ellipsoidal and of constant form. It will be seen, however, that the results obtained in equations (23) to (28) are still valid, if the eccentricities vary, provided the forms of the corpuscles are independent of ρ, that is of their volume. The only difference in such a case is, as seen from (11), that the coefficients λ_s are now equal to the mean value of the λ's determined by equation (12) for each separate combination of eccentricities occurring. Thus in expansions of the form (14) the different powers and product powers of the eccentricities should simply be replaced by the corresponding moments and product moments of e_2 and e_3.

I think it will be practicable to distinguish two different cases of varying eccentricity. The first case to be considered will be one of corpuscles having what may be called *a spherical type*. By this I mean the case where the corpuscles are generally of a spherical or only slightly eccentric form, more eccentric forms only occurring as more or less rare exceptions. In such a case our previous results clearly imply that the observations may safely be treated by the spherical methods of reduction. Because in such a case the coefficients λ_s must be approximately equal to unity, as the major part of the individual λ's, defined for each corpuscle by the equation (12), of which the total λ's are the mean values, are then nearly equal to unity. The greater values of the individual λ's of the exceptional, very eccentric corpuscles, will not be sufficient seriously to disturb this tendency.

The second case will be one of what may be called *an eccentric type*. By this I mean a case where the bulk of the corpuscles are ellipsoidal with a considerable

eccentricity (upwards of or above 0·7) and where deviations from this type are the exceptions. Such cases may be rare in anatomical practice, but in other fields they are probably rather the rule. In studying different granulary minerals the usual method is to grind out a thin plane layer from the mineral, and the problem is, exactly as in the anatomical case, to draw conclusions as to the structure of the material from observations performed in a magnification of this sample. In several minerals, however, the corpuscles, or grains, are of more or less constant eccentric form.

But as the form which dominates in the materials mentioned is generally that of an ellipsoid of revolution, mostly of prolate form, I shall show in the following section, by a very simple device, how the problem may always be treated by the spherical method, in the case where the corpuscles are prolate (or oblate) ellipsoids of revolution.

11. As stated in § 2 the fundamental equation I is also valid if we let x and r denote, for example, any two corresponding principal diameters of the section and central ellipse respectively. If the corpuscles are all prolate ellipsoids of revolution it is evident that the minor diameter σ of the central ellipse will always be equal to the diameter s at right angles to the axis of revolution of the corpuscle. Thus if we observe the minor diameters, ξ, of the section ellipses we have the relation

$$\phi(\xi) = \xi \int_{\xi}^{R} \frac{f(s)\,ds}{s\sqrt{s^2 - \xi^2}},$$

where $f(s)$ is the frequency function of the diameter at right angles to the axis of revolution for the corpuscles contained in the section. The function $\phi(\xi)$, of course, denotes the frequency function of the observed minor diameters in the corresponding section ellipses. Furthermore, we have, as in the general case, the relation II, which here takes the form

$$f(s) = F(s)\frac{h_s}{h_0}.$$

Here $F(s)$ is the frequency function of s for the corpuscles in general and h_s is the regression of the "height" h on s. And h_0 is, as before, the general mean of h. But, denoting by v the angle between the normal to the section plane and the axis of revolution, we have

$$\sqrt{\frac{1 - e^2 \sin^2 v}{1 - e^2}} \quad \text{.............................(29).}$$

Now, as the factor s is not a function of the angle v (which would have been the case if we had used the geometric mean of the principal diameters or, indeed, any diameter but the minor diameter as measure), and assuming as before that the direction parameter v varies independently of the "size" s (and, if variable, also that the eccentricity e varies independently of s) it follows from this equation that we have

$$\frac{h_s}{h_0} = \frac{s}{s_0},$$

and thus that
$$F(s) = \frac{s_0}{s} f(s).$$

It is thus seen that we have exactly the same relations between the functions F and ϕ as in the case of spherical corpuscles. Hence, applying the reductions deduced for the case of spherical corpuscles to the distribution of the *minor* diameters in the section ellipses, we obtain the distribution of the *minor* diameters of the corpuscles, if they are all prolate ellipsoids of revolution. This is completely independent of the way in which the directions of the axes of revolution vary, and also of the way in which the eccentricities may vary, provided only that both are uncorrelated with the minor diameter.

It is evident that the same result will hold if the corpuscles are all *oblate* ellipsoids of revolution, if we then apply the spherical reductions to the distribution of the *major* diameters in the section ellipses. The result obtained will be the distribution of the *major* diameters of the corpuscles.

But there is one respect in which an assumption as to the way in which the directions and eccentricities vary must be made. This is when it is required to find the number N of corpuscles, per unit volume in the material investigated, from the number n of section ellipses, per unit area in the section. Clearly we have

$$h_0 N = n.$$

But from (29) we have

$$h_0 = s_0 M\left(\sqrt{\frac{1 - e^2 \sin^2 v}{1 - e^2}}\right) = s_0 k_0,$$

if the corpuscles are prolate, and s_0 = mean of minor diameter,

and $$h_0 = s_0 M\left(\sqrt{1 - e^2 \cos^2 v}\right)$$

if the corpuscles are oblate, and s_0 = mean of major diameter.

Here s_0 is determined in exactly the same manner as r_0 in the spherical case (see for instance the numerical examples in § 10 of previous paper).

If the directions are haphazard the probability that v falls in the interval v to $v + dv$ is equal to $\sin v\, dv$. Thus, as

$$\int_0^{\frac{\pi}{2}} dv \sin v \sqrt{1 - e^2 \sin^2 v} = \frac{1}{2}\left(1 + \frac{1 - e^2}{2e} \ln \frac{1 + e}{1 - e}\right);$$

$$\int_0^{\frac{\pi}{2}} dv \sin v \sqrt{1 - e^2 \cos^2 v} = \frac{1}{2}\left(\sqrt{1 - e^2} + \frac{1}{e} \arcsin e\right),$$

we have, in the case of constant eccentricity,

$$k_0 = \frac{1}{2}\left(\frac{1}{\sqrt{1 - e^2}} + \frac{\sqrt{1 - e^2}}{2e} \ln \frac{1 + e}{1 - e}\right); \quad k_0' = \frac{1}{2}\left(\sqrt{1 - e^2} + \frac{1}{e} \arcsin e\right),$$

and, if e varies independently of s, the mean values of those functions of e should be taken.

Finally, if it be required to find, from the distribution $F(s)$ of the diameter s at right angles to the axis of revolution, the distribution $P(\rho)$ of the geometric mean diameter ρ, we have, in the case where the corpuscles are all prolate,

$$\rho = s \,.\, (1 - e^2)^{-\frac{1}{6}},$$

and in the case where the corpuscles are all oblate,

$$\rho = s \, . \, (1 - e^2)^{\frac{1}{6}}.$$

Thus, if the eccentricity does not vary, we have only to change the scale of the abscissa in order to pass over from $F(s)$ to $P(\rho)$. In the case where the eccentricity varies (independently of s, as we have already been compelled to assume) the distribution of ρ may be obtained from the distribution of s, by the aid of the following relations between the moments of ρ and of s:

For prolate corpuscles:

$$M(\rho^t) = M(s^t) \, . \, M((1 - e^2)^{-\frac{t}{6}}).$$

For oblate corpuscles:

$$M(\rho^t) = M(s^t) \, . \, M((1 - e^2)^{\frac{t}{6}}).$$

If it be not justifiable to assume the eccentricity to vary independently of s, but it be found more natural to assume it independent of ρ (that is of the volume of the corpuscle), then of course the solution is obtained from the solution given for general tri-axial ellipsoids, by putting $e_2 = e_3 = e$, $e_2 = e$, $e_3 = 0$, respectively. In such cases, however, it will be necessary to know, at least approximately, the distribution of the eccentricity. Having determined this distribution, the mean values of the coefficients λ_s will be readily determined by the aid of Tables I and II.

I have therefore, in the next section, tried to give a method of determining, for the case of ellipsoids of revolution, the distribution of the actual eccentricities of the corpuscles from the distribution of the observed eccentricities of the section ellipses.

12. Supposing the corpuscles to be *prolate* ellipsoids of revolution with a variable eccentricity e of the generating ellipse, it is easily shown that the eccentricity ϵ of the central ellipse is given by the equation

$$\epsilon = e \sin v \qquad \text{.................................(30)},$$

v being, as before, the angle between the axis of revolution of the corpuscle and the normal to the section.

Since in an haphazard arrangement of the directions of the axes of revolution, $\sin v \, dv$ is the probability that a given corpuscle has the angle v in the interval v to $v + dv$, the expression

$$\sin v \frac{dv}{d\epsilon} d\epsilon = \frac{\epsilon \, d\epsilon}{e \sqrt{e^2 - \epsilon^2}},$$

gives the probability that, e being known, ϵ falls in the interval ϵ to $\epsilon + d\epsilon$. Hence, if $\pi(\epsilon)$ is the frequency function of ϵ and $F(e)$ the frequency function of e, we must have

$$\pi(\epsilon) = \epsilon \int_\epsilon^1 \frac{F(e) \, de}{e \sqrt{e^2 - \epsilon^2}} \qquad \text{...........................(31)}.$$

But, although the eccentricity in the section ellipse is always the same as in the central ellipse, the function $\pi(\epsilon)$, which applies to the corpuscles in general,

is not identical to the frequency function $\psi(\epsilon)$ of the eccentricities of the observed section ellipses. The reason for this is, of course, that the section contains corpuscles which are, in a certain way, selected with regard to the eccentricity ϵ.

We have $$h = S\sqrt{1 - e^2 \sin^2 v} = S\sqrt{1-\epsilon^2},$$
if S is the major diameter of the corpuscle. Clearly,
$$n\psi(\epsilon) = N\pi(\epsilon)\,h_\epsilon,$$
where h_ϵ is the regression of h on ϵ. But we have, as before, $n = Nh_0$, h_0 being the total mean of h. Thus we must have
$$\psi(\epsilon) = \frac{h_\epsilon}{h_0}\,\pi(e) \qquad\text{(32)}.$$

We shall now regard three different cases: (1) that e varies independently of the major diameter S, (2) that it varies independently of the minor diameter s, and (3) that it varies independently of the volume, i.e. of the "diameter" ρ. At the same time we shall always suppose that the angle v is independent of e and ρ. It will then also be independent of S and s. Thus the three assumptions considered also imply that ϵ is independent of S, s and ρ, respectively.

Case 1. We here have
$$h_\epsilon = S_0\sqrt{1-\epsilon^2},$$
S_0 being the mean of S. Furthermore
$$h_0 = S_0\,.\,M(\sqrt{1-e^2}) = S_0\delta_0.$$

We consequently get the relation
$$\psi(\epsilon) = \frac{1}{\delta_0}\sqrt{1-\epsilon^2}\,.\,\pi(\epsilon),$$
or
$$\psi(\epsilon) = \frac{1}{\delta_0}\,\epsilon\sqrt{1-\epsilon^2}\int_\epsilon^1 \frac{F(e)\,de}{e\sqrt{e^2-\epsilon^2}} \qquad\text{(33)}.$$

Case 2. We here have
$$h = s(1-e^2)^{-\frac{1}{2}}\sqrt{1-\epsilon^2}.$$

But the correlation function of e and ϵ is clearly given by
$$\frac{\epsilon F(e)}{e\sqrt{e^2-\epsilon^2}}.$$

Hence we find
$$\pi(\epsilon)\,h_\epsilon = s_0\sqrt{1-\epsilon^2}\,.\,\epsilon\int_\epsilon^1 \frac{F(e)(1-e^2)^{-\frac{1}{2}}\,de}{e\sqrt{e^2-\epsilon^2}}.$$

As furthermore we now have
$$h_0 = s_0 M((1-e^2)^{-\frac{1}{2}}\sqrt{1-\epsilon^2}) = s_0\delta_0',$$
we get by the aid of (32):
$$\psi(\epsilon) = \frac{1}{\delta_0'}\,\epsilon\sqrt{1-\epsilon^2}\int_\epsilon^1 \frac{F(e)\,(1-e^2)^{-\frac{1}{2}}\,de}{e\sqrt{e^2-\epsilon^2}}.$$

Case 3. As in this case
$$h = \rho(1-e^2)^{-\frac{1}{3}}\sqrt{1-\epsilon^2},$$

we have the following integral equation:

$$\psi(\epsilon) = \frac{1}{\delta_0''}\,\epsilon\sqrt{1-\epsilon^2}\int_{\epsilon}^{1}\frac{F(e)(1-e^2)^{-\frac{1}{2}}\,de}{e\sqrt{e^2-\epsilon^2}}.$$

Here the constant δ_0'', which, as well as the constants δ_0 and δ_0', is incidentally of no importance as being only a proportional factor, is defined by the relation

$$\delta_0'' = M\left[(1-e^2)^{-\frac{1}{2}}\sqrt{1-e^2}\right].$$

By the so-called "spherical reduction," for which several methods of procedure were discussed in my previous paper, we actually solve an integral equation of the following form:

$$\psi(x) = kx\int_{x}^{R}\frac{F(r)\,dr}{\sqrt{r^2-x^2}}.$$

We thus arrive at the following remarkable rule for determining the distribution of the eccentricity in the case where the corpuscles may be regarded as prolate ellipsoids of revolution: *Dividing the series of ordinates $\psi(\epsilon)$ of the observed distribution of the eccentricities, ϵ, in the section ellipses, by $\sqrt{1-\epsilon^2}$, and applying the "spherical reduction" to the resulting series (for instance, by inserting it in equations (18) of the previous paper), we get a series of reduced "frequencies," which, if multiplied by e, $e(1-e^2)^{\frac{1}{2}}$ and $e(1-e^2)^{\frac{1}{3}}$, respectively, are proportional to the ordinates of the frequency function of e, according as e may be assumed to vary independently of the major diameters, the minor diameters, or the volumes of the corpuscles.*

If the corpuscles are *oblate* ellipsoids of revolution, we get instead, putting

$$\epsilon' = \frac{\epsilon}{\sqrt{1-\epsilon^2}}, \quad e' = \frac{e}{\sqrt{1-e^2}},$$

$$\epsilon' = e'\sin v, \quad h = s\sqrt{1+\epsilon'^2}.$$

Hence, in a similar manner to the above, we here get the integral equation

$$\psi_1(\epsilon') = \frac{1}{\delta_0}\,\epsilon'\sqrt{1+\epsilon'^2}\int_{\epsilon'}^{\infty}\frac{F_1(e')(1+e'^2)^{a}\,de'}{e'\sqrt{e'^2-\epsilon'^2}},$$

where $a=0$ when e is independent of the minor diameter s, $a=-\frac{1}{2}$ when e is independent of the major diameter S, and $a=-\frac{1}{3}$ when e is independent of the volume of the corpuscles.

As the frequency functions $\psi(\epsilon)$ and $F(e)$ of ϵ and e are expressed in terms of the frequency functions $\psi_1(\epsilon')$ and $F_1(e')$ of ϵ' and e' by the simple relations

$$\psi(\epsilon) = \psi_1(\epsilon')(1+\epsilon'^2)^{\frac{3}{2}}; \quad F(e) = F_1(e')(1+e'^2)^{\frac{3}{2}},$$

we see that the procedure of determining the actual eccentricity distribution from the observed eccentricity distribution is as readily performed in the oblate case as in the prolate.

13. For a complete solution of the ellipsoidal corpuscle problem it now only remains to find the number N of corpuscles, per unit volume of the material investigated, from the number n of corpuscle images, per unit area of the section. As in the spherical case we may here use two different methods, both very simple.

First method. As we have

$$n = Nh_0,$$

and

$$h_0 = \rho_0 \lambda_0,$$

we readily get N from n, as ρ_0 is the mean of ρ, which may be determined either from the harmonic mean of the observed x's, or directly from the distribution of ρ, obtained by the reduction.

Second method. Quite independently of any assumption as to the form of the corpuscles we must always have the total volume V of the corpuscles, per unit volume of the material, equal to the total area of the corpuscle contours, per unit area of the section. Hence V may be regarded as known. But the volume of an ellipsoidal corpuscle is equal to

$$\frac{\pi}{6} s_1 s_2 s_3 = \frac{\pi}{6} \rho^3.$$

Thus we have

$$V = \frac{\pi}{6} N . N_3' \qquad \text{.................................(34).}$$

As N_3' is the mean of ρ^3 and may be determined either from the moments v_2' and v'_{-1} of the observed x's, or directly from the distribution of ρ, obtained by the reduction, N is readily found from V.

14. Before concluding this investigation I shall discuss a special problem, which was recently put before me by a civil engineer, Mr R. Wijkander, and which I consider to be of interest in this connection. Mr Wijkander was investigating a material called *perlite*, which consists of pure iron, with corpuscles of *cementite* (Fe_3C). At a certain temperature the corpuscles are dissolved in the iron. The rapidity of this process depends, in the first place, on the total surface of the corpuscles.

The problem to be considered is consequently the following: *To find the total surface of the corpuscles, per unit volume of the material investigated, from observations in a thin sample section.*

When the corpuscles are spherical it follows immediately from equations (11) of my previous paper, that the required relative surface is equal to four times the total length of the observed diameters, per unit area of the section. For the total surface of the corpuscles is $N . N_2'\pi$; as $N_2'\pi = 4N_1'v_1'$, and as $n = N . N_1'$, we have

$$N . N_2'\pi = 4nv_1' = 4X \qquad \text{.............................(35),}$$

where X is the sum of the observed diameters, per unit area of the section.

This result also follows immediately from the fact that the section diameter may be regarded as the "image" in the section of an equatorial plane through the corpuscle, taken at right angles to the section. Thus exactly the same relation holds between the sum of the areas of the great circles of the corpuscles and the sum of the lengths of the section diameters as was pointed out in § 13 with regard to the corpuscle *volumes* and the section *areas*. But the surface of a sphere is

equal to four times the area of its great circle, and thus the same result is arrived at by very simple considerations.

In the general ellipsoidal case the problem may be solved in the following way: The surface of an ellipsoid having the principal diameters s_1, s_2 and s_3 and the principal eccentricities

$$\sqrt{\frac{s_1^2 - s_2^2}{s_1^2}}, \quad e_3:$$

is known to be equal to

$$T = (s_1 s_2 s_3)^{\frac{2}{3}} \pi \,.\, \lambda (e_2, e_3) = \pi \rho^2 \lambda (e_2, e_3),$$

where $\lambda (e_2, e_3)$ is a function of the eccentricities, defined by the integral (compare Serret-Scheffer, *loc. cit.*, § 592),

$$\lambda (e_2, e_3) = [(1 - e_2^2)(1 - e_3^2)]^{\frac{1}{6}} \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\sqrt{1 + e_2^2 x^2 + e_3^2 y^2}}{\sqrt{1 - x^2 - y^2}} \, dx\, dy.$$

Comparing with (12) and (13) it is seen that we have the following expansion of $\lambda (e_2, e_3)$:

$$\lambda (e_2, e_3) = [(1 - e_2^2)(1 - e_3^2)]^{\frac{1}{6}} [(1 + \tfrac{1}{6} p_2 - \tfrac{1}{40} p_4 + \tfrac{1}{112} p_6 - \tfrac{5}{1152} p_8 \dots].$$

It may also, as shown by Legendre, be expressed in terms of normal elliptic integrals of the first and the second orders.

In the case of ellipsoids of revolution the function reduces to:

in the *prolate* case,

$$\lambda (e, e) = (1 - e^2)^{\frac{1}{3}} \frac{1}{2} \left[1 + \frac{1}{e\sqrt{1 - e^2}} \arcsin e \right]$$

= the λ_0 of formula (18 bis), tabulated in Table II,

and in the *oblate* case,

$$\lambda (e, 0) = (1 - e^2)^{-\frac{1}{6}} \frac{1}{2} \left[1 + \frac{1 - e^2}{2e} \ln \frac{1 + e}{1 - e} \right]$$

= the λ_0 of formula (17 bis), tabulated in Table I.

But the total surface, per unit volume of the material investigated, is

$$Y = NM (T) = N\pi N_2' \,.\, \lambda \quad \dots\dots\dots\dots\dots\dots\dots\dots (36),$$

where λ is the mean of the function $\lambda (e_2, e_3)$, and it is assumed that the forms of the corpuscles are independent of their volumes.

In the case of ellipsoids of revolution of constant form it may be shown, in the same way as in the case of spherical corpuscles, that we have

$$Y = 4X \,.\, p (e),$$

where X is the sum per unit area of the section, of the major and minor diameters respectively, of the section ellipses according as the form is prolate or oblate. In the former case we have,

$$p (e) = \frac{1}{2} \left[\sqrt{1 - e^2} + \frac{1}{e} \arcsin e \right],$$

in the latter $$p(e)=\frac{1}{2\sqrt{1-e^2}}\left[1+\frac{1-e^2}{2e}\cdot\ln\frac{1+e}{1-e}\right].$$

In many cases the relative surface required may be approximately determined when we only know n and V. As V in the *perlite* case and similar cases may be found from a chemical analysis of the material, this implies that the surface may be found without any observations at all in the section, except the number n of corpuscle contours per unit area. When the corpuscles are spherical the coefficient λ of equation (36) is equal to 1, and for the same reason as in the case of the functions λ_0, considered in §§ 7, 8 and 10, λ will not deviate much from unity, when the eccentricities are not on the average very great. From (34) and (36) we thus obtain the formula

$$N_2'\sqrt{\frac{6}{N_1'N_3'}}\sqrt{n.V}=K\sqrt{n.V}.$$

If we assume the diameter of the corpuscles to be distributed according to the type xe^{-x^2}, considered in § 6 of my previous paper, we get $K=4$. Only for rather special forms of the size distribution will K differ sensibly from this value.

A FURTHER NOTE ON THE DISTRIBUTION OF RANGE IN SAMPLES TAKEN FROM A NORMAL POPULATION.

BY EGON S. PEARSON, D.Sc.

TOWARDS the end of his paper on "The Extreme Individuals and the Range of Samples taken from a Normal Population*," Mr L. H. C. Tippett has considered a method of obtaining the moments of the range from the moments of the correlation surface of the two extreme individuals of a sample. If u and v are the largest and smallest values of the character found among the individuals of the sample, and w the range which is given by $w = u - v$, then he gives the following perfectly general relations for the moments of the distribution of w,

$$\left.\begin{aligned} \bar{w} &= \bar{u} - \bar{v} \\ {}_w\mu_2 &= 2\,{}_u\mu_2(1-r) \\ {}_w\mu_3 &= 2\,{}_u\mu_3 + 6p_{12} \\ {}_w\mu_4 &= 2\,{}_u\mu_4 - 8p_{13} + 6p_{22} \end{aligned}\right\} \quad \text{.............................(i)},$$

where ${}_u\mu_2$, ${}_u\mu_3$, ${}_u\mu_4$ are the moment coefficients of the u margin†, and r, p_{11}, p_{12}, p_{13} and p_{22} the correlation and cross-product moment coefficients of the u, v frequency surface. Tippett has shown that for samples of size 60 or more the coefficient of correlation between u and v is practically negligible, and on the assumption that the regression will then be approximately linear and the distribution homoscedastic, he makes use of the simple relations (ii) to obtain the moments of the w distribution:

$$\left.\begin{aligned} {}_w\mu_2 &= 2\,{}_u\mu_2 \\ {}_w\mu_3 &= 2\,{}_u\mu_3 \\ {}_w\mu_4 &= 2\,{}_u\mu_4 + 6\,{}_u\mu_2^2 \end{aligned}\right\} \quad \text{.............................(ii)}.$$

In small samples the correlation will not vanish nor will the regression be linear; for this reason and also if possible to test more rigorously the validity of the assumptions made in the case of large samples, it is of interest to consider the general expression for the u, v surface. This can be readily obtained by introducing a geometrical conception.

Suppose that $\phi_n(u, v)$ is the correlation surface for u and v in samples of n; as $u \geqslant v$ this surface must lie wholly to the left-hand side of the line $u = v$, or AOB (Fig. 1). If now samples of $n + 1$ are taken, the surface $\phi_n(U, V)$ will differ from $\phi_n(u, v)$ because the $n + 1$th individual, which we may consider as that last drawn, will

* *Biometrika*, Vol. XVII. p. 364.

† The distributions in the u and v margins are identical but reversed so that

$$\bar{u} = -\bar{v}, \quad {}_u\mu_2 = {}_v\mu_2, \quad {}_u\mu_3 = -{}_v\mu_3, \quad {}_u\mu_4 = {}_v\mu_4, \quad p_{12} = -p_{21}, \quad p_{13} = p_{31}.$$

sometimes be $> u$ and sometimes $< v$. That is to say in a sample of $n+1$ we have n individuals for which the highest and lowest values of the variate follow a frequency distribution $\phi_n(u, v)$ and a single individual (if we are sampling from a normal population with unit standard deviation) with distribution

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

with no correlation between x and u, or x and v. Of all possible samples of $n+1$, consider those only in which the $n+1$th individual lies between x and $x+dx$; three cases occur.

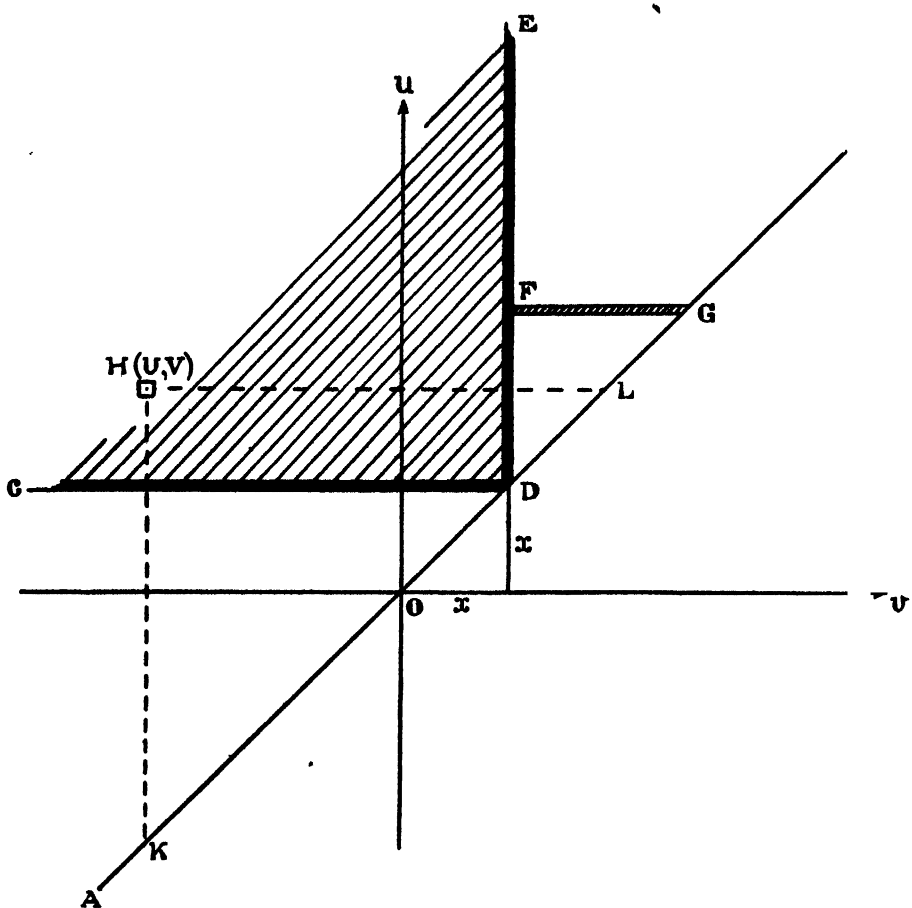


Fig. 1.

1. $u > x > v$, so that $U = u$, $V = v$. Here the frequency distribution of U and V will be as for u and v, or $\phi_{n+1}(U, V)$ will consist of superimposed parts of which one is the "slice" of $\phi_n(u, v)$ standing on the sector CDE which is shaded in the figure, D being the point (x, x).

2. $u > v > x$, so that $U = u$, $V = x$. In this case the contribution to $\phi_{n+1}(U, V)$ is a strip of frequency of width dx lying along the line DE, such that the frequency

standing on an element $dU\,dx$ at any point F equals the frequency of $\phi_n(u, v)$ standing on the strip FG of breadth dU; but this is

$$dU\int_x^U \phi_n(U, v)\,dv.$$

The "slice" of $\phi_n(u, v)$ lying above the sector EDB contributes in fact to $\phi_n(U, V)$ a narrow strip of the same volume lying along DE.

3. $x > u > v$, so that $U = x$, $V = v$. In this case the contribution to $\phi_{n+1}(U, V)$ is a strip of frequency of width dx lying along the line DC, giving at any point V a frequency on the element $dV\,dx$ of

$$dV\int_V^x \phi_n(u, V)\,du.$$

The "slice" over CDA becomes a strip along DC.

To obtain the complete frequency surface $\phi_{n+1}(U, V)$ we have to sum the three contributions corresponding to every value of x possible. Now x varies between $-\infty$ and $+\infty$ with a proportional frequency

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},$$

and as it varies D passes from one end to the other of the diagonal line AOB. Consider the contributions to a block $\phi_{n+1}(U, V)\,dU\,dV$ standing at H on a base $dU\,dV$.

Source 1. Will only contribute frequency when D, the point (x, x), lies between K and L. The total contribution is therefore

$$dU\,dV\int_V^U \phi_n(U, V)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx = \phi_n(U, V)\,dU\,dV\int_V^U \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx.$$

Source 2. Will only provide a contribution when x is such that the line DE passes through H, or when $x = V$; it will then contribute

$$\frac{e^{-\frac{1}{2}V^2}}{\sqrt{2\pi}}\,dV \times dU\int_V^U \phi_n(U, v)\,dv.$$

Source 3. Will similarly only provide a contribution when $x = U$, so that DC passes through H; then the contribution will be

$$\frac{e^{-\frac{1}{2}U^2}}{\sqrt{2\pi}}\,dU \times dV\int_V^U \phi_n(u, V)\,du.$$

Hence it follows that

$$\phi_{n+1}(U, V) = \phi_n(U, V)\int_V^U \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx + \frac{e^{-\frac{1}{2}V^2}}{\sqrt{2\pi}}\int_V^U \phi_n(U, v)\,dv + \frac{e^{-\frac{1}{2}U^2}}{\sqrt{2\pi}}\int_V^U \phi_n(u, V)\,du \quad \text{.....................(iii)}.$$

Equation (iii) provides a reduction formula for obtaining the correlation surface of the values of characters of highest and lowest individuals in samples of n in terms of the equations of surfaces for smaller samples. Take the simplest case

where $n=2$; the frequency surface for the two variates, if there is no restriction as to smaller or greater, is simply the normal surface

$$z=\frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} \quad\text{..............(iv).}$$

We obtain the frequency surface for u and v by doubling this surface back over the line $x=y$, and have

$$z=\phi_2(u,v)=\frac{1}{\pi}e^{-\frac{1}{2}(u^2+v^2)} \quad\text{..............(v)}$$

as the correlation surface for u and v. The frequency is cut off abruptly along the line $u=v$. Now substitute (v) into (iii), and write

$$A_u=\int_{-\infty}^{u}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx \quad\text{..............(vi),}$$

$$\phi_3(U,V)=\frac{e^{-\frac{1}{2}(U^2+V^2)}}{\pi}(A_U-A_V)+\frac{e^{-\frac{1}{2}V^2}}{\sqrt{2\pi}}\int_V^U\frac{e^{-\frac{1}{2}(U^2+v^2)}}{\pi}dv$$
$$+\frac{e^{-\frac{1}{2}U^2}}{\sqrt{2\pi}}\int_V^U\frac{e^{-\frac{1}{2}(u^2+V^2)}}{\pi}du=\frac{3}{\pi}e^{-\frac{1}{2}(U^2+V^2)}(A_U-A_V) \quad\text{...(vii).}$$

On substituting (vii) in (iii) it can be shown that

$$\phi_4(U,V)=\frac{6}{\pi}e^{-\frac{1}{2}(U^2+V^2)}(A_U-A_V)^2 \quad\text{..............(viii).}$$

What then is the general law? Assume

$$\phi_n(U,V)=c_n e^{-\frac{1}{2}(U^2+V^2)}(A_U-A_V)^{n-2} \quad\text{..............(ix).}$$

On substituting in (iii) it follows that

$$\phi_{n+1}(U,V)=c_n e^{-\frac{1}{2}(U^2+V^2)}\left\{(A_U-A_V)^{n-1}+\int_V^U\frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}(A_U-A_v)^{n-2}dv\right.$$
$$\left.+\int_V^U\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}(A_u-A_V)^{n-2}dv\right\}.$$

But
$$\frac{d}{dv}(A_U-A_v)^{n-1}=-(n-1)(A_U-A_v)^{n-2}\frac{dA_v}{dv}$$
$$=-(n-1)(A_U-A_v)^{n-2}\frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}\text{ by (vi),}$$

$$\phi_{n+1}(U,V)=c_n e^{-\frac{1}{2}(U^2+V^2)}\left\{(A_U-A_V)^{n-1}-\int_V^U\frac{d(A_U-A_v)^{n-1}}{dv}\frac{dv}{n-1}\right.$$
$$\left.+\int_V^U\frac{d(A_u-A_V)^{n-1}}{du}\frac{du}{n-1}\right\}$$
$$=c_n e^{-\frac{1}{2}(U^2+V^2)}(A_U-A_V)^{n-1}\left\{1+\frac{1}{n-1}+\frac{1}{n-1}\right\},$$

$$\phi_{n+1}(U,V)=c_n\frac{n+1}{n-1}e^{-\frac{1}{2}(U^2+V^2)}(A_U-A_V)^{n-1} \quad\text{..............(x).}$$

The particular cases (v), (vii) and (viii) can therefore be generalised into (ix) or (x), and it only remains to find c_n. But $c_3 = 3/\pi$. Hence

$$c_n = \frac{3}{\pi} \cdot \frac{4}{2} \cdot \frac{5}{3} \cdots\cdots \frac{n}{n-2} = \frac{n(n-1)}{2\pi}$$

The general correlation surface for u and v in samples of n is therefore finally*

$$z = \frac{n(n-1)}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} (A_u - A_v)^{n-2} \quad \text{....................(xi)},$$

where

$$A_t \quad \int_{-\infty}^{t} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \quad \text{..........................(vi) bis.}$$

The marginal distributions of this surface can be obtained as follows. The frequency for u between u and $u + du$ is

$$\begin{aligned} du \int_{-\infty}^{u} z\,dv &= du \frac{n(n-1)}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \int_{-\infty}^{u} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}} (A_u - A_v)^{n-2} dv \\ &= -du \frac{n(n-1)}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \int_{-\infty}^{u} (A_u - A_v)^{n-2} \frac{d(A_u - A_v)}{dv} dv \\ &= -du \frac{n(n-1)}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \left[\frac{(A_u - A_v)^{n-1}}{n-1}\right]_{A_v=0}^{A_v=A_u} \\ &= du \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} A_u^{n-1}. \end{aligned}$$

Or the frequency distribution of u, the largest individual, is

$$y = \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} A_u^{n-1} \quad \text{............................(xii).}$$

Similarly the frequency distribution of v is

$$y' = \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} (1 - A_v)^{n-1} \quad \text{......................(xiii).}$$

It will be noticed that (xii) satisfies the relation given by Tippett in his equation (1), viz.

$$\int_{-\infty}^{u} y\,dx = A_u^n.$$

To find the distribution of the range, w, from equations (i) we need the moments and product moment coefficients of (xi) and (xii) up to the 4th order. The expressions involved cannot in general be integrated, but it has been possible to obtain a complete solution for the cases $n = 2, 3, 4, 5$ and 6, with the help of the following integrals whose numerical values were obtained by quadrature:

* It will be found, as of course should be the case, that $\int_{-\infty}^{+\infty} \int_{v}^{\infty} z\,du\,dv = 1$.

(a) $\int_{-\infty}^{+\infty} e^{-u^2} A_u^2\, du = \cdot 538\ 9796.$

(b) $\int_{-\infty}^{+\infty} e^{-u^2} A_u^3\, du = \cdot 365\ 3560.$

(c) $\int_{-\infty}^{+\infty} e^{-u^2} A_u^4\, du = \cdot 265\ 4030.$

(d) $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} A_u^2\, du = 1\cdot 297\ 9316.$

(e) $\int_{-\infty}^{+\infty} e^{-\frac{3}{2}u^2} A_u^2\, du = \cdot 420\ 0004.$

(f) $\int_{-\infty}^{+\infty} e^{-\frac{3}{2}u^2} A_u^3\, du = \cdot 268\ 2000.$

(g) $\int_{-\infty}^{+\infty} e^{-2u^2} A_u^2\, du = \cdot 353\ 4936.$

(h) $\int_{-\infty}^{+\infty} e^{-\frac{3}{2}u^2} A_u\, du \int_{-\infty}^{u} e^{-v^2}\, dv = \sqrt{\pi} \int_{-\infty}^{+\infty} e^{-\frac{3}{2}u^2} A_u A_{\sqrt{2}u}\, du = \cdot 772\ 6291.$

(j) $\int_{-\infty}^{+\infty} e^{-u^2} A_u\, du \int_{-\infty}^{u} e^{-\frac{3}{2}v^2}\, dv = \sqrt{\frac{2\pi}{3}} \int_{-\infty}^{+\infty} e^{-u^2} A_u A_{\sqrt{3}u}\, du = \cdot 830\ 5582.$

It will be seen that (xii) gives for the mean u,

$$ {}_n\bar{u} = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} u A_u^{n-1}\, du = \frac{n(n-1)}{2\pi} \int_{-\infty}^{+\infty} e^{-u^2} A_u^{n-2}\, du \quad \ldots\ldots\text{(xiv)}, $$

on integrating by parts. The integrals (*a*), (*b*) and (*c*) above are respectively

(*a*) $\frac{1}{6}\pi\, {}_4\bar{u} = \frac{1}{12}\pi\, {}_4\bar{w}$, (*b*) $\frac{1}{10}\pi\, {}_5\bar{u} = \frac{1}{20}\pi\, {}_5\bar{w}$, (*c*) $\frac{1}{15}\pi\, {}_6\bar{u} = \frac{1}{30}\pi\, {}_6\bar{w}$,

giving values for the mean ranges ${}_4\bar{w}$, ${}_5\bar{w}$, ${}_6\bar{w}$ corresponding exactly with those tabled by Tippett*, although the form of integral which he has used for the mean, viz.

$$ {}_n\bar{w} = \int_{-\infty}^{+\infty} \{1 - (1 - A_u)^n - A_u^n\}\, du = \int_{-\infty}^{+\infty} (1 - 2A_u^n)\, du \quad \ldots\ldots\ldots\text{(xv)}, $$

does not correspond with (xiv)†.

Making use of the numerical values of these integrals, (*a*)...(*j*), the constants of the five distributions can be found, and are given in Table I. From these can be obtained as a first objective, by using equations (i), the constants of the distribution of range in samples of 2, 3, 4, 5 and 6 individuals; they are given in Table II. Those for $n = 2$ were already known. In Figures 2 and 3 I have connected the values of σ_w and r with those of Tippett's for $n = 10$ and 20, which were obtained by computation without any assumptions as to the form of the u, v distribution. Figure 2 reproduces on a much enlarged scale the start of the curve given in Diagram VI of Tippett's paper; it can be readily used for interpolation of values of σ_w.

Our second purpose is to ascertain how nearly in the case of the constants ${}_w\beta_1$ and ${}_w\beta_2$ we have bridged the gap left open by Tippett, that is to say how nearly at $n = 6$ is there justification for assuming linear regression and homo-

* His Table X, p. 386, *loc. cit.*

† The two forms can be made equivalent by showing that in the limit $\Big[u(2A_u^n - 1)\Big]_{u=-\infty}^{u=+\infty} = 0.$

scedasticity in the u, v frequency surface? On these assumptions we should have

$$\left.\begin{aligned} p_{12} &= -\, r_u \mu_3 \\ p_{13} &= \quad r_u \mu_4 \\ p_{22} &= \{1 + r^2 ({}_u\beta_2 - 1)\}\, {}_u\mu_2{}^2 \end{aligned}\right\} \quad \text{......................(xvi)},$$

which give for $n = 6$

$$p_{12} = -\cdot 012\ 079 \text{ against true } + \cdot 014\ 490,$$
$$p_{13} = +\cdot 075\ 781 \quad ,, \quad ,, \quad + \cdot 067\ 983,$$
$$p_{22} = +\cdot 180\ 087 \quad ,, \quad ,, \quad + \cdot 177\ 403,$$

and using equations (i)

$${}_w\beta_1 = \cdot 0301 \text{ against true } \cdot 1892,$$
$${}_w\beta_2 = 3\cdot 0803 \quad ,, \quad ,, \quad 3\cdot 1698,$$

values which are far from satisfactory. In fact at $n = 6$ the regression is not linear nor is the distribution homoscedastic.

TABLE I.

Momental Constants of u, v Distribution.

Size of Sample, $n =$	*2*	*3*	*4*	*5*	*6*
Moments of u margin					
Mean	·564 190	·846 284	1·029 375	1·162 964	1·267 206
σ	·825 645	·747 975	·701 225	·668 981	·644 925
μ_2	·681 690	·559 467	·491 716	·447 535	·415 928
μ_3	·077 079	·089 199	·091 207	·090 585	·089 168
μ_4	1·422 797	·975 522	·764 597	·641 083	·559 440
β_1	·0188	·0154	·0700	·0915+	·1105+
β_2	3·0617	3·1166	3·1623	3·2008	3·2338
Cross-products					
r	·466 942	·294 688	·212 893	·165 831	·135 458
p_{11}	·318 310	·164 868	·104 683	·074 215	·056 341
p_{12}	·077 079	·045 761	·028 954	·019 858	·014 490
p_{13}	·622 273	·266 003	·149 071	·096 337	·067 983
p_{22}	·696 036	·370 644	·262 295	·209 221	·177 403

TABLE II.

Momental Constants of Range w.

Size of Sample, $n =$	*2*	*3*	*4*	*5*	*6*
Mean	1·1284	1·6926	2·0588	2·3259	2·5344
Standard Deviation	·8525+	·8884	·8798	·8641	·8480
β_1	·9906	·4174	·2735−	·2167	·1892
β_2	3·8692	3·2864	3·1884	3·1693	3·1698

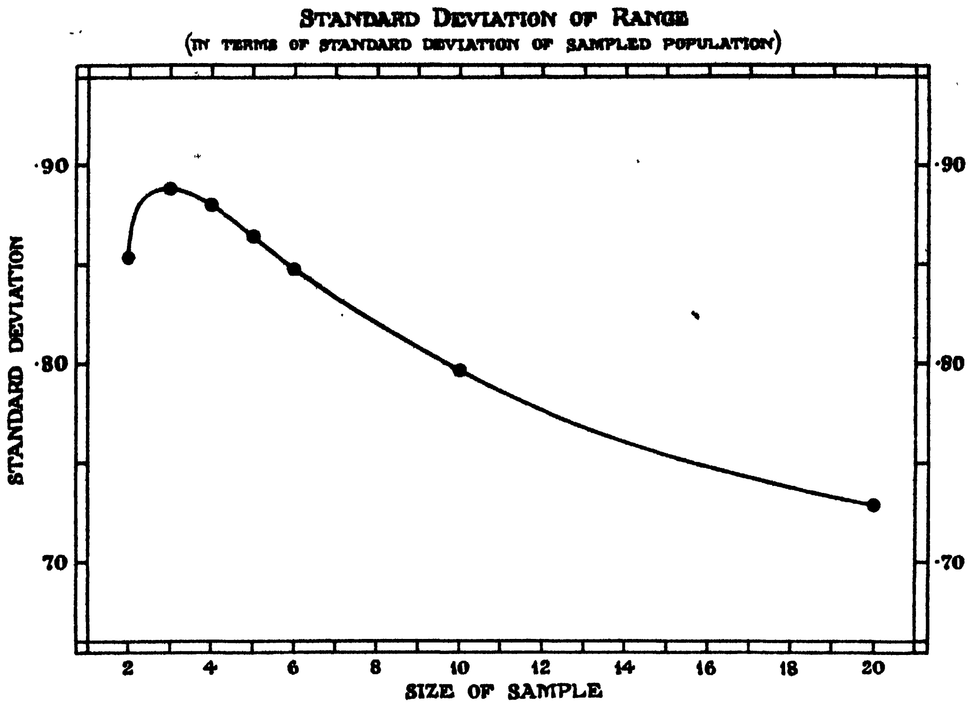

STANDARD DEVIATION OF RANGE
(IN TERMS OF STANDARD DEVIATION OF SAMPLED POPULATION)
STANDARD DEVIATION
·90
·80
70
·70
2
4
6
8
10
12
14
16
18
20
SIZE OF SAMPLE

Fig. 2.

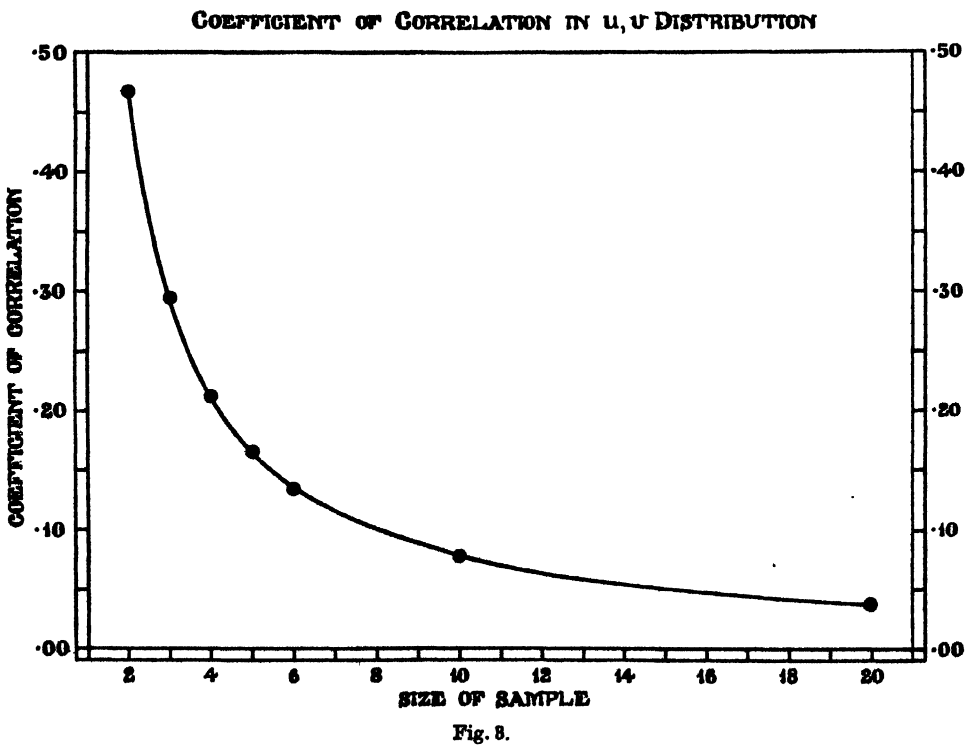

COEFFICIENT OF CORRELATION IN u, u' DISTRIBUTION
COEFFICIENT OF CORRELATION
·50
·40
·30
·20
·10
·00
2
4
6
8
10
12
14
16
18
20
SIZE OF SAMPLE

Fig. 3.

From (xi) and (xiii) we have that the mean value of u for a given v is

$$ {}_n\bar{u}_v = \frac{n-1}{\sqrt{2\pi}} \frac{1}{(1-A_v)^{n-1}} \int_v^\infty u e^{-\frac{1}{2}u^2} (A_u - A_v)^{n-2} du $$

$$ = \frac{(n-1)(n-2)}{2\pi} \frac{1}{(1-A_v)^{n-1}} \int_v^\infty e^{-u^2} (A_u - A_v)^{n-3} du \quad \text{......(xvii)} $$

on reduction.

Similarly the mean value of u^2 for a given v is, after reduction,

$$ {}_n\sigma^2_{u_v} + ({}_n\bar{u}_v)^2 = 1 + \frac{(n-1)(n-2)(n-3)}{4\pi\sqrt{2\pi}} \frac{1}{(1-A_v)^{n-1}} \int_v^\infty e^{-\frac{3}{2}u^2} (A_u - A_v)^{n-4} du $$

$$ \text{..............(xviii).} $$

From the computations already made in obtaining the numerical values of the integrals (a)...(j), ${}_n\bar{u}_v$ and ${}_n\sigma^2_{u_v}$ were readily obtained from (xvii) and (xviii) for the case $n = 6$, and their values at intervals of v of 0·5* are given in Table III. The distribution within the range $\bar{v} \pm 3\sigma_v$ is illustrated in Figure 4; it will be seen that towards the left-hand side of the diagram there is a condition approaching linearity of regression and homoscedasticity, but that as v increases, the regression

TABLE III.

Regression and Scedasticity of u, v Distribution in a Sample of 6.

	$\bar{u}_v$	$\sigma^2_{u_v}$	σ_{u_v}	Parabola (xxvii) $\bar{u}_v$	Expansions in A_v of p. 185 $\bar{u}_v$	σ_{u_v}
+1·0	2·100	·2191	·468	1·951	1·950+	·396
+0·5	1·809	·2573	·507	1·730	1·765−	·476
0·0	1·570	·3094	·556	1·545	1·562	·547
−0·5	1·391	·3613	·601	1·396	1·391	·598
−1·0	1·274	·4009	·633	1·283	1·274	·633
−1·5	1·208	·4274	·654	1·206	1·208	·654
−2·0	1·178	·4404	·664	1·165+	1·178	·664
−2·5	1·167	·4457	·668	1·161	1·167	·668
−3·0	1·164	·4470	·669	1·193	1·164	·669

line begins to rise rapidly and the scedasticity to decrease. To understand the general significance of these features, let us consider the limits of ${}_n\bar{u}_v$ and ${}_n\sigma^2_{u_v}$.

(1) As $v \to -\infty$ and $A_v \to 0$,

$$ {}_n\bar{u}_v \to \frac{(n-1)(n-2)}{2\pi} \int_{-\infty}^{+\infty} e^{-u^2} A_u^{n-3} du, $$

$$ \to {}_{n-1}\bar{u} \quad \text{from (xiv),} $$

or the array means tend to the mean value of u in a sample of $n-1$ individuals.

(2) As $v \to +\infty$ and $A_v \to 1$, (xvii) becomes zero in numerator and denominator. But after $n-1$ differentiations it is found that $\underset{v\to+\infty}{\text{Lt}} ({}_n\bar{u}_v) = v$, or the regression line is asymptotic to the line $u = v$.

* The unit of u and v is, as always, the standard deviation of the original normal population from which the sample has been drawn.

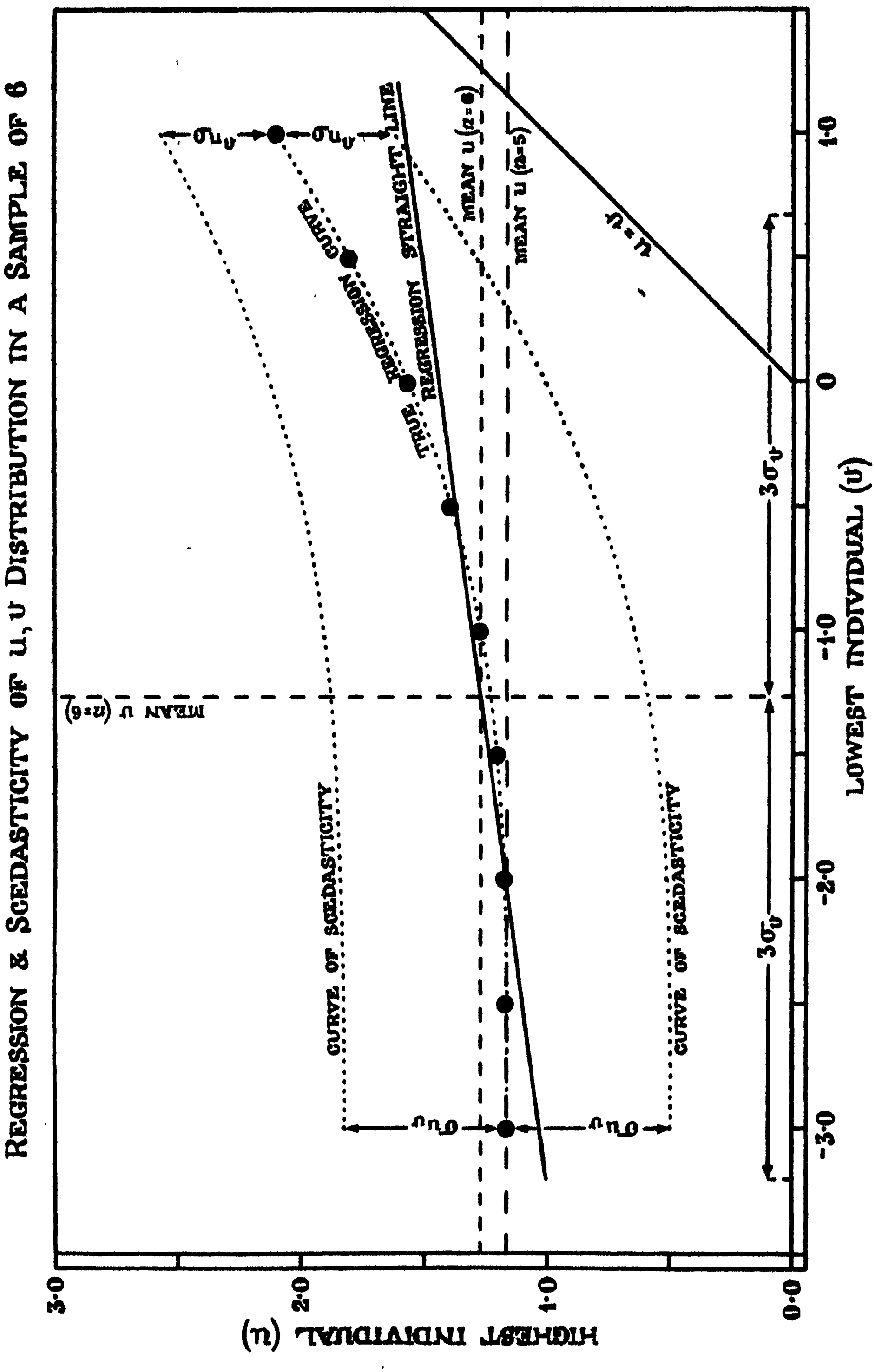

REGRESSION & SCEDASTICITY OF u, υ DISTRIBUTION IN A SAMPLE OF 6
CURVE OF SCEDASTICITY
CURVE OF SCEDASTICITY
TRUE REGRESSION CURVE
REGRESSION STRAIGHT LINE
MEAN u (n=6)
MEAN u (n=5)
MEAN υ (n=6)
u = υ
σuυ
3συ
3συ
LOWEST INDIVIDUAL (υ)
HIGHEST INDIVIDUAL (u)
-3·0
-2·0
-1·0
0
1·0
0·0
1·0
2·0
3·0

Fig. 4.

${}_n\bar{u}_v$ approaches ${}_{n-1}\bar{u}$ far more rapidly in the negative direction than it approaches $u = v$ in the positive direction, as is illustrated in Figure 4, but it is clear that however great n may be, the regression can never be strictly linear. r, the coefficient of correlation between u and v, is found to tend to zero as n increases, simply because within the range of significant frequency—let us say $u = \bar{u} \pm 3\sigma_u$, $v = \bar{v} \pm 3\sigma_v$—the regression line is effectively a horizontal straight line.

To find the limits of ${}_n\sigma^2_{u_v}$ we note that the second moment of u (or v) about the origin is

$$_n\mu_2' = 1 + \frac{n(n-1)(n-2)}{4\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{3}{2}u^2} A_u^{n-3}\, du \quad \text{.............(xix).}$$

Then,

(1) As $v \rightarrow -\infty$ and $A_v \rightarrow 0$,

$$_n\sigma^2_{u_v} + ({}_n\bar{u}_v)^2 \rightarrow 1 + \frac{(n-1)(n-2)(n-3)}{4\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{3}{2}u^2} A_u^{n-4}$$

$$\rightarrow {}_{n-1}\mu_2' \text{ or } {}_{n-1}\sigma_u^2 + {}_{n-1}\bar{u}^2.$$

Hence as

$$_n\bar{u}_v \rightarrow {}_{n-1}\bar{u},$$

$$_n\sigma_{u_v} \rightarrow {}_{n-1}\sigma_u,$$

or the standard deviation in the array tends to the standard deviation of u in a sample of $n-1$.

(2) By $n-1$ successive differentiations of numerator and denominator of (xviii) it is found that,

As $v \rightarrow +\infty$ and $A_v \rightarrow 1$,

$$_n\sigma^2_{u_v} + ({}_n\bar{u}_v)^2 \rightarrow 1 + (v^2 - 1)$$

$$\rightarrow v^2.$$

Hence $\underset{v \rightarrow +\infty}{\mathrm{Lt}}\ {}_n\sigma^2_{u_v} = 0$, or for high values of v, the scedasticity tends to vanish.

Here again we see that however large the sample, we shall not obtain theoretical homoscedasticity over the whole surface, but as in the case of the regression it may be possible to show that this holds effectively in the region of significant frequency. If we take $\bar{v} + 3\sigma_v$ as the maximum value of the character of the lowest individual likely to be met with in a sample, and therefore as a practical limit on one side to the u, v surface, we obtain the values given in Table IV*. For

TABLE IV.

Values of A_v, etc. at Limit of Significant Frequency.

Size of Sample	$v = \bar{v} + 3\sigma_v$	A_v	A_v^2	A_v^3
6	+ ·67	·7486	·5604	·4195
20	− ·29	·3859	·1489	·0575
60	− ·96	·1685	·0284	·0048
100	− 1·22	·1112	·0124	·0014
500	− 1·93	·0268	·0007	·0000

* Using values of σ_v given in Tippett's Table I, except for $n=6$.

samples of 60 or more, A_v is seen to be very small in the range of significant frequency, and this suggests that we should expand (xvii) and (xviii) in powers of A_v.

Write $$F_n(A_v) = \int_v^\infty e^{-ku^2}(A_u - A_v)^n\,du \quad \text{.....................(xx).}$$

Then

$$\frac{\partial F}{\partial v} = \frac{\partial}{\partial v}\left\{\int_v^\infty e^{-ku^2} A_u{}^n\,du - nA_v\int_v^\infty e^{-ku^2} A_u{}^{n-1}\,du \right.$$
$$\left. + \frac{n(n-1)}{2!}A_v{}^2\int_v^\infty e^{-ku^2} A_u{}^{n-2}\,du \ldots (-1)^n A_v{}^n \int_v^\infty e^{-ku^2}\,du\right\}$$
$$= -e^{-kv^2}A_v{}^n\left(1 - n + \frac{n(n-1)}{2!} - \ldots (-1)^n\right)$$
$$-n\frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}\left\{\int_v^\infty e^{-ku^2}A_u{}^{n-1}\,du - (n-1)A_v\int_v^\infty e^{-ku^2}A_u{}^{n-2}\,du\right.$$
$$\left. + \frac{(n-1)(n-2)}{2!}A_v{}^2\int_v^\infty e^{-ku^2}A_u{}^{n-3}\,du \ldots (-1)^{n-1}A_v{}^{n-1}\int_v^\infty e^{-ku^2}\,du\right\}$$
$$= 0 - n\frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}F_{n-1}(A_v) = -n\frac{dA_v}{dv}F_{n-1}(A_v).$$

But $$\frac{\partial F}{\partial A_v} = \frac{\partial F}{\partial v}\Big/\frac{dA_v}{dv} = -nF_{n-1}(A_v),$$

and similarly $$\frac{\partial^2 F}{\partial A_v{}^2} = n(n-1)F_{n-2}(A_v),$$
$$\frac{\partial^3 F}{\partial A_v{}^3} = n(n-1)(n-2)F_{n-3}(A_v), \text{ etc.}$$

It follows that if $F_n(A_v)$ may be expanded in a Maclaurin series, then

$$\int_v^\infty e^{-ku^2}(A_u - A_v)^n\,du = \int_{-\infty}^{+\infty} e^{-ku^2}A_u{}^n\,du - nA_v\int_{-\infty}^{+\infty} e^{-ku^2}A_u{}^{n-1}\,du$$
$$+ \frac{n(n-1)}{2!}A_v{}^2\int_{-\infty}^{+\infty} e^{-ku^2}A_u{}^{n-2}\,du - \ldots \quad \text{......(xxi).}$$

Expanding $\dfrac{1}{(1-A_v)^{n-1}}$ in ascending powers of A_v, and remembering that the integrals in (xxi), when multiplied by the appropriate factors,

(*a*) become the successive marginal means, ${}_n\bar{u}, \ldots$ when $k = 1$ (see (xiv)),

(*b*) " " " second moments, ${}_n\mu_2', \ldots$ when $k = \frac{3}{2}$ (see (xix)),

it can be shown after some reduction of (xvii) and (xviii) that

$${}_n\bar{u}_v = {}_{n-1}\bar{u} + (n-1)A_v({}_{n-1}\bar{u} - {}_{n-2}\bar{u}) + \frac{(n-1)}{2!}A_v{}^2\{(n-2)_{n-3}\bar{u} - 2(n-1)_{n-2}\bar{u} + n\,{}_{n-1}\bar{u}\}$$
$$+ \frac{n-1}{3!}A_v{}^3\{n(n+1)\,{}_{n-1}\bar{u} - 3n(n-1)\,{}_{n-2}\bar{u}$$
$$+ 3(n-1)(n-2)\,{}_{n-3}\bar{u} - (n-2)(n-3)\,{}_{n-4}\bar{u}\} + \ldots \quad \text{...(xxii),}$$

$$\begin{aligned}{}_{n}\sigma^2_{u_v} = {}_{n-1}\sigma_u^2 &+ (n-1) A_v \{{}_{n-1}\sigma_u^2 - {}_{n-2}\sigma_u^2 - ({}_{n-1}\bar{u} - {}_{n-2}\bar{u})^2\} \\ &+ \frac{n-1}{2!} A_v^2 \{(n-2)\, {}_{n-3}\sigma_u^2 - 2(n-1)\, {}_{n-2}\sigma_u^2 + n\, {}_{n-1}\sigma_u^2 \\ &\qquad + (n-2)({}_{n-1}\bar{u} - {}_{n-3}\bar{u})^2 - 4(n-1)({}_{n-1}\bar{u} - {}_{n-2}\bar{u})^2\} \\ &+ \frac{n-1}{3!} A_v^3 \{n(n+1)\, {}_{n-1}\sigma_u^2 - 3n(n-1)\, {}_{n-2}\sigma_u^2 + 3(n-1)(n-2)\, {}_{n-3}\sigma_u^2 \\ &\qquad - (n-2)(n-3)\, {}_{n-4}\sigma_u^2 - 6(n-1)(2n-1)({}_{n-1}\bar{u} - {}_{n-2}\bar{u})^2 \\ &\qquad + 6(n-1)(n-2)({}_{n-1}\bar{u} - {}_{n-3}\bar{u})^2 - 3(n-1)(n-2)({}_{n-2}\bar{u} - {}_{n-3}\bar{u})^2 \\ &\qquad - (n-2)(n-3)({}_{n-1}\bar{u} - {}_{n-4}\bar{u})^2\} + \ldots \qquad \text{(xxiii).}\end{aligned}$$

The convergence of these two series depends on the values of the coefficients of the successive powers of A_v; while these contain functions of n which increase as the size of the sample is increased, they also contain the differences of the first and second moments of the distribution of u, which steadily decrease. The following numerical values have been found for equations (xxii) and (xxiii):

$$\begin{aligned}
n &= 6 \\
{}_{6}\bar{u}_v &= 1{\cdot}1630 + {\cdot}6679 A_v + {\cdot}1729 A_v^2 + {\cdot}1729 A_v^3, \\
{}_{6}\sigma^2_{u_v} &= {\cdot}4475 - {\cdot}3101 A_v + {\cdot}1254 A_v^2 - {\cdot}1989 A_v^3 \\
n &= 7 \\
{}_{7}\bar{u}_v &= 1{\cdot}2672 + {\cdot}6255 A_v + {\cdot}1852 A_v^2 + {\cdot}1481 A_v^3, \\
{}_{7}\sigma^2_{u_v} &= {\cdot}4159 - {\cdot}2548 A_v + {\cdot}0651 A_v^2 - {\cdot}0859 A_v^3. \\
n &= 20 \\
{}_{20}\bar{u}_v &= 1{\cdot}844 + {\cdot}465 A_v + {\cdot}184 A_v^2 + {\cdot}12 A_v^3. \\
n &= 60 \\
{}_{60}\bar{u}_v &= 2{\cdot}313 + {\cdot}385 A_v + {\cdot}16 A_v^2, \\
{}_{60}\sigma^2_{u_v} &= {\cdot}2077 - {\cdot}065 A_v + \ldots \text{ approximately.} \\
n &= 100 \\
{}_{100}\bar{u}_v &= 2{\cdot}504 + {\cdot}360 A_v + {\cdot}14 A_v^2, \\
{}_{100}\sigma^2_{u_v} &= {\cdot}1848 - {\cdot}05 A_v + \ldots \text{ approximately.} \\
n &= 500 \\
{}_{500}\bar{u}_v &= 3{\cdot}036 + {\cdot}30 A_v.
\end{aligned}$$

For $n=6$ the values of the mean and of the array standard deviation have been inserted in the last columns of Table III; it will be seen that as far as $v=0$ or within the range $v=-\infty$ to $v=\bar{v}+2\,{}_{6}\sigma_v$, four terms of the expansions give a very satisfactory fit to the true values of the regression and scedasticity. For larger samples we have only available complete tabled values of the means of the u distribution, with a few isolated values of σ_u*.

The coefficients in (xxii) contain 1st, 2nd and 3rd differences of ${}_{n}\bar{u}$, and these become so small that when we reach a sample of 500 it becomes impossible to ascertain even the coefficient of A_v^2. For the same reason we should not be able

* Tippett's Table X giving $\bar{w}=2\bar{u}$, and his Table I giving σ_u at $n=10, 20, 60, 100, 200, 500, 1000$.

to obtain sufficiently accurate values for the differences of ${}_n\sigma_u{}^2$ by interpolating in Tippett's Table I. Approximate values for the coefficient of A_v in the expansion of ${}_n\sigma_u{}^2$ have however been obtained for $n = 60$ and 100*. An examination of these equations suggests the following points:

(*a*) There is some irregularity for $n = 6$ and 7, but otherwise the numerical coefficients appear to decrease both within the series as the index of A_v increases, and from series to series as n increases.

TABLE V.

Regression and Scedasticity at Limits of Significant Frequency.

n		Values from Expansions in A_v of p. 185			Values on Assumptions of Linearity of Regression and Homoscedasticity	
		Asymptotic Limit	At $v = \bar{v} + 2\sigma$	At $v = \bar{v} + 3\sigma$	At $v = \bar{v} - 3\sigma$	At $v = \bar{v} + 3\sigma$
60	$\bar{u}_v$ σ_{u_v}	2·31 ·456	2·34 ·450	2·38 ·444	2·30 ·454	2·34 ·454
100	$\bar{u}_v$ σ_{u_v}	2·50+ ·430	2·52 ·427	2·55 ·423	2·50− ·429	2·52 ·429

(*b*) For $n = 60$ and 100, we have the comparisons of Table V. Assuming that the expansions of page 185 are adequate, we see that while the assumptions of linear regression and homoscedasticity are not strictly accurate, even in the range $\bar{v} \pm 3\sigma$, the error at the limit of significant frequency is now reduced to something of the order of $2\,°/_\circ$, while in the case of $n = 6$ it was of the order of $20\,°/_\circ$†. That is to say for samples of size 60 or more we can hardly be involved in any serious error in accepting Tippett's values for the constants β_1 and β_2 of the distribution of range, which have been based on the assumption of linear regression and homoscedasticity in the u, v surface. It remains therefore to bridge over the gap between $n = 6$ and $n = 60$. Figure 7 shows that this is rather a critical stretch of the $\beta_1 \beta_2$ curve, because it is here that the skewness of the distribution of range after starting to decrease as n increases from 0 to 6 begins to increase.

Turning back to p. 179, we see that the values of p_{12} and p_{13} found for $n = 6$ on the assumption of linear regression are most unsatisfactory, particularly the former. Suppose that a parabola is fitted to the regression line by Least Squares or by the method of Orthogonal Functions‡, then to the second order its equation can be put in the form

$$\bar{u}_v - \bar{u} = r(v - \bar{v}) + \frac{p_{12} + \mu_3 r}{\mu_4 \mu_2 - \mu_3{}^2 - \mu_2{}^3}\{(v - \bar{v})^2 \mu_2 + (v - \bar{v})\mu_3 - \mu_2{}^2\} \ldots \text{(xxiv)},$$

* Obtained by fitting parabolae through ${}_{20}\sigma$, ${}_{60}\sigma$, ${}_{100}\sigma$, and ${}_{60}\sigma$, ${}_{100}\sigma$, ${}_{200}\sigma$ respectively.

† For $n=6$, at $v = +0{\cdot}5 = \bar{v} + 2{\cdot}7\sigma$, the true values of $\bar{u}_v$ and σ_{u_v} are 1·81 and ·507 respectively, while for linear regression and homoscedasticity we should have 1·51 and ·689.

‡ *Biometrika*, Vol. XIII. p. 296 *et seq.*

where the momental constants μ_2, μ_3, μ_4 are those for the distribution of u (e.g. ${}_u\mu_3 = -{}_v\mu_3$). Now at $n = 10$, 20, 60, etc., all the constants in this equation are known except p_{12}; if then we can find by some other method the equation to the second order parabola "best fitting" the regression line in these cases, we shall be able to determine the values of p_{12}, whether such a curve gives an adequate description of the regression line or not.

The parabola of (xxiv) cuts the regression straight line

$$\tilde{u}_v - \bar{u} = r(v - \bar{v}) \quad \text{.............................(xxv)},$$

where
$$(v - \bar{v})^2 \mu_2 + (v - \bar{v}) \mu_3 - \mu_2^2 = 0,$$

or
$$v = \bar{v} + \tfrac{1}{2}\{-\sqrt{\beta_1} \pm \sqrt{\beta_1 + 4}\}\, \sigma_v{}^* \quad \text{....................(xxvi)}.$$

But these two points are known for $n = 10$, 20, 60, etc., and it is therefore only necessary to fix one other point on the parabola to determine its equation. For $n = 6$ the equation to the regression parabola given by (xxiv) is

$$\tilde{u}_v = 1{\cdot}544\,571 + {\cdot}334\,276 v + {\cdot}072\,329 v^2 \quad \text{.............(xxvii)},$$

ordinates of which are given in Table III; they will be seen to correspond closely with the true regression points in the central region only. The vertex of this parabola is at

$$\left.\begin{aligned} v &= -2{\cdot}311 = {}_6\bar{v} - 1{\cdot}618\,{}_6\sigma_v \\ u &= 1{\cdot}158 \end{aligned}\right\},$$

which is slightly below the true asymptote

$$u = {}_{n-1}\bar{u} = 1{\cdot}163.$$

The vertices of the regression parabolae for $n = 2$, 3, 4, 5 are found to lie at

$$\begin{aligned} n &= 2, \quad v = -2{\cdot}581 = {}_2\bar{v} - 2{\cdot}443\,{}_2\sigma_v, \\ n &= 3, \quad v = -2{\cdot}252 = {}_3\bar{v} - 1{\cdot}879\,{}_3\sigma_v, \\ n &= 4, \quad v = -2{\cdot}235 = {}_4\bar{v} - 1{\cdot}720\,{}_4\sigma_v, \\ n &= 5, \quad v = -2{\cdot}268 = {}_5\bar{v} - 1{\cdot}652\,{}_5\sigma_v. \end{aligned}$$

These values suggest that as n increases the vertex of the regression parabola will lie at a point about $1{\cdot}6\,{}_n\sigma_v$ to the left (in the sense of Fig. 4) of the mean of the margin (${}_n\bar{v}$). We cannot of course be sure that the multipliers of ${}_n\sigma_v$, as shown above, will continue to converge on $-1{\cdot}6$, but some form of extrapolation is necessary at this stage and this appears a reasonable hypothesis†. Giving the regression parabolae vertices at points with $v = {}_n\bar{v} - 1{\cdot}6\,{}_n\sigma_v$ and making them pass through the two known points of (xxvi), I have calculated their equations, which are as follows:

$$\begin{aligned} n &= 10, \quad {}_{10}\tilde{u}_v = 1{\cdot}7728 + {\cdot}2361 v + {\cdot}04764 v^2, \\ n &= 20, \quad {}_{20}\tilde{u}_v = 2{\cdot}0327 + {\cdot}1413 v + {\cdot}0261 v^2, \\ n &= 60, \quad {}_{60}\tilde{u}_v = 2{\cdot}4066 + {\cdot}0622 v + {\cdot}0102 v^2, \\ n &= 100, \quad {}_{100}\tilde{u}_v = 2{\cdot}5688 + {\cdot}0409 v + {\cdot}0064 v^2. \end{aligned}$$

* β_1 is for the u distribution, i.e. $\sqrt{{}_u\beta_1} = -\sqrt{{}_v\beta_1}$. Of course $\sigma_u = \sigma_v$.

† Slight variations in the position chosen for the vertex of the parabola do not make a very serious difference to the resulting value calculated for ${}_w\beta_1$. E.g. for $n = 20$ if vertex is at

$$v = {}_n\bar{v} - 1{\cdot}5\,{}_n\sigma_v, \text{ then } {}_w\beta_1 = {\cdot}165,$$

and at $v = {}_n\bar{v} - 1{\cdot}7\,{}_n\sigma_v$, then ${}_w\beta_1 = {\cdot}157$.

Assuming these equations to correspond very closely to the "best fitting" regression parabolae, we obtain the following values of p_{12} and ${}_w\beta_1$, which are compared in the last two columns with the results given by Tippett in his Table IV.

TABLE VI. *Values for β_1 of Range.*

n	p_{12}	${}_w\beta_1$	${}_w\beta_1$ (if r were zero)	${}_w\beta_1$ (if regression linear)
10	·00576	·156	·084	·063
20	·00172	·161	·125	·111
60	·00031	·201	·188	·181
100	·00014	·223	·215	·210
200	—	·247	as in preceding column	

Various other methods of estimating ${}_w\beta_1$ were considered, but it is believed that the values given in the third column of Table VI are the most satisfactory that can be obtained without the very great labour of direct computation. It seems unlikely that they can be more than 2 or 3 °/₀ in error and they connect up satisfactorily the known values for $n = 2$ to 6, with those given by Tippett for large samples where the assumption of linear regression is certainly justified. Figure 5 shows graphically the relation between these different values of ${}_w\beta_1$ in the interval $n = 2$ to 20.

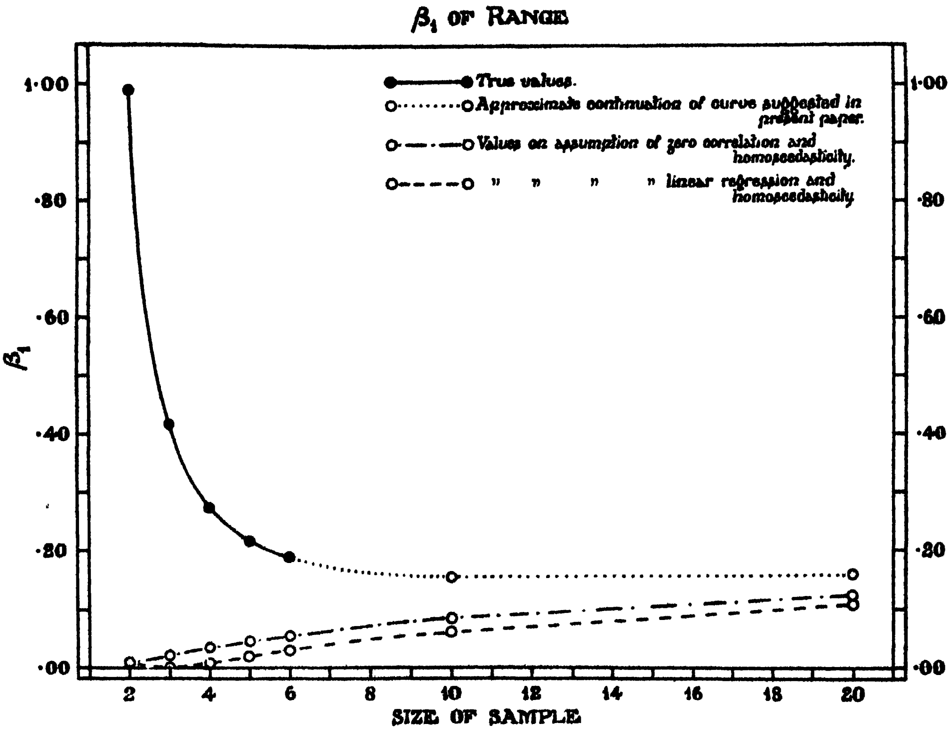


Fig. 5.

The value of ${}_w\beta_2$ depends on that of ${}_w\mu_4$ and for this we require p_{13} and p_{22}. If a second order parabola gave an adequate description of the regression line, it can be shown that

$$p_{13} = r\mu_4 + \frac{p_{12} + r\mu_3}{\mu_4\mu_2 - \mu_3^2 - \mu_2^3}\,\mu_3\mu_2^2\left\{-\frac{\beta_3}{\beta_1} + \beta_2 + 1\right\}^* \quad \text{.........(xxviii)}.$$

In the case of $n = 6$, this leads to

$$p_{13} = \cdot075\ 781 - \cdot017\ 469 = \cdot058\ 312,$$

whereas the true value is ·067 983; that is to say the correction to the value obtained on the assumption of linear regression is in the right direction but double the required amount. In fact for $n = 10$, 20, etc. we cannot hope to obtain a satisfactory value for p_{13} from the second order regression parabolae of p. 187; for they do not adequately represent the regression. Now p_{13} could be obtained from the "best fitting" cubic regression line which could be determined to pass through 3 points on the known regression straight line, and to touch the asymptote $u = {}_{n-1}\bar{u}$. Again an improved value for p_{22} could be obtained by using $\eta^2_{u_v}$ in the third equation of (xvi) and by applying some correction for scedasticity, but when it is remembered that the basic constants on which we should need to build up this

TABLE VII.

Values for β_2 of Range.

n	${}_w\beta_2$	${}_w\beta_2$ (if r were zero)	${}_w\beta_2$ (if regression linear)
10	3·22	3·17	3·14
20	3·26	3·23	3·22
60	3·35	3·34	3·33
100	3·39	3·38	3·38
200	3·44	as in preceding column	

corrective superstructure (σ_u, ${}_u\beta_1$ and ${}_u\beta_2$ of Tippett's Table I) are known to only 3 or at most 4 significant figures, it seems very doubtful whether the final results so obtained would be at all reliable. After trying various cruder methods of approximation, the following values of ${}_w\beta_2$ are therefore suggested as probably lying within 1 or 2°/₀ of the true values. They are again compared with Tippett's values, and the whole represented graphically in Figure 6.

As in the case of Figure 5 it is seen how the approximate curves approach the curve of more accurate values from below, the assumption of zero correlation giving slightly more satisfactory results than that of linear regression. No importance can, I think, be attached to this fact; below $n = 20$ neither assumption can be justified, but the combination of several inaccurate terms in equations (i) are such

* Again as in (xxiv) the momental constants are for the u distribution and $\beta_3 = \frac{\mu_5\mu_3}{\mu_2^4}$, and can be obtained approximately by assuming that this distribution follows a Pearson Type curve, when

$$\beta_3 = \frac{2\beta_1(\beta_2^2 + 7\beta_2 - 3\beta_1)}{9 + 3\beta_1 - \beta_2}.$$

as to cause the former to give to the ratios ${}_w\beta_1$ and ${}_w\beta_2$ values nearer the truth than the latter.

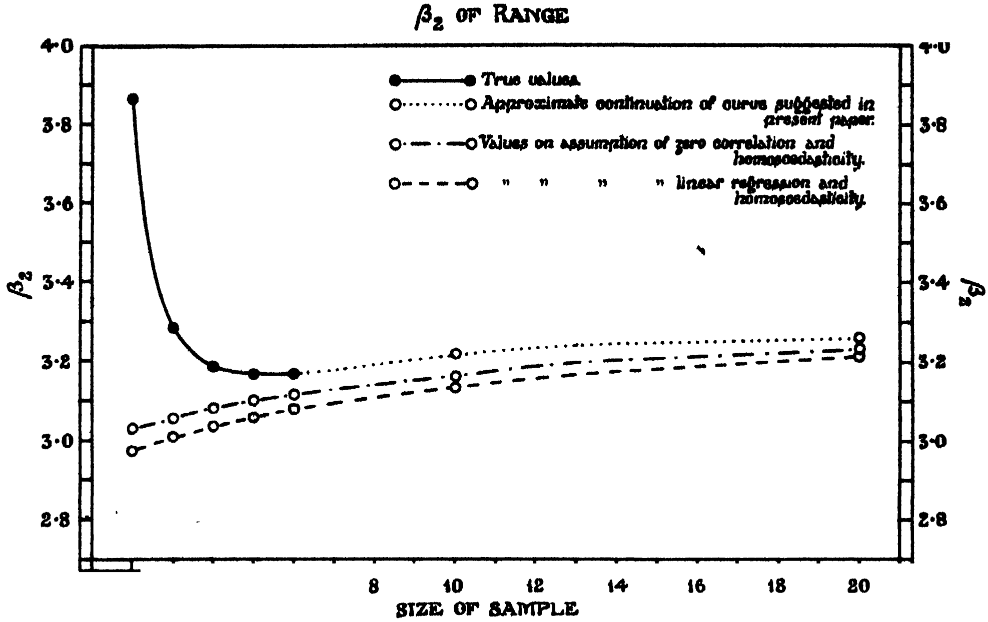


Fig. 6.

Figure 7 is a diagram of the $\beta_1\beta_2$ field, showing the resulting curve followed by the point ${}_w\beta_1$, ${}_w\beta_2$ as n increases; the less certain portion between $n = 6$ and 100 is dotted. The diagram also contains the curve for ${}_u\beta_1$, ${}_u\beta_2$, or the constants of the distribution of the largest individual in the sample.

The peculiar forms of these curves deserve perhaps special consideration. Taking first that for the distribution of u, the largest individual, we start with the case of a sample of one where the distribution is simply that of the original normal population; then as n increases the corresponding β_1, β_2 point moves at a decreasing rate along a curve which lies first in the Type IV and later the Type VI area*. The physical interpretation of this increasing skewness would seem to be as follows. For a given size of sample there is a certain modal value of u; above this modal value u may take freely any of the population values extending towards $+\infty$; but below it, the range becomes more and more restricted as n increases and there are more and more individuals in the sample whose values must lie below u. Thus the distribution of u is skew, and positively skew with the steeper slope on the side corresponding to the lower values of the character. The distribution of v, the lowest individual, is identical but with negative skewness.

* Strictly u may lie anywhere between $+\infty$ and $-\infty$, but the larger the sample the greater becomes the improbability of u taking large negative values. This tendency is reflected in the $\beta_1\beta_2$ diagram by the curve passing near $n = 10$ from the Type IV area (unlimited in both directions) to the Type VI area (limited in one direction).

DIAGRAM SHOWING THE β_1, β_2 POINTS CORRESPONDING TO THE FREQUENCY DISTRIBUTIONS OF THE EXTREME INDIVIDUAL (u or v) & THE RANGE (w) IN SAMPLES OF 12

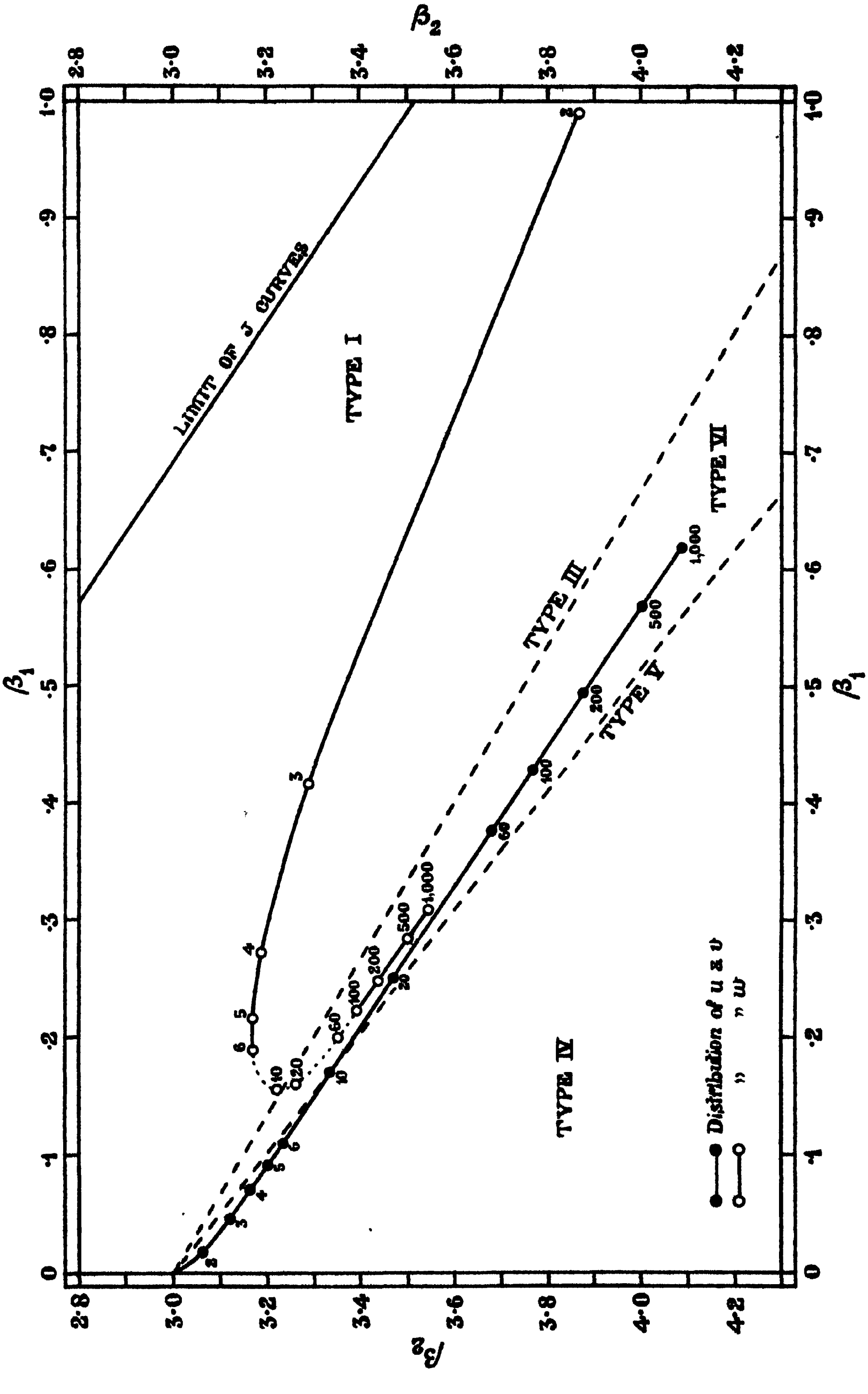


Fig. 7.

Turning to the range, w, in the sample we start with the case of two individuals only, when the distribution is half a normal curve with a precipitous end at $w=0$. This half curve cannot of course be represented exactly by a complete Pearson curve, but the curve most closely representing it would be very skew ($\beta_1 = \cdot 9906$, $\beta_2 = 3\cdot 8692$, without corrections for abruptness). For a sample of three, the modal value of w will still be low, lying near the limiting value of 0, but the curve is far less skew ($\beta_1 = \cdot 4174$, $\beta_2 = 3\cdot 2864$). The skewness decreases until n is in the neighbourhood of 10. This stretch of the curve corresponds to that in which there is a correlation between the highest and lowest individuals in the sample. The backward bend of the curve comes at the transition period when the influence of the position of v on that of u is vanishing. When they are effectively independent, we reach the stretch of curve from $n=100$ to 1000 and beyond, which follows closely the curve for the distribution of u. Here there is no upper limit to the range but an increasingly sharply defined lower limit resulting from the fact that more and more observations have to be fitted into the space between u and v as n increases. This again implies positive skewness, and the appropriate curve is a Type VI with a limiting range on the lower side, but extending to infinity in the direction of increasing range. We have from equations (i) and (ii), as a very close approximation,

$$_w\beta_1 = \tfrac{1}{2}\,_u\beta_1, \quad _w\beta_2 = \tfrac{1}{2}\,_u\beta_2 + 1\cdot 5.$$

In any particular problem where the distribution of range is required, a worker may proceed as follows:

Mean; values to be taken from Table X at the end of Tippett's paper.

Standard Deviation; for large samples use Tippett's Diagram VI or for n less than 20, Figure 2 of the present paper.

$_w\beta_1$ *and* $_w\beta_2$; for $n=200$ or more, use Diagrams VII and VIII of Tippett's paper; otherwise Figures 5, 6, or 7 of this paper.

TABLE VIII.

Summary of Constants of Distribution of Range.

n	Mean	Standard Deviation	β_1	β_2
2	1·12838	·8525	·9906	3·8692
3	1·69257	·8884	·4174	3·2864
4	2·05875	·8798	·2735	3·1884
5	2·32593	·8641	·2167	3·1693
6	2·53441	·8480	·1892	3·1698
10	3·07751	·797	·156	3·22
20	3·73495	·729	·161	3·26
60	4·63856	·639	·201	3·35
100	5·01519	·605	·223	3·39
200	5·49209	·566	·247	3·44
500	6·07340	·524	·285	3·50
1000	6·48287	·497	·309	3·54

As the results are somewhat scattered, it is perhaps desirable to put together a framework of values in one table. These are given in Table VIII (p. 192). It is hoped that the present paper will have cleared up certain points left doubtful in the earlier work; but at the same time the writer wishes to make it clear that he has had the easier task, and that it would have been impossible for him to bridge over the gap of medium sized samples without the heavy spadework which Mr Tippett carried out in computing the values of the means and standard deviations of range. The diagrams are as usual the excellent work of Miss Ida McLearn.

Note on the Relation between $\bar{w}_n$ and $\chi_{n,p}$.

In the notation of the preceding paper, and of Irwin's paper on "Francis Galton's Individual Difference Problem*," we may write, in samples of n drawn at random from a normal population, $\bar{w}_n$ = mean range; $\chi_{n,p}$ = mean difference between the characters of the pth and $(p+1)$th individuals when the sample is arranged in the order of magnitude of the characters.

The existence of the following general relation,

$$\chi_{n,p} = \frac{1}{2}\frac{n!}{p!\,n-p!}(-1)^{p+1}\Delta^p\bar{w}_{n-p} \quad \text{.........................(i)},$$

between $\chi_{n,p}$ and the forward differences of $\bar{w}_{n-p}$ has been recently indicated to me by "Student," who was able to prove that it held true for $p = 1$, 2 and 3, and also to obtain numerical confirmation by using the special values of $\chi_{n,p}$ given by Irwin (*loc. cit.* p. 107), and the series of tabled values of $\bar{w}_n$ given by Tippett†. Given a complete table of ranges, this equation should enable one to obtain any desired value of $\chi_{n,p}$, but in practice its use will be limited to fairly low values of n by the fact that the differences of $\bar{w}_n$ decrease very rapidly. For example, taking Tippett's table of $\bar{w}_n$, which is calculated to 5 decimal places, it is found that $\Delta^3\bar{w}_n$ has practically vanished by the time that $n = 40$. On the other hand, for a sample of 10, we can obtain the following series of values of $\chi_{10,p}$, which have errors of not more than one unit in the third decimal place:

$p=$	1	2	3	4	5	6	7	8	9
	·537	·345	·280	·256	·237	·256	·280	·345	·537

These give a total range of 3·073 against the true $\bar{w}_{10} = 3·07751$.

A general proof of equation (i) may be obtained as follows. If we draw repeated samples of n from the population, these will give a representative distribution of samples of $n-s$ supposing that in each case the first s individuals in the draw are excluded. Denote by λ the constitution of a sample of n with a particular set of character values for the n individuals, and suppose that the chance of obtaining λ is p_λ. The distribution λ may be represented diagrammatically as follows, the n observations lying at $A_1, A_2, \ldots A_n$ as below:

A_1 A_2 A_3 .... A_r A_{r+1} .... A_{n-2} A_{n-1} A_n

| x_1 | x_2 | .... | x_r | .... | x_{n-2} | x_{n-1} |

* *Biometrika*, Vol. XVII. pp. 100—128.

† *Biometrika*, Vol. XVII. Table X, pp. 386—387.

being placed in order of character value; $x_1, x_2, \ldots x_{n-1}$ are the distances between successive values. The s first drawn individuals will be distributed in a random manner among the A's. For the sample of n, we have the range

$$_\lambda w_n = x_1 + x_2 + \ldots + x_{n-1}.$$

For the sample of $n - s$, the range will be short by at least x_1 whenever one of the s individuals lies at A_1; the chance of this happening is s/n. Similarly the range will be short by at least x_{n-1} whenever one of the s individuals lies at A_n; the chance of this is also s/n.

The range will be short by at least $x_1 + x_2$ whenever two of the s individuals lie one at A_1 and one at A_2; the chance of this occurring is $s(s-1)/n(n-1)$. This is also the chance of the range being short by at least $x_{n-2} + x_{n-1}$.

In general the range for the sample of $n - s$ will be short of the range for a sample of n by an amount which includes x_r, whenever the positions $A_1, A_2, A_3, \ldots A_r$ are all occupied by r of the s first drawn individuals; the chance of this occurring is

$$\frac{s(s-1)\ldots(s-r+1)}{n(n-1)\ldots(n-r+1)}.$$

When none of the s lie at either A_1 or A_n, the range in the sample of $n - s$ is the same as in the sample of n. Now taking all possible constitutions, λ, of samples of n, and remembering that p_λ is quite independent of the distribution of the s individuals among the points $A_1, \ldots A_n$, we find for the mean range

$$\bar{w}_n = \bar{w}_{n-s} + \frac{s}{n}(\bar{x}_1 + \bar{x}_{n-1}) + \frac{s(s-1)}{n(n-1)}(\bar{x}_2 + \bar{x}_{n-2}) + \frac{s(s-1)(s-2)}{n(n-1)(n-2)}(\bar{x}_3 + \bar{x}_{n-3})$$
$$+ \ldots + \frac{s(s-1)\ldots\ldots 1}{n(n-1)\ldots(n-s+1)}(\bar{x}_s + \bar{x}_{n-s}) \ \ldots\ldots\ldots\ldots\text{(ii)}.$$

This relation holds because, in order to make up w_n, w_{n-s} must be supplemented

(1) by x_1 on every occasion when A_1, the extreme, is occupied by one of the s first drawn individuals;

(2) by x_2 in addition whenever A_2, the second place, is occupied as well as the first;

(3) by x_3 also, whenever A_1, A_2 and A_3 are all occupied, etc. etc.

Hence remembering that $\bar{x}_r = \bar{x}_{n-r} = \chi_{n,r}$, we have

$$\bar{w}_n = \bar{w}_{n-s} + 2\frac{s}{n}\chi_{n,1} + 2\frac{s(s-1)}{n(n-1)}\chi_{n,2} + \ldots + \frac{2(s!)}{n(n-1)\ldots(n-s+1)}\chi_{n,s} \ \ldots\text{(iii)}.$$

If now we take the p equations obtained by putting s equal successively to $p, p-1, \ldots 3, 2, 1$, we may eliminate all the χ's except $\chi_{n,p}$ by taking p differences, and have finally

$$0 = \Delta^p \bar{w}_{n-p} + (-1)^p \frac{2(p!)}{n(n-1)\ldots(n-p+1)}\chi_{n,p},$$

or

$$\chi_{n,p} = \tfrac{1}{2}\frac{n!}{p!\,\overline{n-p}!}(-1)^{p+1}\Delta^p \bar{w}_{n-p} \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i) bis}.$$

If n be odd this relation holds for all values of p between 1 and $\frac{1}{2}(n-1)$; if even from $p = 1$ to $\frac{1}{2}n$.

ON THE CORRELATION OF HEAD MEASUREMENTS AND MENTAL AGILITY. WOMEN.

BY J. R. MUSSELMAN, PH.D., Johns Hopkins University.

THE question as to the correlation of head measurements and intelligence has been discussed before. Observations would appear to give somewhat conflicting results; all those of the Biometric Laboratory* show only the smallest correlation between intelligence and head shape or size. Other investigators, especially in Germany, without using any adequate correlation method have asserted there is such a relation. It is not denied there may be some association; the problem is to find the intensity of that relationship, and without a measure of this intensity any such assertions are of little value. The biometricians have not denied some association but have asserted it is so small that it can be of no prognostic value.

It seemed worth while to consider whether on the data of Francis Galton's Second Anthropometric Laboratory Series any relation could be found between head size or shape and mental agility, as represented by reaction times to sight and sound.

The data for women in Francis Galton's Second Series cover about 1850 individuals between the ages of 3 and 76. At that time (1880–90) measurements were taken in inches, so the tables and computations were made in that unit and the final results then expressed in millimetres. At the beginning of his series the head lengths were measured from the nasion to the occipital point, consequently these have been discarded and I have taken only the data for head lengths from glabella to occipital point which cover about 1700 individuals. It is necessary to correct both the head lengths and head breadths for age. Diagrams I and II (pp. 197–8) show the curves of means for lengths and breadths by years. Above the age 53 the curves are dotted as the number of individuals is relatively small in these age groups and it appears that we have mostly large-headed women in them†. Consequently this part of the curve does not portray wider actual experience as well as it might. On the same diagrams are plotted the standard deviations by years. The total numbers in the age groups are too small to make any correction for the standard deviation of arrays of real service. The average results appear to be adequately represented by a horizontal straight line. We find that the mean head breadth of 1852 women is 145·68 mm., and the mean head length of 1693 women is 184·38 mm.; with 4·855 mm. and 6·232 mm. as standard deviations respectively.

* See especially *Biometrika*, Vol. v. p. 105 *et seq.*

† It is possible that a certain amount of vanity in size of head led to a selection of the larger-headed old ladies; or the curves may suggest that the head does continue to grow till extreme old age.

The following comparison of head measurements before and after correction to a standard age of 40·5 years may be of interest:

Character	Before Correction for Age			After Correction for Age		
	Mean	Standard Deviation	Coefficient of Variation	Mean	Standard Deviation	Coefficient of Variation
Breadth ...	145·68 mm.	4·855 mm.	3·33	146·19 mm.	4·667 mm.	3·19
Length... ...	184·38 mm.	6·232 mm.	3·38	184·97 mm.	5·923 mm.	3·20
Cephalic Index	79·07	3·036	—	[79·03*]	—	—

The raising of the means may be naturally expected as the mean value at 40·5 years is greater than that of earlier years which contain the bulk of the material. It would be still greater had childhood been more heavily represented. The slightly decreased variability is also to be expected, as the variability at a given age must be less than the total variability for all ages. As the standard deviations of the arrays have been treated as sensibly constant (see Diagrams I and II) the reduction to age 40·5 as standard was simple, the deviation plus or minus of any individual from the mean of his age was merely added or subtracted from the mean of age 40·5 for the same character.

For those women for whom head breadths were available (1834 and 1835) we have for reaction times the following values:

Reaction Times in Women in $\frac{1}{100}$'s of a second.

	Mean	Standard Deviation	Coefficient of Variation
Reaction Time to Sound	15·822	3·1962	20·20
„ „ Sight	19·350	3·6796	19·02

It is clear accordingly that there is much variability in reaction time, that to sight being absolutely greater, but relatively less than that to sound.

Tables I and II give the data from which Diagrams I and II were made. Table III (p. 199) gives in inches and millimetres the head breadth and head length by ages as read from the curve fitted by aid of a spline to the means. As it seems improbable that the large-headed women in later ages are in all these cases (30 in all) a true sample of existent population for ages above 53, the same head length and head breadth reductions were used. Table IV (p. 200) gives the standard deviations by years of the head breadths and head lengths in inches and millimetres; from these data the upper series of points on Diagrams I and II were plotted.

* Ratio of corrected means, showing cephalic index needs no correction for age.

TABLE I.

Galton's Anthropometric Data, Second Laboratory Series, Females. Head Breadth and Age.

Central Ages in Years.

Head Breadth in Inches. Central Values.	5·73	8	10	11·5	12·5	13·5	14·5	15·5	16·5	17·5	18·5	19·5	20·5	21·5	22·5	23·5	24·5	25·5	26·5	27·5	28·5	29·5	30·5	31·5	32·5	33·5	34·5	35·5	36·5	37·5	39	41	43	45	47	49·5	52·5	55·5	59·5	67·63	Totals
5·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5
5·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5
5·2	—	2	1	—	—	1·5	—	—	—	2	—	—	—	—	1	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9·5
5·3	4	1	2·5	3	2·5	2	1	—	3	1·5	1·5	·5	—	1	·5	1	3	—	2	—	—	—	—	—	—	—	—	·5	1	—	—	—	1	—	—	—	—	—	—	—	32·5
5·4	2	4·5	4	1·5	5	·5	1	1	2·5	2·5	1	5·5	4·5	2	4	4·5	2	3	3	2	·5	1·5	1	1·5	1	2	—	1·5	—	—	1	—	1	2	·5	—	—	—	—	—	69·5
5·5	4	5	5·5	5	5·5	9·5	3	6·5	7·5	11·5	6·5	9	10·5	13·5	8	7	6·5	12	7	6·5	4·5	3·5	7·5	2	2	5	—	2	·5	2·5	2	2	1	4	3·5	1	—	—	—	—	192·5
5·6	1	4	5·5	9·5	7	10	7·5	9·5	13·5	13·5	11·5	11·5	18	14·5	18·5	13	17·5	11	5	4	8	6	10·5	5	6	3	5	1·5	8·5	2·5	4	4	4·5	1·5	4·5	·5	2·5	3·5	—	—	286·5
5·7	1	1·5	10	5	10	8	12·5	6	20	20·5	25·5	24	21	18·5	27	22	11·5	15·5	12	12·5	15	15	11	9·5	4	5·5	5	7	4	5	6	7·5	5	4	5·5	4	3·5	4	1	—	405·5
5·8	1	2·5	3	—	6	4	6·5	18·5	10·5	10	14	21	18	24	18	16	12·5	11·5	14	10	15	8·5	7	6	4	4·5	6·5	4·5	4·5	6·5	7·5	10·5	7	3·5	5·5	5·5	3·5	2·5	2·5	1	337
5·9	—	1·5	3·5	1	5	3·5	6	5·5	10·5	10·5	13	18·5	18·5	16	14·5	24	16·5	11	13	8	8	3·5	8·5	8·5	4	5·5	6·5	5	·5	5	6·5	11·5	3·5	4	2·5	4·5	4	1·5	·5	2	295·5
6·0	—	—	—	1	2	4	3·5	1·5	4·5	5	5·5	11	9·5	5·5	6·5	2·5	12·5	9	3	8	4	3	5	5·5	3·5	2·5	3	1·5	—	3	4·5	3	4	2	1	1·5	2·5	3·5	2	3	152·5
6·1	—	—	—	1	—	—	—	1·5	4·5	1	2	2·5	3	1	1	1	4·5	3	—	2	4	—	·5	1	·5	—	2	2·5	—	·5	·5	1·5	—	—	—	—	1	—	1	1	44
6·2	—	—	—	—	—	—	—	—	·5	—	·5	2	—	—	1	2	·5	1	2	—	1	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	1	13·5
6·3	—	—	—	—	—	—	—	—	—	—	1	·5	—	—	1	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	·5	1	—	1	—	—	—	7
6·4	—	—	—	—	—	—	—	·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	—	—	—	—	—	1
6·5	—	—	—	—	—	—	—	·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5
6·6	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
6·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
6·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
6·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	13	22	35	27	44	43	42	51	77	78	82	106	103	96	101	94	88	78	61	54	60	41	53	39	25	28	29	26	19	25	32	41	28	22	24	17	18	15	7	8	1852

Mean Head Breadth 5·736 inches or 145·68 mm. Standard Deviation ·1911 inches or 4·8549 mm.

TABLE II.

Galton's Anthropometric Data, Second Laboratory Series, Females. Head Length and Age.

Central Ages in Years.

Head Length in Inches. Central Values.	5·73	8	10	11·5	12·5	13·5	14·5	15·5	16·5	17·5	18·5	19·5	20·5	21·5	22·5	23·5	24·5	25·5	26·5	27·5	28·5	29·5	30·5	31·5	32·5	33·5	34·5	35·5	36·5	37·5	39	41	43	45	47	49·5	52·5	55·5	59·5	67·63	Totals
6·4	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
6·5	—	1·5	—	1	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5·5
6·6	1	1·5	·5	—	—	1	—	—	—	—	—	—	—	1	—	—	—	·5	—	—	·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
6·7	4	1	·5	1	—	—	—	—	3	1	—	1	1	—	·5	2	1	·5	·5	1	·5	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	20·5
6·8	2	1	3	2·5	1	—	2	2	2·5	1	1·5	3	1	1·5	2·5	—	3	·5	·5	2	2	·5	—	—	1	—	2	1	1	3	1	1	1	—	1	—	—	—	—	—	47
6·9	1	5·5	3·5	1	1	3·5	·5	·5	4·5	2	6·5	1·5	2	4·5	6	4	3·5	1·5	2	2·5	1	2·5	5	3	2	2	2	2	1	—	—	1	1	—	—	—	1	—	—	—	80·5
7·0	1·5	3	11·5	3·5	4	6·5	3·5	8	4·5	6·5	7·5	8	7	7·5	9·5	7·5	7·5	11	3	4	1	3·5	3·5	1	1	4·5	1	—	1	1·5	1	4	4·5	2	3	1	—	—	—	—	158·5
7·1	1·5	1·5	3·5	4·5	10·5	8·5	9	5	7·5	8·5	6·5	13	13	9·5	15	9	11·5	7·5	8·5	2	9	4·5	4	4	3·5	4	3·5	2	2·5	1·5	2·5	7·5	1·5	2	2·5	1	2	—	—	1	214
7·2	1	3	5	4·5	9·5	7·5	10·5	5	16	10·5	9	12·5	16	20	17	17	13	14	7·5	7·5	8	6·5	7	9·5	6	1·5	4	5	3	2	6	7·5	4	1·5	6·5	3	3	2	—	1	293
7·3	—	2	3·5	1·5	6	4·5	4	13	8	15	8·5	19·5	20·5	13·5	9·5	19	13	9·5	12·5	12	10·5	5·5	6·5	7·5	2·5	4	1·5	5·5	4·5	5	11	7	4	4·5	4·5	2·5	1·5	·5	—	1	284·5
7·4	—	1	1	2·5	3	5·5	5	8	8·5	12·5	14	10·5	9·5	11	10	11·5	15·5	12	7·5	5·5	7	9	9·5	4	3·5	4	7	3·5	1·5	5·5	6	6	3	4	·5	3·5	3·5	1·5	1	—	237·5
7·5	—	—	1	3	3·5	2·5	1·5	1·5	7	3·5	11·5	8·5	9	10	7	7	4	3·5	3·5	2·5	8·5	3	4·5	4·5	3·5	2	2	3	3·5	3	·5	1	4	3	1	4	3·5	2·5	2·5	1	150·5
7·6	—	—	—	—	2·5	2	1	2	4·5	3·5	5·5	3·5	2·5	3·5	5·5	5	4	5·5	3·5	2·5	4	3	5	2	1	1·5	1	2·5	—	1·5	3	3·5	—	2	1	—	·5	5	2	2·5	97·5
7·7	—	—	—	—	—	·5	2	—	1	3	2·5	6	5·5	3	·5	1	4	—	4	1·5	3	1	1·5	·5	—	1	—	·5	—	2	—	1	3	—	1	1	—	·5	·5	1·5	52·5
7·8	—	—	—	—	—	—	1	—	1	—	—	6·5	·5	1	—	—	—	3	—	—	2	1	·5	—	—	2·5	2	—	—	—	—	·5	1	—	—	—	1	1	—	—	24·5
7·9	—	—	—	—	—	—	—	—	·5	—	—	·5	·5	1	—	1	2	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	1	—	1	2	—	—	11·5
8·0	—	—	—	—	—	—	—	—	·5	—	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	—	—	—	—	4
8·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	—	—	—	—	2·5
8·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	·5
8·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·5	—	·5
Totals	12	21	34	26	41	42	40	46	69	67	74	95	88	88	84	86	82	69	53	44	57	40	48	37	25	27	27	25	18	25	31	40	27	19	23	16	17	15	7	8	1693

Mean Head Length 7·259 inches or 184·38 mm. Standard Deviation ·2454 inches or 6·2320 mm.

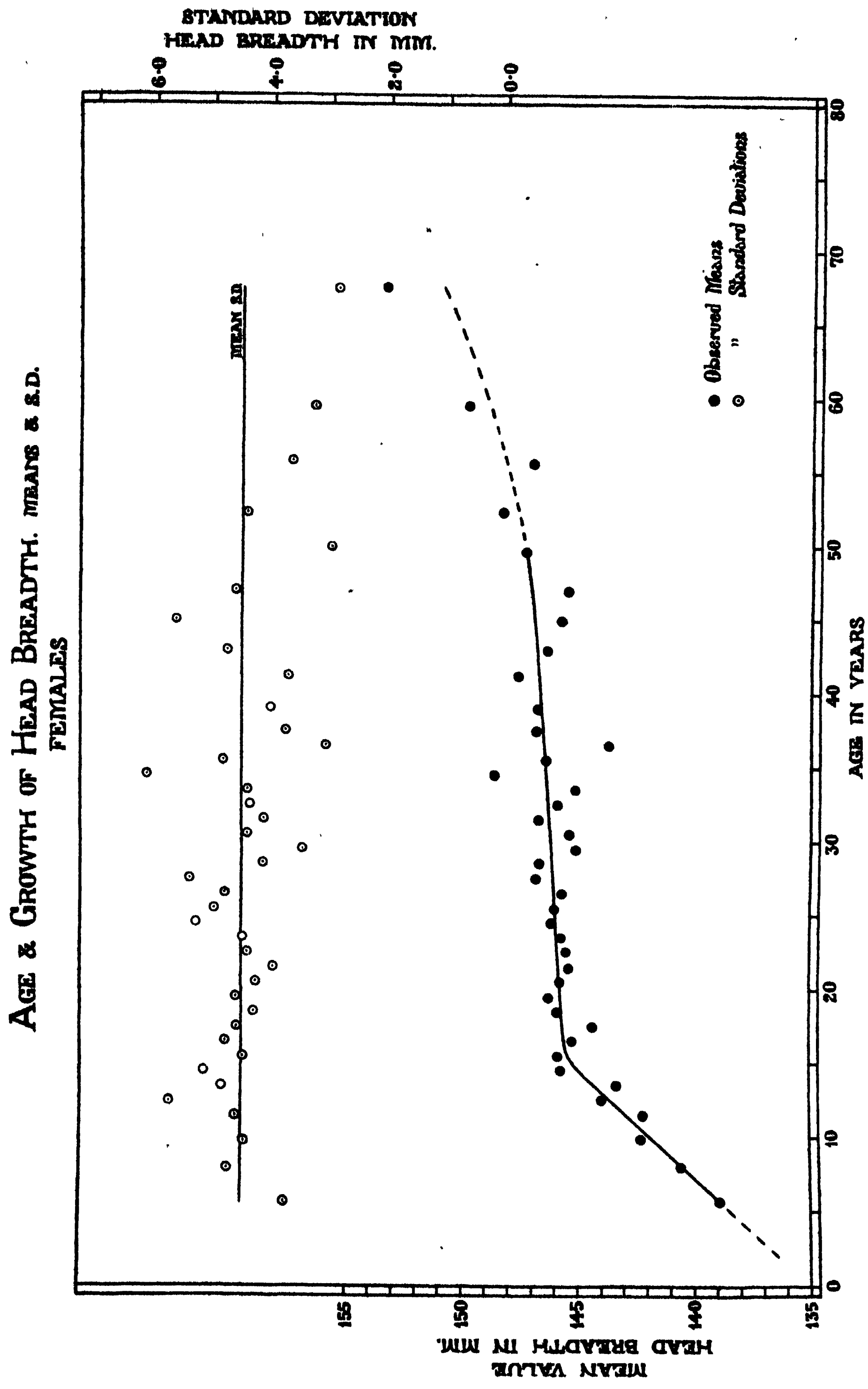

AGE & GROWTH OF HEAD BREADTH. MEANS & S.D.
FEMALES
STANDARD DEVIATION
HEAD BREADTH IN MM.
6·0
4·0
2·0
0·0
MEAN S.D.
● Observed Means
⊙ " Standard Deviations
MEAN VALUE
HEAD BREADTH IN MM.
155
150
145
140
135
0
10
20
30
40
50
60
70
80
AGE IN YEARS

Diagram I.

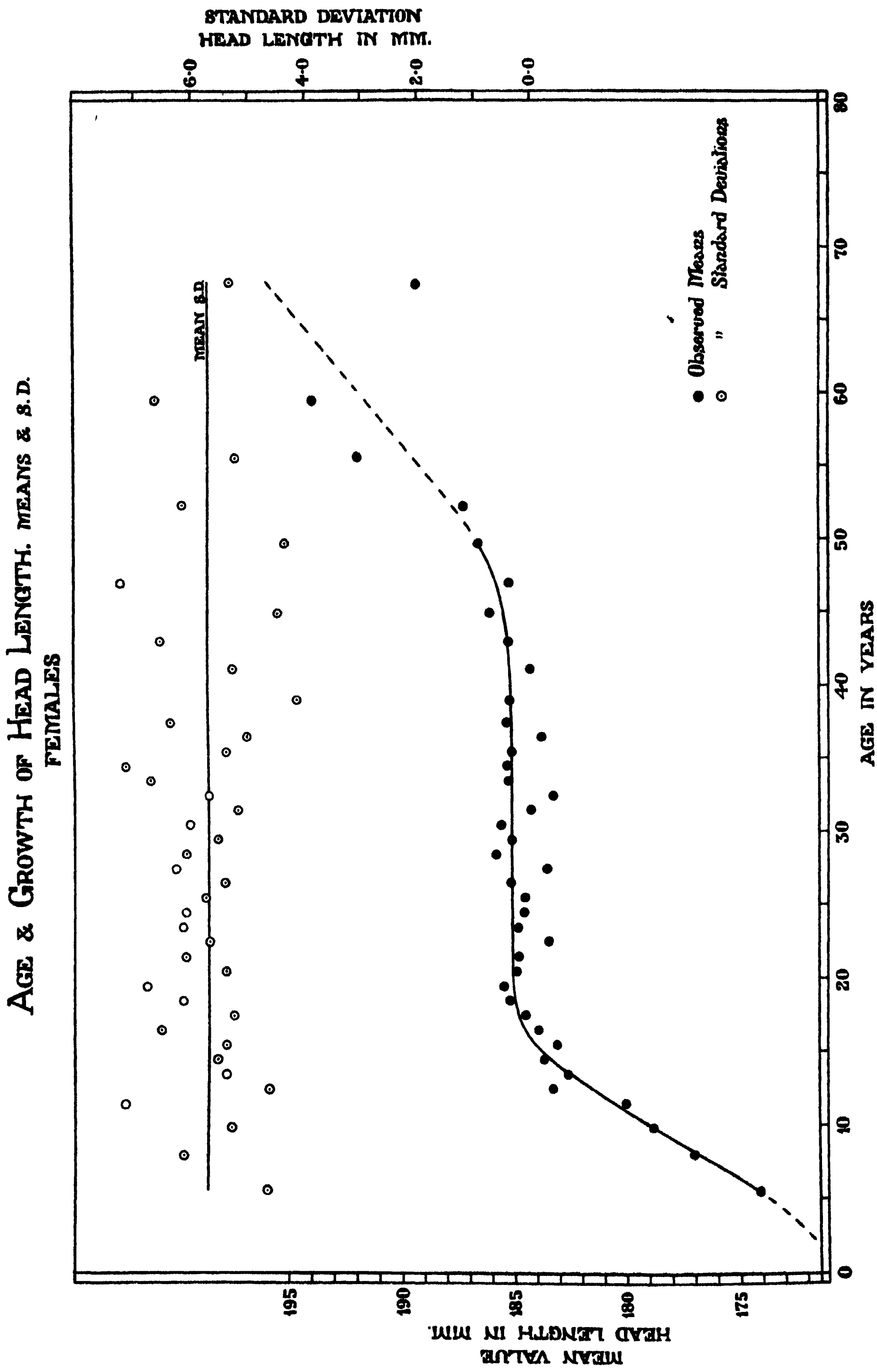

AGE & GROWTH OF HEAD LENGTH. MEANS & S.D.
FEMALES
STANDARD DEVIATION
HEAD LENGTH IN MM.
6·0
4·0
2·0
0·0
MEAN S.D.
● Observed Means
⊙ " Standard Deviations
0
10
20
30
40
50
60
70
80
AGE IN YEARS
195
190
185
180
175
MEAN VALUE
HEAD LENGTH IN MM.

Diagram II.

TABLE III.

Head Breadth and Head Length: read from Curve for each Age.

Central Age	Inches		Millimetres		Central Age	Inches		Millimetres	
	Breadth	Length	Breadth	Length		Breadth	Length	Breadth	Length
3·5	5·41	6·73	137·4	170·9	*29·5*	5·76	7·29	146·3	185·2
4·5	5·44	6·80	138·2	172·7	*30·5*	5·76	7·29	146·4	185·2
5·5	5·46	6·84	138·7	173·7	*31·5*	5·76	7·29	146·4	185·2
6·5	5·49	6·89	139·4	175·0	*32·5*	5·76	7·29	146·4	185·2
7·5	5·52	6·93	140·2	176·0	*33·5*	5·76	7·29	146·5	185·2
8·5	5·55	6·98	141·0	177·3	*34·5*	5·76	7·29	146·5	185·2
9·5	5·57	7·02	141·5	178·3	*35·5*	5·77	7·29	146·6	185·2
10·5	5·60	7·07	142·2	179·6	*36·5*	5·77	7·29	146·6	185·2
11·5	5·63	7·11	143·0	180·6	*37·5*	5·77	7·29	146·6	185·2
12·5	5·65	7·14	143·5	181·4	*38·5*	5·77	7·29	146·6	185·2
13·5	5·68	7·18	144·3	182·4	*39·5*	5·77	7·29	146·6	185·2
14·5	5·71	7·22	145·0	183·4	*40·5*	5·77	7·29	146·7	185·3
15·5	5·73	7·24	145·5	183·9	*41·5*	5·78	7·29	146·8	185·3
16·5	5·74	7·26	145·8	184·4	*42·5*	5·78	7·29	146·8	185·3
17·5	5·74	7·27	145·8	184·7	*43·5*	5·78	7·29	146·9	185·4
18·5	5·74	7·28	145·9	184·9	*44·5*	5·78	7·30	147·0	185·4
19·5	5·74	7·28	146·0	184·9	*45·5*	5·78	7·30	147·0	185·4
20·5	5·74	7·28	146·1	184·9	*46·5*	5·79	7·31	147·1	185·7
21·5	5·75	7·28	146·1	185·0	*47·5*	5·79	7·32	147·1	185·9
22·5	5·75	7·28	146·1	185·0	*48·5*	5·79	7·33	147·2	186·2
23·5	5·75	7·28	146·1	185·0	*49·5*	5·80	7·34	147·3	186·4
24·5	5·75	7·28	146·2	185·1	*50·5*	5·80	7·36	147·4	186·9
25·5	5·75	7·28	146·2	185·1	*51·5*	5·81	7·38	147·6	187·5
26·5	5·75	7·28	146·2	185·1	*52·5*	5·81	7·39	147·7	187·7
27·5	5·75	7·29	146·3	185·2	*53·5*	5·82	7·41	147·8	188·2
28·5	5·75	7·29	146·3	185·2	Above *53·5*	5·86	7·55	148·8	191·8

Tables V to X inclusive give the correlation tables of head size and shape with reaction to sound and sight. The method of determining this reaction is described by Koga and Morant*. We find the following values for the coefficients of correlation and the correlation ratios.

Head Breadth, Reaction Time to Sound	− ·0787 ± ·0156	·1193	·014223	·005995
Head Length ,, ,, ,,	− ·0188 ± ·0165	·1058	·011202	·008353
Cephalic Index ,, ,, ,,	− ·0112 ± ·0165	·0608	·003699	·010180
Head Breadth, Reaction Time to Sight	− ·0353 ± ·0157	·0649	·004211	·005998
Head Length ,, ,, ,,	− ·0377 ± ·0165	·1049	·011003	·008358
Cephalic Index ,, ,, ,,	·0161 ± ·0165	·0567	·003214	·010186

There is a small negative but significant correlation of head breadth and reaction time to sound; with a smaller and insignificant value for head breadth and reaction time to sight. There is also a small correlation, negative between head length and head breadth and reaction to sight, but in all cases with the

* *Biometrika*, Vol. xv. p. 346.

TABLE IV.

Standard Deviations by Years of Head Breadths and Lengths in Inches and Millimetres.

Central Age	Inches		Millimetres	
	Breadth	Length	Breadth	Length
5·73	·151	·184	3·84	4·67
8	·189	·243	4·80	6·17
10	·178	·211	4·52	5·36
11·5	·183	·284	4·65	7·21
12·5	·230	·182	5·84	4·62
13·5	·193	·214	4·90	5·44
14·5	·205	·219	5·21	5·56
15·5	·180	·213	4·57	5·41
16·5	·191	·258	4·85	6·55
17·5	·183	·207	4·65	5·26
18·5	·171	·243	4·34	6·17
19·5	·184	·268	4·67	6·81
20·5	·171	·213	4·34	5·41
21·5	·159	·241	4·04	6·12
22·5	·177	·225	4·50	5·72
23·5	·179	·243	4·55	6·17
24·5	·211	·242	5·36	6·15
25·5	·198	·227	5·03	5·77
26·5	·190	·214	4·83	5·44
27·5	·216	·250	5·49	6·35
28·5	·165	·241	4·19	6·12
29·5	·138	·220	3·51	5·59
30·5	·177	·240	4·50	6·10
31·5	165	·205	4·19	5·21
32·5	173	·226	4·39	5·74
33·5	·176	·266	4·47	6·76
34·5	·244	·283	6·20	7·19
35·5	·192	·213	4·88	5·41
36·5	·121	·197	3·07	5·00
37·5	·150	·251	3·81	6·38
39	·159	·163	4·04	4·14
41	·149	·208	3·78	5·28
43	·191	·259	4·85	6·58
45	·224	·176	5·69	4·47
47	·182	·286	4·62	7·26
49·5	·119	·171	3·02	4·34
52·5	·176	·243	4·47	6·17
55·5	·146	·206	3·71	5·23
59·5	·130	·262	3·30	6·65
67·63	·113	·210	2·87	5·33

exception of head breadth and reaction to sound the value of r is not significant having regard to its probable error, and is therefore negligible.

Turning to the correlation ratio, we see that η^2 is only significant compared with $\bar{\eta}^2$ (the mean value of η^2 in samples from a population with zero correlation) in the case of head breadth and reaction time to sound. As far as cephalic

index is concerned the correlation coefficient is zero and the correlation ratio is insignificant for both reaction times, or mental agility is not associated with either brachycephaly or dolichocephaly.

In conclusion, while there is a small amount of correlation between head breadth and reaction time to sound, its intensity is so slight as to be of no prognostic value. In all other cases the intensity of correlation is of no significance, and the reaction times in this agree with the measurements of other mental characters.

I wish to express my gratitude to Prof. Karl Pearson not only for the use of this material, but also for his valuable suggestions; and to Miss Ida McLearn for making Diagrams I and II.

TABLE V.

Galton's Anthropometric Data, Second Laboratory Series, Females. Reaction Time to Sound and Head Breadth, both Corrected for Age.

Central Values. Head Breadth in Inches.

Central Values of Reaction Time to Sound in $\frac{1}{100}''$.

	5·105	5·205	5·305	5·405	5·505	5·605	5·705	5·805	5·905	6·005	6·105	6·205	6·305	6·405	6·505	6·605	6·705	6·805	6·905	Totals
4·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
5·45	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
6·45	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7·45	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	2
8·45	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	2
9·45	—	—	1	—	3	1	4	6	—	2	1	1	—	—	—	—	—	—	—	19
10·45	—	—	—	1	6	10	8	10	7	5	1	1	—	—	—	—	—	—	—	49
11·45	—	—	1	3	7	14	22	13	11	14	4	—	—	—	—	—	—	—	—	89
12·45	—	—	1	4	6	11	35	25	23	9	4	—	—	—	—	—	—	—	—	118
13·45	—	—	1	3	10	28	33	34	28	10	4	2	—	—	—	—	—	—	—	153
14·45	—	—	1	8	27	42	76	70	54	24	9	3	4	—	1	—	—	—	—	319
15·45	—	1	3	14	35	41	57	54	62	38	8	1	2	—	—	—	—	—	—	316
16·45	—	1	1	7	22	44	58	47	37	18	3	7	1	—	—	—	—	—	—	246
17·45	—	—	1	5	16	23	30	27	26	15	7	—	—	—	—	—	—	—	1	151
18·45	—	—	1	4	10	23	27	14	18	9	6	2	—	—	—	—	—	—	—	114
19·45	—	—	2	3	11	9	19	24	16	4	3	—	—	—	—	1	—	—	—	92
20·45	—	1	1	3	9	10	17	12	16	6	—	1	—	—	—	—	—	—	—	76
21·45	—	—	—	—	2	4	2	3	5	1	—	—	—	—	—	—	—	—	—	17
22·45	—	1	1	—	2	1	8	5	1	—	—	—	—	—	—	—	—	—	—	19
23·45	—	1	—	2	1	2	1	5	1	1	—	—	—	—	—	—	1	—	—	15
24·45	1	—	—	1	1	2	4	2	2	—	—	—	—	—	—	—	—	—	—	13
25·45	—	—	—	1	—	2	2	—	1	—	—	—	—	—	—	—	—	—	—	6
26·45	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	2
27·45	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	2
28·45	—	—	—	—	1	—	2	1	1	—	—	—	—	—	—	—	—	—	—	5
29·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
30·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
31·45	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
32·45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
33·45	—	—	1	—	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	4
34·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35·45	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	1	5	16	60	172	271	409	354	312	156	50	18	7	0	1	1	1	0	1	1835

TABLE VI.

Galton's Anthropometric Data, Second Laboratory Series, Females.
Reaction Time to Sound and Head Length, both Corrected for Age.

Central Values. Head Length in Inches.

Central Values of Reaction Time to Sound in $\frac{1''}{100}$.

	6·505	6·605	6·705	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	7·905	8·005	8·105	Totals
4·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
5·45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
6·45	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
7·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
8·45	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	2
9·45	—	—	—	—	—	3	4	4	4	—	—	1	—	1	—	—	—	17
10·45	—	1	—	—	2	5	2	8	5	8	3	1	1	1	—	—	—	37
11·45	—	—	—	—	5	7	14	15	17	9	8	4	2	2	—	—	—	83
12·45	—	—	2	1	1	6	14	20	25	10	13	15	3	2	—	1	—	113
13·45	—	—	1	3	2	3	21	28	33	27	15	7	5	1	—	1	—	147
14·45	2	—	3	10	8	17	41	50	57	44	30	14	11	8	1	1	—	297
15·45	1	1	3	9	20	26	26	47	65	41	23	18	10	2	3	—	1	296
16·45	—	—	—	3	16	24	25	29	50	41	24	14	8	2	4	1	1	242
17·45	—	1	1	1	6	15	22	27	20	23	11	10	6	3	—	—	—	146
18·45	—	—	—	1	6	9	11	21	20	13	8	6	4	1	—	1	—	101
19·45	—	—	—	4	4	9	6	19	6	12	7	3	5	2	-	1	—	78
20·45	—	—	4	2	3	4	9	8	9	14	5	2	2	—	1	—	—	63
21·45	—	—	1	—	—	—	1	4	3	3	1	1	—	—	—	—	—	14
22·45	—	—	—	—	1	1	2	3	3	3	—	2	—	—	—	—	—	15
23·45	—	—	—	—	—	1	1	2	2	—	—	1	—	—	—	—	—	7
24·45	—	—	—	—	—	—	1	—	4	1	—	—	—	—	1	—	—	7
25·45	—	—	—	1	—	—	—	2	—	—	—	—	—	—	—	—	—	3
26·45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
27·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
28·45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
29·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
30·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
31·45	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
32·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
33·45	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	2
34·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35·45	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	3	3	16	35	75	131	200	290	325	249	149	99	57	26	10	6	2	1676

TABLE VII.

Galton's Anthropometric Data, Second Laboratory Series, Females.
Reaction Time to Sight and Head Breadth, both Corrected for Age.

Central Values. Head Breadth in Inches.

Central Values of Reaction Time to Sight in $\frac{1}{100}''$.	5·105	5·205	5·305	5·405	5·505	5·605	5·705	5·805	5·905	6·005	6·105	6·205	6·305	6·405	6·505	6·605	6·705	6·805	6·905	Totals
4·45	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
5·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
6·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
7·45	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
8·45	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
9·45	—	—	—	—	—	1	3	3	—	1	—	—	—	—	—	—	—	—	—	8
10·45	—	—	—	1	4	5	3	3	4	3	1	1	1	—	—	—	—	—	—	26
11·45	—	—	—	—	2	1	4	3	3	2	1	—	—	—	—	—	—	—	—	16
12·45	—	—	1	—	2	1	5	9	1	2	1	—	—	—	—	—	—	—	—	22
13·45	—	—	—	1	—	5	2	3	2	4	1	—	—	—	—	—	—	—	—	18
14·45	—	—	2	2	7	8	14	14	7	7	1	—	—	—	—	—	—	—	—	62
15·45	—	1	2	3	4	17	27	27	12	8	1	—	—	—	—	—	—	—	—	102
16·45	—	1	—	—	14	24	42	24	33	10	7	—	—	—	1	—	—	—	—	156
17·45	—	—	—	9	20	21	36	42	40	20	11	5	2	—	—	—	—	—	—	206
18·45	—	1	3	7	20	32	49	46	36	18	6	5	1	—	—	—	—	—	—	224
19·45	1	—	1	6	23	39	57	51	43	21	5	2	2	—	—	—	—	—	—	251
20·45	—	—	4	13	23	40	63	40	45	30	6	—	1	—	—	—	—	—	—	265
21·45	—	—	—	3	11	21	30	22	23	8	2	2	—	—	—	—	—	—	—	122
22·45	—	—	1	2	9	11	16	14	23	4	2	—	—	—	—	1	—	—	—	83
23·45	—	—	—	4	10	10	16	15	10	1	—	2	—	—	—	—	—	—	—	68
24·45	—	1	1	4	10	12	15	12	17	8	3	—	—	—	—	—	—	—	—	83
25·45	—	—	—	1	6	10	12	10	4	6	2	1	—	—	—	—	1	—	—	53
26·45	—	—	—	—	2	5	2	2	3	1	—	—	—	—	—	—	—	—	1	16
27·45	—	—	—	1	—	2	3	3	4	—	—	—	—	—	—	—	—	—	—	13
28·45	—	—	—	—	—	1	1	4	—	1	—	—	—	—	—	—	—	—	—	7
29·45	—	1	—	1	—	2	2	2	—	1	—	—	—	—	—	—	—	—	—	9
30·45	—	—	—	—	1	2	1	2	1	—	—	—	—	—	—	—	—	—	—	7
31·45	—	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	3
32·45	—	—	—	—	1	—	2	2	—	—	—	—	—	—	—	—	—	—	—	5
33·45	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
34·45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
35·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
36·45	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	2
37·45	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	5	16	60	172	271	408	354	312	156	50	18	7	0	1	1	1	0	1	1834

TABLE VIII.

Galton's Anthropometric Data, Second Laboratory Series, Females.
Reaction Time to Sight and Head Length, both Corrected for Age.

Central Values. Head Length in Inches.

Central Values of Reaction Time to Sight in $\frac{1}{100}''$.	6·505	6·605	6·705	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	7·905	8·005	8·105	Totals
4·45	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
5·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
6·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
7·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
8·45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
9·45	—	—	—	1	—	1	2	1	—	—	2	—	—	—	—	—	—	7
10·45	—	—	—	—	2	1	2	4	2	4	1	—	2	—	—	—	—	18
11·45	—	—	—	—	1	1	2	1	6	2	—	—	1	—	1	—	—	15
12·45	—	—	1	—	—	1	3	7	3	3	—	—	1	—	—	—	—	19
13·45	—	—	—	—	—	3	2	6	2	3	—	1	—	—	—	—	—	17
14·45	—	1	1	3	—	4	7	10	9	5	6	4	1	—	—	—	—	51
15·45	—	—	—	4	2	2	16	13	21	12	7	5	1	2	—	1	—	86
16·45	—	—	1	2	8	6	16	33	34	26	13	8	4	1	—	—	—	152
17·45	1	—	1	1	6	12	17	30	40	30	27	15	9	6	1	1	—	197
18·45	1	1	—	3	5	22	26	26	45	34	18	14	7	4	4	1	1	212
19·45	1	1	2	4	14	22	34	35	46	27	29	16	9	2	1	1	—	244
20·45	—	—	2	4	11	17	28	44	44	42	21	13	11	4	2	1	—	244
21·45	—	—	1	3	4	12	8	24	24	25	9	4	3	—	—	—	1	118
22·45	—	—	3	2	8	6	11	13	12	15	3	1	2	2	—	1	—	79
23·45	—	—	1	5	4	3	8	10	10	4	7	7	2	1	—	—	—	62
24·45	—	—	2	2	4	5	7	12	13	9	1	6	3	2	1	—	—	67
25·45	—	—	—	1	2	8	5	8	8	5	3	3	1	1	—	—	—	45
26·45	—	—	—	—	1	3	1	5	2	2	1	—	—	—	—	—	—	15
27·45	—	—	—	—	—	—	2	2	2	1	1	—	—	—	—	—	—	8
28·45	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	2
29·45	—	—	—	—	1	1	1	1	—	—	—	1	—	—	—	—	—	5
30·45	—	—	—	—	—	—	1	1	—	—	—	1	—	—	—	—	—	3
31·45	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	2
32·45	—	—	—	—	—	—	—	3	—	—	—	—	—	—	—	—	—	3
33·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
34·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
36·45	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
37·45	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
Totals	3	3	16	35	75	131	199	290	325	249	149	99	57	26	10	6	2	1675

TABLE IX.

Galton's Anthropometric Data, Second Laboratory Series, Females. Cephalic Index and Reaction Time to Sound, the latter Corrected for Age.

Central Values. Cephalic Index.

Central Values of Reaction Time to Sound in $\frac{1}{100}''$.

	69·05	70·05	71·05	72·05	73·05	74·05	75·05	76·05	77·05	78·05	79·05	80·05	81·05	82·05	83·05	84·05	85·05	86·05	87·05	88·05	89·05	90·05	91·05	92·05	93·05	94·05	95·05	96·05	97·05	98·05	99·05	Totals
5·45	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
6·45	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
8·45	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
9·45	—	—	—	—	—	—	2	—	—	2	5	2	2	—	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
10·45	—	—	—	—	—	2	4	3	2	7	3	3	6	2	3	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37
11·45	—	—	1	—	—	1	4	7	9	13	13	10	11	8	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	83
12·45	—	1	—	—	2	3	7	17	13	14	16	9	14	8	3	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	113
13·45	—	—	1	1	3	8	4	10	15	26	27	17	9	16	2	4	1	1	1	—	—	—	—	—	1	—	—	—	—	—	—	147
14·45	—	—	1	1	5	15	10	24	32	37	46	28	26	24	19	15	5	4	2	1	1	—	—	—	—	—	—	—	—	—	—	296
15·45	—	—	1	2	7	7	13	26	32	33	44	34	33	17	20	9	8	4	4	1	—	—	—	—	—	—	—	—	—	—	—	295
16·45	—	—	—	2	6	7	14	35	29	23	23	29	30	14	10	9	4	3	2	—	—	—	—	—	—	—	—	—	—	—	—	240
17·45	—	—	1	—	1	2	12	16	14	14	24	16	15	14	5	7	4	—	—	—	—	—	—	—	—	—	—	—	—	—	1	146
18·45	1	—	1	—	2	4	2	5	11	22	14	11	10	6	4	4	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	101
19·45	—	—	2	—	—	5	5	5	10	5	15	6	11	4	5	1	2	1	—	—	—	—	—	1	—	—	—	—	—	—	—	78
20·45	—	1	—	—	—	1	2	4	9	10	11	3	9	4	6	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
21·45	—	—	—	1	—	—	1	1	—	2	2	4	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
22·45	—	—	—	—	—	—	—	2	1	3	6	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
23·45	—	—	—	—	1	—	—	1	1	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
24·45	—	—	—	—	—	—	—	1	1	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
25·45	—	—	—	—	—	—	—	—	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
26·45	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
27·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
28·45	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
29·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
30·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
31·45	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
32·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
33·45	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
34·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	2	8	7	27	56	81	157	179	217	255	173	180	119	85	58	34	15	10	2	1	0	0	1	1	0	0	0	0	0	1	1670

TABLE X.

Galton's Anthropometric Data, Second Laboratory Series, Females. Cephalic Index and Reaction Time to Sight, the latter Corrected for Age.

Central Values. Cephalic Index.

Central Values of Reaction Time to Sight in $\frac{1}{100}''$.

	69·05	70·05	71·05	72·05	73·05	74·05	75·05	76·05	77·05	78·05	79·05	80·05	81·05	82·05	83·05	84·05	85·05	86·05	87·05	88·05	89·05	90·05	91·05	92·05	93·05	94·05	95·05	96·05	97·05	98·05	99·05	Totals
4·45	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
5·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
6·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
7·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
8·45	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
9·45	—	—	—	—	—	—	1	—	—	1	—	2	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
10·45	—	—	—	—	—	1	—	2	1	4	4	4	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18
11·45	—	—	—	—	—	—	3	—	1	2	1	3	2	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
12·45	—	—	1	—	—	1	1	—	—	4	2	2	4	2	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	19
13·45	—	—	—	—	—	1	—	1	—	1	4	3	3	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
14·45	—	—	—	—	1	—	3	10	2	7	7	3	4	6	6	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	51
15·45	—	1	—	—	—	2	3	9	16	9	19	2	7	8	5	4	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	86
16·45	—	—	—	—	1	4	6	13	17	22	25	15	21	15	5	6	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	152
17·45	—	—	—	2	8	8	11	16	25	28	29	15	20	10	9	2	5	4	2	2	—	—	—	—	—	—	—	—	—	—	—	196
18·45	1	—	—	1	5	9	9	17	25	31	32	22	22	13	12	7	2	2	1	—	—	—	—	—	—	—	—	—	—	—	—	211
19·45	—	—	2	1	4	6	12	30	26	23	34	32	22	14	12	12	8	2	3	—	1	—	—	—	—	—	—	—	—	—	—	244
20·45	—	—	4	—	4	11	8	24	28	38	33	28	22	17	8	12	1	2	1	—	—	—	—	—	—	—	—	—	—	—	—	242
21·45	—	—	1	—	3	3	7	12	9	12	18	18	16	10	5	1	3	—	—	—	—	—	—	—	1	—	—	—	—	—	—	118
22·45	—	1	—	1	—	2	3	7	4	10	12	8	10	3	7	4	4	2	—	—	—	—	—	—	—	—	—	—	—	—	—	79
23·45	—	—	—	2	—	4	4	4	6	7	9	5	7	6	4	1	3	—	—	—	—	—	—	1	—	—	—	—	—	—	—	62
24·45	—	—	—	—	1	2	6	4	12	8	5	5	7	5	5	3	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	66
25·45	—	—	—	—	—	1	3	3	4	4	12	2	9	2	2	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	44
26·45	—	—	—	—	—	—	—	1	1	2	4	2	1	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
27·45	—	—	—	—	—	—	—	2	—	2	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	8
28·45	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
29·45	—	—	—	—	—	—	—	1	1	—	—	—	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
30·45	—	—	—	—	—	—	—	—	1	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
31·45	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
32·45	—	—	—	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
33·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
34·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
36·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
37·45	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	2	8	7	27	56	81	157	179	217	255	173	180	118	85	58	34	15	10	2	1	0	0	1	1	0	0	0	0	0	1	1669

ON THE DEGREE OF RELATIONSHIP BETWEEN HEAD MEASUREMENTS AND REACTION TIME TO SIGHT AND SOUND.

By G. E. HARMON, M.D., School of Medicine of Western Reserve University.

PREVIOUS investigations from the Biometric Laboratory* have indicated that only the smallest relationship between head measurements and intelligence exists. For women it has also been shown by Musselman† that the relationship between head measurements and reaction times to sight and sound is very small. Indeed the degree of association in the cases studied was too small to be of any real value for prognostic purposes.

The present study deals with the data for males in Francis Galton's Second Anthropometric Laboratory Series which were put at my disposal by the Director of the Galton Laboratory, University of London. About 5600 individuals are included in these data. The measurements were taken in inches, and though the computations involved in this paper were made in this unit, I have for convenience expressed many of the constants in the now more usual millimetres. In the case of head length only those individuals have been included on whom the measurement was taken from the glabella to the occipital point. At the beginning of the series head length was measured from the nasion to the occipital point. All individuals so measured have been excluded.

Both head measurements have been corrected for age by the method described by Holzinger (*Biometrika*, Vol. XVI. p. 140), taking age 40·5 as the standard age to which all other ages were reduced. In Diagrams I and II will be found curves of the means and the standard deviations for head breadths and lengths by years. The dotted portion of the curve of means indicates the years for which the numbers of observations were quite small. In the case of breadth the formula used in making the corrections was $y_s = \bar{y}_s + (y_t - \bar{y}_t)\dfrac{\sigma_s}{\sigma_t}$, in which s refers to the standard age 40·5 and t to any other age, while y_s is the value of any observation y_t reduced to the standard age. The values for σ_s and values for σ_t are given in Table VII and for $\bar{y}_s$ and $\bar{y}_t$ in Table VI. The standard deviations of head lengths were found to be well represented by a horizontal straight line corresponding to ·244 inches, the value of the weighted average of the standard deviations. For this case $\dfrac{\sigma_s}{\sigma_t}$ becomes unity, and the formula for correcting for age becomes $y_s = \bar{y}_s + (y_t - \bar{y}_t)$. The values for $\bar{y}_s$ and $\bar{y}_t$ may be found in Table VI.

* *Biometrika*, Vol. v. p. 105.

† See present issue of *Biometrika*, pp. 195—206.

GROWTH OF HEAD BREADTH WITH AGE, MEANS & S.D.
MALES

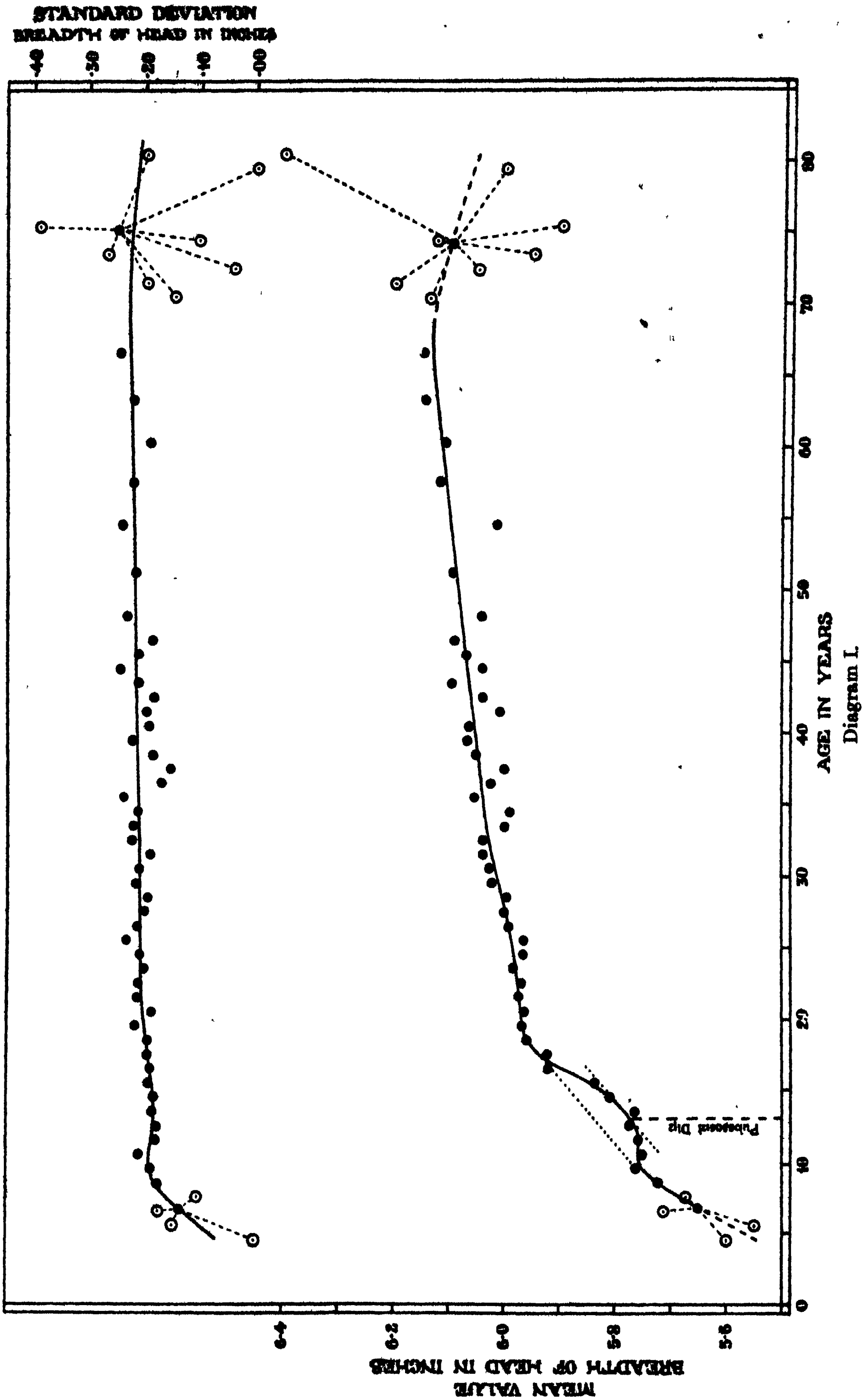
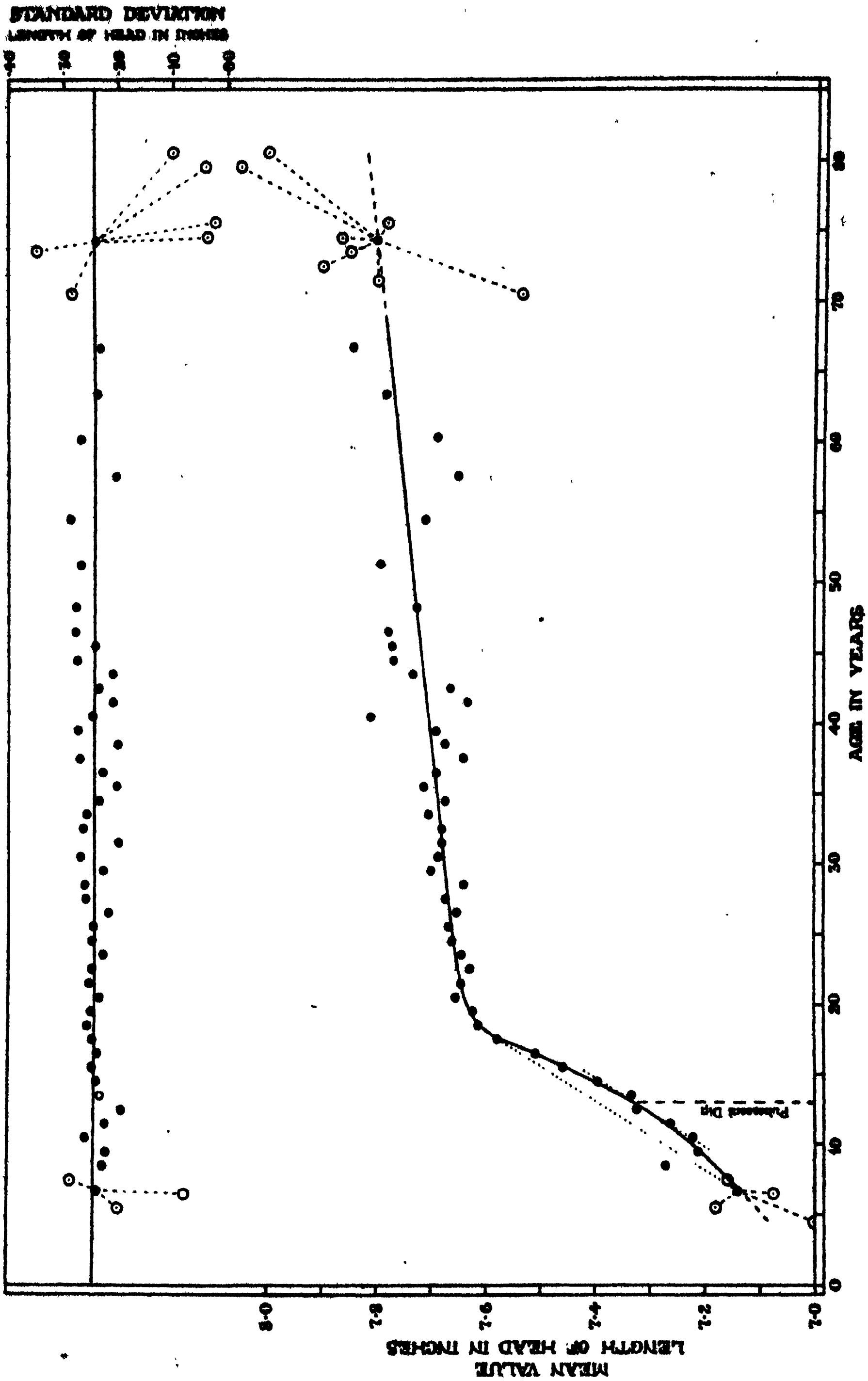


Diagram I.



Growth of Head Length with Age, Means & S.D. Males

Diagram II.

Before correction the mean head breadth of 5615 individuals was 5·951 inches or 151·164 mm. and the mean head length of 4721 individuals was 7·594 inches or 192·898 mm. In Table I will be found a comparison of head measurements before and after correction.

TABLE I.

*Head Measurements before and after Correction for Age**.

Character	Before Correction for Age			After Correction for Age		
	Mean	Standard Deviation	Coefficient of Variation	Mean	Standard Deviation	Coefficient of Variation
Breadth	151·164 mm.	5·659 mm.	3·744	153·808 mm.	5·460 mm.	3·550
Length	192·898 mm.	7·062 mm.	3·661	195·563 mm.	6·261 mm.	3·202
Cephalic Index	78·346	2·975	3·797	[78·649]†	—	—

It will be noticed that the means after correction for age are higher, owing to the frequency being greater before than after age 40·5. The variability is decreased by the reduction of the measurements to a standard age as we should naturally expect.

In Table II will be found the means, standard deviations, and coefficients of variation for reaction times in men based upon the results obtained in the case of 5563 and 5564 individuals.

TABLE II.

*Reaction Times in Males in $\frac{1}{100}$ths of a Second**.

	Mean	Standard Deviation	Coefficient of Variation
Reaction Time to Sight	18·774	3·570	19·016
,, ,, Sound	15·583	2·984	19·149

It will be observed that the reaction time to sight is greater than that to sound and there is considerable variability exhibited by these characters. The method of determining these reaction times has been described by Koga and Morant‡. Comparing with Musselman's results, we see that there is very little difference between male and female reaction times, the male being slightly quicker than the female. In relative variability, both sexes have identical coefficients for

* Constants in each case were calculated from the distribution with the greatest frequency for that variate.

† Ratio of reduced means.

‡ *Biometrika*, Vol. xv. p. 347.

TABLE III. *Galton's Anthropometric Data, Second Laboratory Series, Males. Head Breadth and Age.*

Age in Years (with the same meaning as in Table V.)

Head Breadth in Inches. Central Values.	4–7	8	9	10	11	12	13	14	15	16	17	18	19	20
5·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5·2	—	—	—	—	—	—	2	—	—	—	—	0·5	0·5	—
5·3	—	—	2	3	1	2	—	—	0·5	0·5	—	0·5	0·5	1
5·4	2	0·5	0·5	4·5	2·5	1	6	3·5	3·5	1·5	0·5	2	3·5	0·5
5·5	4·5	4·5	3	2·5	8·5	6·5	19·5	16	12·5	5·5	8·5	4	7·5	3
5·6	4	4·5	4·5	9·5	10	17	22·5	24·5	16	13	20·5	11	11	15·5
5·7	6·5	4	8	10	22	31·5	34	34·5	50	23·5	45	41	35·5	30
5·8	4·5	4·5	8	12	23	20·5	36·5	49·5	41·5	45·5	43·5	60·5	62	56
5·9	1·5	3	8·5	7	10	18	34	35	40·5	64	68	83	73	82
6·0	—	2	5	6·5	7	11·5	23·5	35	33·5	48·5	71	95	89·5	100·5
6·1	—	1	1·5	4	4	4	6·5	11	17·5	36	44·5	60	70	49·5
6·2	—	—	—	—	1·5	4	0·5	3	7·5	13	26	33·5	42	33
6·3	—	—	—	—	0·5	—	1	2	3	8·5	10	22	16·5	23·5
6·4	—	—	—	—	—	—	—	0·5	1	4·5	3·5	2	8·5	6·5
6·5	—	—	—	—	—	—	—	0·5	1	1	2	4	2	3
6·6	—	—	—	1	—	—	—	—	—	—	—	0·5	1·5	—
6·7	—	—	—	—	—	—	—	—	—	—	—	1·5	0·5	—
6·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·6	—	—	—	—	—	—	—	—	—	—	—	—	1	—
Totals	23	24	41	60	90	116	186	215	228	265	343	421	425	404

Head Breadth in Inches. Central Values.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
5·0	—	—	—	—	—	0·5	—	—	—	—	—	—	—	—	—	—
5·1	—	—	—	—	—	0·5	—	—	—	—	—	—	—	—	—	—
5·2	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—
5·3	2	0·5	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—
5·4	3	0·5	—	1·5	—	—	—	1	0·5	—	0·5	0·5	—	1·5	0·5	—
5·5	6	5·5	3	4	3	1·5	1	—	1·5	1·5	—	2·5	1	1·5	0·5	—
5·6	5	6·5	6·5	2·5	6·5	4·5	6	2·5	—	2	1	—	2·5	2	1·5	—
5·7	28·5	28·5	15	14·5	15	11·5	9	8·5	12	3·5	1·5	4·5	4	1·5	2·5	2
5·8	40	35	26	24	18	20·5	13	14	6·5	10·5	8	4·5	5·5	11·5	6	9·5
5·9	56·5	36·5	46	30·5	26	30	21	20·5	14·5	18	12·5	9	13	7·5	14·5	8
6·0	61	63	47·5	36·5	25	36·5	33·5	20	13·5	14	19	13	10·5	16·5	10·5	6·5
6·1	46	37	24·5	21	22·5	26·5	23·5	22·5	19	15·5	18	15	10·5	12	8	11
6·2	30	22·5	23·5	15	9·5	20	14·5	12·5	15	11	10·5	6	6·5	9	7	9·5
6·3	20·5	10·5	14·5	7·5	7·5	7·5	8	6	5·5	6	3·5	7·5	5·5	5·5	6·5	5·5
6·4	9	6	5	4	5	4·5	2	2·5	4	5·5	4	3	2	0·5	3·5	—
6·5	3·5	4	1·5	1	3	—	2·5	2	3	2·5	2	2·5	—	—	4	—
6·6	—	1	—	2	—	1	—	—	—	—	—	—	—	1	0·5	—
6·7	—	0·5	—	—	—	2	1	—	—	—	—	—	—	—	0·5	—
6·8	—	0·5	0·5	—	1	—	—	—	—	—	—	—	1	—	—	—
6·9	—	—	0·5	—	—	—	—	—	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	311	258	214	164	143	167	135	112	95	90	81	68	62	70	66	52

Head Breadth in Inches. Central Values.	37	38	39	40	41	42	43	44	45	46	47–49	50–52	53–55	56–58	59–61	62–64	65–69	70–80	Totals
5·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5
5·1	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1·5
5·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4·0
5·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	14·5
5·4	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	0·5	43·0
5·5	1	—	—	—	1	—	—	1	—	—	1	—	0·5	—	—	—	—	—	143·0
5·6	—	—	1·5	—	1	0·5	—	2	—	—	4	—	1·5	—	—	—	—	—	243·0
5·7	1	3	1·5	1	1	2	2	1	0·5	1·5	5·5	2	2·5	2	1	—	3	1	569·0
5·8	3·5	2·5	1	6	5·5	5·5	1·5	2	2·5	1·5	5·5	6·5	3·5	2	2	4	—	—	775·0
5·9	11·5	5	4	4·5	8·5	3	2·5	4·5	4·5	2·5	7·5	6	4·5	5	3·5	3	2·5	—	974·0
6·0	13·5	9	6	8	8	10·5	3	8	4·5	6·5	16·5	15	6·5	4	5·5	4	3	6	1092·5
6·1	8·5	6·5	7	8	7	7	6·5	4	5·5	5·5	10	16	13	5·5	6·5	4·5	6	3·5	772·5
6·2	6	5·5	3·5	5·5	7	5·5	5·5	7·5	7·5	5	12·5	5·5	5	7·5	4·5	6·5	4·5	4·5	495·0
6·3	1	5	2	3	2	4	0·5	0·5	3	2·5	7·5	3	4	4·5	4	5	0·5	1·5	268·0
6·4	1	0·5	1·5	2·5	1	2	1·5	1·5	1	2	3	2·5	1·5	2	3	1	3·5	1·5	119·5
6·5	—	1	1	1·5	1	—	1·5	1	1	1	4	6	0·5	1·5	1	2·5	3	—	72·5
6·6	—	—	1	—	—	—	0·5	—	—	—	—	0·5	—	—	—	1·5	1	1	14·0
6·7	—	—	—	—	—	—	—	1	—	—	—	1	—	1	—	—	—	—	9·0
6·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3·0
6·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5
7·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1·0
Totals	47	38	30	40	43	40	25	34	31	28	77	64	44	35	31	32	27	20	5615·0

Mean Head Breadth 5·951 inches or 151·164 mm. Standard Deviation ·223 inches or 5·659 mm.

TABLE IV. *Galton's Anthropometric Data, Second Laboratory Series, Males. Head Length and Age.*

Age in Years (with the same meaning as in Table V.)

Head Length in Inches. Central Values.	4–7	8	9	10	11	12	13	14	15	16	17	18	19	20
6·6	—	—	—	—	1	—	1	—	—	—	—	—	—	—
6·7	—	—	1	0·5	—	—	—	1	1	0·5	—	0·5	—	—
6·8	1·5	—	1	1	2	0·5	2	—	—	1	—	1·5	1	—
6·9	2·5	1·5	2	4	3	1·5	6·5	4·5	1·5	3	2	2	1	—
7·0	5	4	4	10	7·5	5·5	9·5	10·5	8·5	1	5·5	3·5	2	1
7·1	4	2·5	4	7	9	10·5	16	16·5	15·5	10·5	8·5	7	5·5	4
7·2	2	2·5	4	8·5	9	22·5	25	23·5	16·5	14·5	10	12·5	13·5	10
7·3	3	3·5	8	5	14	18	27	23	19·5	31·5	22	21·5	22	17·5
7·4	—	4	4	3	15·5	16	26	25	30·5	29	38	42·5	30	36·5
7·5	2	1	2	1	9	14	23	42	36	35	46	52	51·5	41·5
7·6	—	3	1	5	1	7	16	24	29·5	40·5	53	47·5	59·5	44
7·7	—	1	1	2	0·5	5·5	11	10	18·5	32	42	55·5	57·5	57
7·8	1	—	—	2	2·5	1	3	6	10·5	19	31	44·5	35	55·5
7·9	—	—	—	—	—	—	—	3·5	11·5	6	15·5	29·5	23·5	28·5
8·0	—	—	—	—	—	—	1	3·5	1	5·5	14·5	20	23·5	20·5
8·1	—	—	—	—	—	—	—	—	—	3	3	9·5	6	9
8·2	—	—	—	—	—	—	—	—	—	—	4	3	2·5	3
8·3	—	—	—	—	—	—	—	—	1	—	0·5	1·5	1·5	1
8·4	—	—	—	—	—	—	—	—	—	—	0·5	—	1·5	1
8·5	—	—	—	—	—	—	—	—	—	—	—	—	1	—
8·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	21	23	32	49	74	102	167	193	201	232	296	354	338	330

Head Length in Inches. Central Values.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
6·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6·8	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6·9	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—
7·0	—	2·5	—	1	—	—	1	1	—	1·5	—	—	1	0·5	—	1
7·1	2·5	3·5	1·5	4·5	1·5	—	2	0·5	1	1·5	—	1	—	1·5	—	—
7·2	16·5	4·5	5·5	1	5	3·5	2	3·5	1	2	2	4·5	0·5	1	—	—
7·3	13·5	6·5	12·5	4·5	8·5	5·5	7·5	5·5	2·5	2	3	1·5	3	2	3	—
7·4	18	25·5	14·5	9·5	10	16·5	11	10·5	5·5	8·5	3·5	3	4·5	4·5	4	4
7·5	39	38·5	23·5	25	14	22	14·5	7	7	8	8·5	4	6	6·5	5	5·5
7·6	42	31	31·5	14·5	14	27·5	13·5	15·5	16	7	13·5	9	7·5	8	10	10·5
7·7	44	34·5	22	13	21	16·5	15·5	16·5	12·5	12	8·5	12	9	7·5	12	5·5
7·8	24	25	23	23·5	13	21	17	10	12	19	15	8	8·5	11	10	4
7·9	16·5	14	14·5	13·5	19	14	13	6·5	7	7·5	8	3	6·5	6·5	6	5
8·0	15	15	12	7	9·5	7·5	5	6·5	6	4	4·5	2	3·5	5·5	5·5	3
8·1	10·5	4	6·5	4	2·5	3	9	2·5	2·5	2·5	1·5	4·5	1	1·5	3·5	3·5
8·2	6·5	2·5	—	2	1	1	1	1·5	3	2·5	—	2·5	3	—	—	—
8·3	1	2	—	1	1	—	0·5	1	—	—	—	—	1	—	—	—
8·4	0·5	—	—	—	—	1	0·5	—	—	—	—	—	—	—	—	—
8·5	0·5	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—
8·6	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—
Totals	250	211	167	124	120	139	113	89	76	79	68	55	55	56	59	42

Head Length in Inches. Central Values.	37	38	39	40	41	42	43	44	45	46	47–49	50–52	53–55	56–58	59–61	62–64	65–69	70–80	Totals
6·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2·0
6·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4·5
6·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12·5
6·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37·0
7·0	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	1	—	—	—	88·5
7·1	1	—	—	—	—	2	—	1	—	—	1	1·5	0·5	1	—	—	—	1	150·5
7·2	1	—	1	2	1·5	—	—	—	—	1	2·5	0·5	1	—	0·5	—	—	—	237·5
7·3	2	1·5	2	—	0·5	—	—	—	0·5	1	2·5	1	1·5	—	1·5	1	1	—	332·5
7·4	6	5·5	2	1·5	9	3·5	2	1·5	1·5	2	6·5	3·5	8	1	3	1	—	—	510·5
7·5	4	3	3	3·5	3	3	3	2·5	3	2	6	2	1	6	3·5	3	2	1·5	645·0
7·6	7	3·5	3	—	4·5	11·5	1·5	7	4	4·5	8·5	10	3	8·5	3	5·5	1	1·5	689·5
7·7	6	6	3·5	3	8·5	5	6·5	7	3·5	1	8	10	4·5	4·5	4	3·5	3	0·5	643·5
7·8	5	6	3	8	4·5	2	1·5	1	2	4	9·5	8·5	4·5	3	3·5	2·5	5	4·5	533·0
7·9	2	4·5	1·5	6·5	6	4	3·5	5·5	3	1·5	4·5	5·5	8	—	5	3·5	3	6	352·0
8·0	—	3	2	4	0·5	3	1	0·5	2·5	4·5	9·5	8·5	2	2·5	2	4	4	0·5	255·0
8·1	0·5	—	1	5·5	1	1·5	1	3	2	4	4	6	3	1·5	3	3·5	1	1·5	136·0
8·2	2	—	2	1	—	0·5	1	1	2	1	4	3	2·5	—	—	1·5	2·5	1	64·0
8·3	1·5	—	—	—	—	—	—	1	—	0·5	0·5	1	—	—	—	—	0·5	—	18·0
8·4	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	—	—	—	5·5
8·5	—	—	—	—	—	—	—	0·5	—	—	—	1	—	—	—	—	—	—	3·5
8·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5
Totals	38	33	24	35	39	36	21	32	24	27	67	62	40	28	30	29	23	18	4721·0

Mean Head Length 7·594 inches or 192·898 mm. Standard Deviation ·278 inches or 7·062 mm.

TABLE V. *Mean and Standard Deviation by Age of Head Breadth and Length in Inches and Millimetres.*

Age*	Means				Standard Deviations			
	Inches		Millimetres		Inches		Millimetres	
	Breadth	Length	Breadth	Length	Breadth	Length	Breadth	Length
4—7	5·65	7·14	143·51	181·35	·136	·237	3·45	6·02
8	5·72	7·27	145·29	184·66	·176	·228	4·47	5·79
9	5·76	7·21	146·30	183·13	·187	·223	4·72	5·66
10	5·75	7·22	146·05	183·39	·235†	·261	5·97	6·63
11	5·76	7·26	146·30	184·40	·180	·225	4·57	5·71
12	5·77	7·32	146·56	185·93	·179	·194	4·55	4·93
13	5·76	7·33	146·30	186·18	·187	·233	4·75	5·92
14	5·81	7·40	147·57	187·96	·184	·240	4·67	6·10
15	5·84	7·46	148·33	189·48	·194	·250	4·93	6·35
16	5·92	7·51	150·37	190·75	·190	·239	4·83	6·07
17	5·92	7·58	150·37	192·53	·196	·249	4·98	6·32
18	5·96	7·61	151·38	193·29	·195	·258	4·95	6·55
19	5·97	7·62	151·64	193·55	·218	·251	5·54	6·38
20	5·96	7·65	151·38	194·31	·190	·235	4·83	5·97
21	5·97	7·65	151·64	194·31	·215	·256	5·46	6·50
22	5·97	7·63	151·64	193·80	·214	·250	5·44	6·35
23	5·98	7·64	151·89	194·05	·203	·229	5·16	5·82
24	5·97	7·66	151·64	194·56	·211	·249	5·36	6·32
25	5·97	7·67	151·64	194·82	·236	·246	5·99	6·25
26	5·99	7·65	152·14	194·31	·216	·219	5·49	5·56
27	6·00	7·67	152·40	194·82	·204	·261	5·18	6·63
28	6·00	7·64	152·40	194·05	·197	·263	5·00	6·68
29	6·02	7·70	152·91	195·58	·220	·229	5·59	5·82
30	6·03	7·69	153·16	195·32	·214	·270	5·44	6·86
31	6·04	7·68	153·41	195·07	·193	·203	4·90	5·16
32	6·04	7·68	153·41	195·07	·226	·267	5·74	6·78
33	6·01	7·70	152·65	195·58	·225	·259	5·71	6·58
34	5·99	7·68	152·14	195·07	·216	·236	5·49	5·99
35	6·06	7·71	153·92	195·83	·241	·205	6·12	5·21
36	6·03	7·69	153·16	195·32	·172	·229	4·37	5·82
37	6·00	7·64	152·40	194·05	·155	·271	3·94	6·88
38	6·05	7·68	153·69	195·07	·188	·203	4·78	5·16
39	6·07	7·69	154·18	195·32	·226	·274	5·74	6·96
40	6·07	7·81	154·18	198·37	·195	·248	4·95	6·30
41	6·01	7·64	152·65	194·05	·199	·210	5·05	5·33
42	6·04	7·67	153·41	194·82	·186	·236	4·72	5·99
43	6·10	7·73	154·94	196·34	·214	·210	5·44	5·33
44	6·04	7·77	153·41	197·36	·246	·275	6·25	6·98
45	6·07	7·77	154·18	197·36	·213	·243	5·41	6·17
46	6·09	7·78	154·68	197·61	·189	·279	4·80	7·09
47—49	6·04	7·73	153·41	196·34	·234	·277	5·94	7·04
50—52	6·09	7·79	154·68	197·86	·220	·270	5·59	6·86
53—55	6·02	7·71	152·91	195·83	·243	·288	6·17	7·32
56—58	6·12	7·65	155·45	194·31	·223	·206	5·66	5·23
59—61	6·11	7·69	155·19	195·32	·192	·268	4·88	6·81
62—64	6·15	7·78	156·21	197·61	·220	·239	5·59	6·07
65—69	6·15	7·85	156·21	199·39	·246	·232	6·25	5·89
70—80	6·10	7·80	154·94	198·12	·251	·242	6·38	6·15

* Age has the following meaning: group 4—7 includes all those who are at least 4 but not 8; 8 signifies those who are 8 but not 9 and would therefore correspond to a central age of 8·5, and so on.

† When one extreme observation of the array for age 10·5 is omitted this becomes ·2099, the value used in plotting the standard deviations by years.

sight, but the female is slightly more variable than the male in response to sounds.

Tables III, IV, and V present the data upon which the curves of Diagrams I and II are based.

An inspection of the curves of means in Diagrams I and II indicate that there is an increase in the size of the skull from age 27 to age 66 at least. As growth is ordinarily considered to cease at about the former age the tendency observed calls for some comment. It may be due to the fact that for some reason the

TABLE VI.

Head Breadth and Head Length: read from Curve by Ages.

Central Age	Inches		Millimetres		Central Age	Inches		Millimetres	
	Breadth	Length	Breadth	Length		Breadth	Length	Breadth	Length
4·5	5·55	7·09	140·97	180·08	*43·5*	6·07	7·71	154·18	195·83
5·5	5·59	7·11	141·98	180·59	*44·5*	6·07	7·72	154·18	196·09
6·5	5·64	7·13	143·25	181·10	*45·5*	6·07	7·72	154·18	196·09
7·5	5·68	7·16	144·27	181·86	*46·5*	6·07	7·72	154·18	196·09
8·5	5·72	7·18	145·29	182·37	*47·5*	6·08	7·72	154·43	196·09
9·5	5·75	7·21	146·05	183·13	*48·5*	6·08	7·73	154·43	196·34
10·5	5·76	7 24	146·30	183·89	*49·5*	6·08	7·73	154·43	196·34
11·5	5·76	7·27	146·30	184·66	*50·5*	6 09	7·73	154·68	196·34
12·5	5·77	7·31	146·56	185·67	*51·5*	6·09	7·74	154·68	196·59
13·5	5·78	7·35	146·81	186·69	*52·5*	6·09	7·74	154·68	196·59
14·5	5·81	7·40	147·57	187·96	*53·5*	6·10	7·74	154·94	196·59
15·5	5·84	7·45	148·33	189·23	*54·5*	6·10	7·74	154·94	196·59
16·5	5·89	7·51	149·60	190·75	*55·5*	6·10	7·75	154·94	196·85
17·5	5·94	7·58	150·87	192·53	*56·5*	6·11	7·75	155·19	196·85
18·5	5·96	7·61	151·38	193·29	*57·5*	6·11	7·75	155·19	196·85
19·5	5·97	7·63	151·64	193·80	*58·5*	6·11	7·76	155·19	197·10
20·5	5·97	7·64	151·64	194·05	*59·5*	6·11	7·76	155·19	197·10
21·5	5·97	7·65	151·64	191·31	*60·5*	6·12	7·76	155·45	197·10
22·5	5·98	7·65	151·89	194·31	*61·5*	6·12	7·76	155·45	197·10
23·5	5·98	7·65	151·89	194·31	*62·5*	6·12	7·77	155·45	197·36
24·5	5·98	7·66	151·89	194·56	*63·5*	6·13	7·77	155·70	197·36
25·5	5·99	7·66	152·14	194·56	*64·5*	6·13	7·77	155·70	197·36
26·5	5·99	7·66	152·14	194·56	*65·5*	6·13	7·78	155·70	197·61
27·5	6·00	7·67	152·40	194·82	*66·5*	6·13	7·78	155·70	197·61
28·5	6·01	7·67	152·65	194·82	*67·5*	6·13	7·78	155·70	197·61
29·5	6·01	7·67	152·65	194·82	*68·5*	6·13	7·78	155·70	197·61
30·5	6·02	7·68	152·91	195·07	*69·5*	6·13	7·79	155·70	197·86
31·5	6·02	7·68	152·91	195 07	*70·5*	6 13	7·79	155·70	197·86
32·5	6·03	7·68	153·16	195·07	*71·5*	6·12	7·79	155·45	197·86
33·5	6·03	7·69	153·16	195·32	*72·5*	6·11	7·80	155·19	198·12
34·5	6·04	7·69	153·41	195·32	*73·5*	6·11	7·80	155·19	198·12
35·5	6·04	7·69	153·41	195·32	*74·5*	6·10	7·80	154·94	198·12
36·5	6·04	7·69	153·41	195·32	*75·5*	6·09	7·81	154·68	198·37
37·5	6·05	7·70	153·67	195·58	*76·5*	6·08	7·81	154·43	198·37
38·5	6·05	7·70	153·67	195·58	*77·5*	6·08	7·81	154·43	198·37
39·5	6·05	7·70	153·67	195·58	*78·5*	6·07	7·82	154·18	198·63
40·5	6·06	7·70	153·92	195·58	*79·5*	6·06	7·82	153·92	198·63
41·5	6·06	7·71	153·92	195·83	*80·5*	6·05	7·82	153·67	198·63
42·5	6·06	7·71	153·92	195·83					

larger headed people tended to present themselves in greater numbers for examination. Or the result may be due to the effect of selection by death, the smaller headed people being possibly less robust on the average and so tending to die at a younger age. If, however, neither of these suppositions be accepted as explaining adequately the upward trend of the curves, and that trend is based on considerable numbers, then it seems necessary to conclude that the skull continues to increase in size up to quite advanced ages*.

In Table VI there are given by ages the values for head breadth and head length read from the curves fitted to the computed means and plotted in Diagrams I and II. The standard deviations of head breadth as read by ages from the curve fitted to the computed standard deviations and plotted in Diagram I will be found in Table VII. The standard deviation of head length read from the fitted line plotted in Diagram II, as has already been mentioned, is ·244 inches for all ages.

TABLE VII.

Standard Deviation of Head Breadth: read from Curve by Ages.

Central Age	Inches	Millimetres	Central Age	Inches	Millimetres	Central Age	Inches	Millimetres
4·5	·068	1·73	*30·5*	·211	5·36	*56·5*	·222	5·64
5·5	·103	2·62	*31·5*	·211	5·36	*57·5*	·223	5·66
6·5	·131	3·40	*32·5*	·212	5·38	*58·5*	·223	5·66
7·5	·162	4·11	*33·5*	·212	5·38	*59·5*	·224	5·69
8·5	·178	4·52	*34·5*	·213	5·41	*60·5*	·224	5·69
9·5	·188	4·78	*35·5*	·214	5·44	*61·5*	·225	5·71
10·5	·189	4·80	*36·5*	·214	5·44	*62·5*	·225	5·71
11·5	·187	4·75	*37·5*	·214	5·44	*63·5*	·226	5·74
12·5	·184	4·67	*38·5*	·215	5·46	*64·5*	·226	5·74
13·5	·184	4·67	*39·5*	·215	5·46	*65·5*	·227	5·77
14·5	·185	4·70	*40·5*	·215	5·46	*66·5*	·228	5·79
15·5	·188	4·78	*41·5*	·215	5·46	*67·5*	·229	5·82
16·5	·192	4·88	*42·5*	·216	5·49	*68·5*	·230	5·84
17·5	·195	4·95	*43·5*	·217	5·51	*69·5*	·230	5·84
18·5	·199	5·05	*44·5*	·217	5·51	*70·5*	·230	5·84
19·5	·202	5·13	*45·5*	·218	5·54	*71·5*	·229	5·82
20·5	·204	5·18	*46·5*	·219	5·56	*72·5*	·228	5·79
21·5	·206	5·23	*47·5*	·219	5·56	*73·5*	·227	5·77
22·5	·207	5·26	*48·5*	·220	5·59	*74·5*	·226	5·74
23·5	·208	5·28	*49·5*	·220	5·59	*75·5*	·224	5·69
24·5	·209	5·31	*50·5*	·221	5·61	*76·5*	·222	5·64
25·5	·209	5·31	*51·5*	·221	5·61	*77·5*	·219	5·56
26·5	·210	5·33	*52·5*	·221	5·61	*78·5*	·216	5·49
27·5	·210	5·33	*53·5*	·221	5·61	*79·5*	·213	5·41
28·5	·210	5·33	*54·5*	·222	5·64	*80·5*	·211	5·36
29·5	·210	5·33	*55·5*	·222	5·64			

Tables VIII to XIII inclusive present the data concerning the correlation of head measurements and cephalic index with reaction times to both sight and sound.

* Hatters have claimed that the heads of their customers increase in size throughout life.

TABLE VIII.

Galton's Anthropometric Data, Second Laboratory Series, Males. Reaction Time to Sight and Head Breadth both Corrected for Age.

Central Values. Head Breadth in Inches.

Central Values. Reaction Time to Sight in $\frac{1}{100}''$.	5·105	5·205	5·305	5·405	5·505	5·605	5·705	5·805	5·905	6·005	6·105	6·205	6·305	6·405	6·505	6·605	6·705	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	Totals
5·995	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7·995	—	—	—	—	—	—	—	3	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
9·995	1	—	—	—	1	3	8	15	15	15	29	15	6	3	3	—	—	1	—	—	—	—	—	—	—	—	—	—	115
11·995	—	—	—	—	1	2	2	14	18	21	30	24	14	9	1	4	1	—	—	—	—	—	—	—	—	—	—	—	141
13·995	—	—	—	—	—	6	14	27	42	62	59	46	37	15	8	4	2	—	1	—	—	—	—	—	—	—	—	—	323
15·995	1	—	1	—	5	18	54	93	145	197	223	158	101	50	23	5	2	—	1	—	—	—	—	—	—	—	—	—	1077
17·995	—	—	1	2	13	20	37	112	183	265	312	161	115	48	24	14	4	4	—	1	—	—	—	—	—	—	—	1	1317
19·995	—	1	3	5	13	24	68	145	237	241	263	221	114	66	23	15	4	1	—	—	—	—	—	—	—	—	—	—	1444
21·995	—	—	—	2	5	11	27	74	83	70	106	71	43	30	12	4	—	—	1	1	—	—	—	—	—	—	—	—	540
23·995	—	—	—	1	2	9	15	29	45	63	59	43	36	23	8	2	—	—	—	—	—	—	—	—	—	—	—	—	335
25·995	—	—	1	—	4	—	4	19	28	30	23	21	14	3	2	—	—	1	—	—	—	—	—	—	—	—	—	—	150
27·995	—	—	—	—	—	1	1	5	9	9	11	6	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	45
29·995	—	—	—	—	—	4	5	6	6	6	11	2	5	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	47
31·995	—	—	—	1	—	—	—	—	3	2	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
33·995	—	—	—	—	—	—	—	1	1	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	4
35·995	—	—	—	—	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
37·995	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
39·995	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
41·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
43·995	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	2	1	6	11	44	99	235	546	815	985	1132	770	487	249	106	50	13	7	3	2	—	—	—	—	—	—	—	1	5564

TABLE IX.

Galton's Anthropometric Data, Second Laboratory Series, Males. Reaction Time to Sight and Head Length, both Corrected for Age.

Central Values. Head Length in Inches.

Central Values. Reaction Time to Sight in $\frac{1}{100}$".	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	7·905	8·005	8·105	8·205	8·305	8·405	8·505	8·605	Totals
5·995	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
7·995	—	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	3
9·995	—	—	1	2	1	5	5	10	10	12	12	7	6	4	1	—	—	—	—	76
11·995	—	—	—	1	1	1	5	9	10	15	15	5	8	6	4	2	1	—	—	83
13·995	—	—	2	4	6	10	23	29	30	49	39	27	19	9	7	1	—	—	—	255
15·995	—	—	4	2	14	37	67	115	131	138	145	121	71	39	17	6	1	2	—	910
17·995	—	5	4	11	22	43	76	143	191	201	177	146	83	52	22	11	1	2	—	1190
19·995	2	1	5	13	37	58	122	148	176	196	215	138	81	63	31	9	3	—	2	1300
21·995	—	1	1	7	11	25	41	53	76	56	78	49	28	25	8	5	—	1	—	465
23·995	—	—	1	—	7	10	24	22	58	45	42	24	19	11	5	1	—	1	—	270
25·995	—	—	—	3	3	5	13	13	13	12	14	5	5	6	2	—	—	—	—	94
27·995	—	—	1	—	1	—	1	5	5	2	4	2	2	2	—	—	—	—	—	25
29·995	—	—	—	—	1	—	—	—	1	4	2	1	2	1	—	—	—	—	—	12
31·995	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	2
33·995	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
35·995	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
37·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·995	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
41·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
43·995	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
Totals	2	7	19	43	104	196	378	550	702	731	745	525	324	218	97	35	6	6	2	4690

TABLE X.

Galton's Anthropometric Data, Second Laboratory Series, Males. Cephalic Index and Reaction Time to Sight, the latter Corrected for Age.

Central Values. Cephalic Index.

Central Values. Reaction Time to Sight in $\frac{1}{100}''$.	67·05	68·05	69·05	70·05	71·05	72·05	73·05	74·05	75·05	76·05	77·05	78·05	79·05	80·05	81·05	82·05	83·05	84·05	85·05	86·05	87·05	88·05	89·05	90·05	91·05	92·05	93·05	94·05	95·05	96·05	97·05	98·05	Totals
5·995	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7·995	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	3
9·995	—	—	—	—	—	—	—	5	9	9	7	7	14	10	5	5	4	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	76
11·995	—	—	—	—	1	—	3	2	8	7	13	17	3	13	9	4	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	83
13·995	—	—	—	—	3	2	4	6	15	21	35	35	45	27	22	17	12	2	4	3	2	—	—	—	—	—	—	—	—	—	—	—	255
15·995	—	—	2	1	4	5	19	48	70	96	123	151	130	95	49	48	34	21	6	6	1	—	1	—	—	—	—	—	—	—	—	—	910
17·995	—	—	2	—	4	10	41	60	90	118	151	171	151	133	90	65	41	32	13	6	7	1	1	1	—	—	—	—	—	1	—	1	1190
19·995	—	4	1	3	5	21	26	67	107	140	169	177	157	146	109	65	42	32	9	14	2	1	2	1	—	—	—	—	—	—	—	—	1300
21·995	—	1	—	1	1	9	11	24	42	52	54	59	54	58	33	26	12	16	4	2	1	3	2	—	—	—	—	—	—	—	—	—	465
23·995	—	—	—	—	1	2	7	12	13	28	24	37	42	29	30	19	14	8	3	1	—	—	—	—	—	—	—	—	—	—	—	—	270
25·995	1	—	—	—	1	1	3	3	6	8	11	12	17	15	7	6	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	94
27·995	—	—	—	—	—	—	2	1	3	2	3	2	3	2	3	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
29·995	—	—	—	—	1	—	—	1	1	3	1	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
31·995	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
33·995	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
35·995	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
37·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
41·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
43·995	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	5	5	5	21	51	116	230	364	484	591	671	619	531	358	255	168	114	40	33	13	5	6	2	—	—	—	—	—	1	—	1	4690

TABLE XI.

Galton's Anthropometric Data, Second Laboratory Series, Males. Reaction Time to Sound and Head Breadth, both Corrected for Age.

Central Values. Head Breadth in Inches.

Central Values. Reaction Time to Sound in $\frac{1}{100}''$.	5·105	5·205	5·305	5·405	5·505	5·605	5·705	5·805	5·905	6·005	6·105	6·205	6·305	6·405	6·505	6·605	6·705	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	Totals
3·995	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
5·995	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7·995	—	—	—	—	—	—	—	—	—	2	—	—	1	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	6
9·995	—	—	—	—	—	4	6	21	18	25	45	28	18	6	7	1	—	—	—	—	—	—	—	—	—	—	—	—	179
11·995	2	—	1	—	4	8	38	67	96	152	165	100	85	30	20	10	3	—	1	—	—	—	—	—	—	—	—	—	782
13·995	—	—	—	1	14	31	47	132	203	290	322	229	133	68	20	11	5	3	1	—	—	—	—	—	—	—	—	—	1510
15·995	—	—	4	5	15	28	67	173	276	306	343	227	143	82	25	17	1	1	—	1	—	—	—	—	—	—	—	1	1715
17·995	—	—	—	3	4	13	39	78	113	121	131	102	55	35	11	5	3	2	—	—	—	—	—	—	—	—	—	—	715
19·995	—	1	—	1	2	8	27	48	74	63	75	57	32	18	15	2	1	—	1	1	—	—	—	—	—	—	—	—	426
21·995	—	—	—	—	—	3	4	8	12	13	26	10	8	3	2	2	—	—	—	—	—	—	—	—	—	—	—	—	91
23·995	—	—	—	1	2	4	4	10	18	7	14	8	6	4	3	1	—	1	—	—	—	—	—	—	—	—	—	—	83
25·995	—	—	1	—	1	—	2	1	3	3	7	5	3	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	28
27·995	—	—	—	—	1	—	—	4	—	1	3	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	12
29·995	—	—	—	—	1	—	—	1	2	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
31·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
33·995	—	—	—	—	—	—	—	1	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
35·995	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
37·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·995	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	2	1	6	11	44	99	234	545	815	985	1133	769	488	249	106	50	13	7	3	2	—	—	—	—	—	—	—	1	5563

TABLE XII.

Galton's Anthropometric Data, Second Laboratory Series, Males. Reaction Time to Sound and Head Length, both Corrected for Age.

Central Values. Head Length in Inches.

Central Values. Reaction Time to Sound in $\frac{1}{100}''$.	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	7·905	8·005	8·105	8·205	8·305	8·405	8·505	8·605	Totals
3·995	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
5·995	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
7·995	—	—	—	—	—	—	—	—	2	1	1	—	—	—	—	—	—	—	—	4
9·995	—	—	—	2	2	6	13	17	16	21	23	19	6	8	3	1	—	—	—	137
11·995	—	1	—	1	11	25	54	91	95	129	104	75	62	34	16	6	—	—	1	705
13·995	—	1	7	12	23	53	100	166	197	215	227	159	102	67	31	9	3	3	—	1375
15·995	2	4	7	14	32	67	133	170	243	235	227	166	99	64	27	13	2	2	1	1508
17·995	—	1	2	6	17	24	48	66	111	66	99	66	29	27	15	3	1	—	—	581
19·995	—	—	2	6	13	14	22	30	24	44	49	31	20	12	4	3	—	—	—	274
21·995	—	—	1	—	4	3	1	7	9	10	9	6	1	2	1	—	—	1	—	55
23·995	—	—	—	2	2	1	4	1	3	6	4	1	3	2	—	—	—	—	—	29
25·995	—	—	—	—	—	2	—	1	—	2	1	—	1	2	—	—	—	—	—	9
27·995	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	2
29·995	—	—	—	—	—	—	1	1	1	—	—	—	—	—	—	—	—	—	—	3
31·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
33·995	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	2
Totals	2	7	19	43	104	196	378	550	701	730	744	524	324	218	97	35	6	6	2	4686

TABLE XIII.

Galton's Anthropometric Data, Second Laboratory Series, Males. Cephalic Index and Reaction Time to Sound, the latter Corrected for Age.

Central Values. Cephalic Index.

Central Values. Reaction Time to Sound in $\frac{1}{100}''$.

	6·705	6·805	6·905	7·005	7·105	7·205	7·305	7·405	7·505	7·605	7·705	7·805	7·905	8·005	8·105	8·205	8·305	8·405	8·505	8·605	8·705	8·805	8·905	9·005	9·105	9·205	9·305	9·405	9·505	9·605	9·705	9·805	Totals
3·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
5·995	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7·995	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
9·995	—	—	—	—	2	2	4	4	9	13	15	25	18	15	8	8	7	2	1	2	2	—	—	—	—	—	—	—	—	—	—	—	137
11·995	—	1	2	1	3	6	14	40	46	61	91	120	91	77	58	39	29	13	7	3	1	—	2	—	—	—	—	—	—	—	—	—	705
13·995	1	—	1	2	5	14	36	71	121	136	175	197	177	160	92	75	49	36	7	13	4	1	1	1	—	—	—	—	—	—	—	—	1375
15·995	—	2	2	2	6	12	37	71	114	161	194	210	197	183	126	75	46	38	14	7	5	2	1	1	—	—	—	—	—	1	—	1	1508
17·995	—	2	—	—	2	9	12	31	46	64	79	72	81	59	40	37	19	15	7	4	—	1	1	—	—	—	—	—	—	—	—	—	581
19·995	—	—	—	—	3	5	8	10	21	38	27	36	40	23	23	14	10	8	2	3	1	1	1	—	—	—	—	—	—	—	—	—	274
21·995	—	—	—	—	—	1	1	3	3	5	5	6	7	9	8	2	3	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	55
23·995	—	—	—	—	—	2	2	—	2	4	3	2	4	2	2	3	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	29
25·995	—	—	—	—	—	—	1	—	1	1	—	2	—	1	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
27·995	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
29·995	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
31·995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
33·995	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	1	5	5	5	21	51	116	230	363	483	590	671	618	531	358	255	168	114	40	33	13	5	6	2	—	—	—	—	—	1	—	1	4686

The measures of relationship determined from the tables just mentioned are set forth in Table XIV.

TABLE XIV.

Constants of Relationship.

	r	$\eta_{t.h}$*	$\eta^2_{t.h}$	$\bar{\eta}^2_{t.h}$
Head Breadth: Reaction Time to Sight	$-\cdot0420 \pm \cdot00903$	·0743	·005520	$\cdot003595 \pm \cdot00077$
Head Length: " " "	$-\cdot0337 \pm \cdot00984$	·0677	·004583	$\cdot003838 \pm \cdot00086$
Cephalic Index: " " "	$-\cdot0079 \pm \cdot00984$	·0828	·006856	$\cdot005330 \pm \cdot00101$
Head Breadth: Reaction Time to Sound	$-\cdot0486 \pm \cdot00902$	·0929	·008630	$\cdot003595 \pm \cdot00077$
Head Length: " " "	$-\cdot0526 \pm \cdot00983$	·0862	·007430	$\cdot003841 \pm \cdot00086$
Cephalic Index: " " "	$-\cdot0049 \pm \cdot00985$	·0708	·005013	$\cdot005335 \pm \cdot00102$

It will be observed that the correlation coefficients are all negative and very small. Having regard to their probable errors all the coefficients, except those for cephalic index and reaction times to both sight and sound, may be considered to differ significantly from zero. In each case the correlation ratio is quite small and in only two cases, namely for head breadth and reaction time to sound and for head length and reaction time to sound, is the difference between η^2 and $\bar{\eta}^2$† significant when the probable errors of $\bar{\eta}^2$ are taken into account. While r and η are distinctly too small to be of any value for prognostic purposes, they do indicate that there is a very slight tendency indeed for the larger headed individuals to exhibit shorter reaction times. Similarly, the results for cephalic index and reaction time to both sight and sound indicate that there is a very slight tendency for individuals with rounder heads to show the shorter reaction time, but the correlations are not significant having regard to their probable errors.

From the analysis of the data here considered it may be concluded that for males the degree of association between head breadth and head length and reaction time to both sight and sound in each instance is too small to be of any service for purposes of estimation. Or, as far as the mental element in reaction time is concerned, the view taken by previous biometric writers is confirmed, namely that the size and shape of the head have little if any association with the working of the brain.

* $\eta_{t.h}$ means the correlation ratio of reaction time on the corresponding head measurements or index.

† $\bar{\eta}^2$ is the figure that would be obtained if there were no correlation.

$\bar{\eta}^2 = \frac{C-1}{N}$ in which C refers to the number of columns containing observations of the correlation table used in calculating η. The probable error of $\bar{\eta}^2 = \cdot67449\,\sigma_{\bar{\eta}^2}$ has been taken

$$= \cdot67449 \sqrt{\frac{2\bar{\eta}^2(1-\bar{\eta}^2)}{N+1}}$$

MISCELLANEA.

I. On Romanovsky's Generalised Frequency Curves.

BY JOHN WISHART, M.A., B.Sc.

IN *Biometrika*, Vol. XVI. pp. 106—116, Professor Romanovsky has shown how the Pearson Frequency Curves may be generalised in much the same way as the normal curve has been generalised by Thiele and Charlier. In particular he is able to write his equation to the frequency curve as

$$y = A_0 u_0 + A_1 u_1 + A_2 u_2 + \dots \qquad (1),$$

where A_0, A_1, $A_2 \dots$ are constants, and u_0, u_1, $u_2 \dots$ are certain definite one-valued functions of x. Using the first term only of this series leads us to

$$y = A_0 u_0,$$

which represents one or other of the Pearson frequency curves, according to the nature of the functions u_0, u_1, $u_2 \dots$. He then asserts that, according to the number of terms retained in the series (1), equations of the form

$$y = A_0 u_0 + A_1 u_1 + \dots + A_s u_s$$

will approximate more closely to the law of distribution as s increases. The functions u_k are obtained from the relation that if (to take the example of Type III)

$$u_0 = (a+x)^{\alpha} e^{-\nu x}, \text{ then } u_k = D^k (a+x)^{\alpha+k} e^{-\nu x},$$

where $\alpha > -1$ and ν is positive.

Now, apart from the doubtful expediency of using higher moments than the customary first four, owing to their large probable errors, it would appear that the value of the results will depend upon the convergence or otherwise of the series that results from an application of the method. It has long been known that it is possible to expand any function of which we can determine the successive moment coefficients in a series of tetrachoric functions. In particular Henderson, in *Biometrika*, Vol. XIV. pp. 157—185, expanded certain functions which adequately represent the statistical distributions of experience, i.e. the probability integrals of Types I and III, but was reluctantly forced to the conclusion that the convergence of the resultant series was so slight as to render the method useless.

A doubt as to the convergency of the parallel series obtained by Professor Romanovsky may reasonably exist until it can be shown to be without foundation, and although our author was satisfied that his expansions existed and were admissible for functions that are continuous in the interval covered by the common functions of statistical theory, it will appear in the sequel that the convergence of the series is another matter.

A suitable test of the theory would be to take a distribution of which we know the moments, find the appropriate Romanovsky generalised curve, and see whether successive approximations tend on integration to approach closer and closer to the observed frequencies. Since, theoretically, any distribution is possible within limits, we may adopt a theoretical distribution as given by the equation

$$y = y_0 x^{p-1} (1-x)^{q-1} \qquad (2),$$

i.e. a distribution (supposed possible) whose moments are known, and which is fitted perfectly by the application of the customary analysis of Type I curves. This leads us to a possible application of Romanovsky's method, should it be found to yield profitable results. It is known that the equation of Type III,

$$y = y_0 \left(1 + \frac{x}{a}\right)^{\nu a} e^{-\nu x} \qquad (3),$$

represents frequency distributions whose moments are such that the relation $2\beta_2-3\beta_1-6=0$ holds, or, in other words, whose representative points on the β_1, β_2 diagram lie along the line $2\beta_2-3\beta_1-6=0$. Now suppose we have a distribution whose β_1, β_2 nearly satisfy this relation but not quite, so that the representative point falls close to the Type III line. Then it is reasonable to suppose that fitting the distribution with a Type III curve will produce a rough approximation to the actual frequencies. Again, bringing in the fourth moment, and using it to determine the additional constant in the second term of the Generalised Type III curve, ought to improve the fit and give us as good a description of the distribution as fitting in the first instance the appropriate Pearson curve by the aid of four moments. Such an application of the method under discussion has, in fact, been made, not in the first instance to test Romanovsky's method, but as part of a general enquiry into determining approximately the probability integrals of certain Type I curves where one index, say p, is low, but the other, q, is high, so that the integral in question falls outside the *Tables of the Incomplete Beta-function*, which, it is hoped, will shortly be published.

Let, then, $p=4$ and $q=49$ in equation (2) and we shall consider the frequency distribution whose equation is

$$y=y_0x^3(1-x)^{48} \qquad\qquad (4).$$

Whole numbers are taken for p and q for ease in checking, but no loss in generality is suffered by so doing.

The constants of this curve are, by the ordinary formulae,

$$\beta_1=\cdot 73773,$$

$$\beta_2=3\cdot 97969,$$

so we have

$$2\beta_2-3\beta_1-6=-\cdot 25381,$$

that is, the point $\beta_1\,\beta_2$ lies close up to the Type III line. Now we have

$$I_x(4,\,49)=\frac{\Gamma(53)}{\Gamma(4)\,\Gamma(49)}\int_0^x x^3(1-x)^{48}\,dx \qquad\qquad (5),$$

where $I_x(p,\,q)$ is the notation used for the Probability Integral of Type I. This was then worked out for two typical values of x, one on either side of the mode, by interpolating into the unpublished *Tables of the Incomplete Beta-function*, and checking by a binomial expansion.

Thus for $\quad x=\cdot 027\,7159 \quad$ we have: $I_x(4,\,49)=\cdot 055\,8876$,

and for $\quad x'=\cdot 075\,4717$ (mean) we have: $I_{x'}(4,\,49)=\cdot 558\,8886$.

So just over half the total area of the curve is contained between the corresponding ordinates, the curve, and the axis of x. It remains now to compare these values with those found by fitting a Generalised Type III curve.

The equation is

$$y=y_0\left(1+\frac{x}{a}\right)^{\nu a}e^{-\nu x}\left[1+\sum_{k=1}^{s}\frac{S_k}{k!\,(a+1)_k\,S_0}\,\phi_k\right] \qquad\qquad (6),$$

where y_0, a, and ν are the constants of the ordinary Type III curve and $S_0=N$ the total population, while $S_k=\int_{-a}^{\infty}y\phi_k\,dx$, ϕ_k being the following expression

$$\phi_k=(\alpha+1)_k-\frac{k}{1!}(\alpha+2)_k\,\nu\,(a+x)+\frac{k(k-1)}{2!}(\alpha+3)_k\,\nu^2(a+x)^2-\ldots+(-1)^k\nu^k(a+x)^k,\quad \alpha=\nu a \qquad\qquad (7),$$

where $(\alpha+i)_k$ is an abridgment for $(\alpha+i)(\alpha+i+1)\ldots(\alpha+k)$.

Transferring to the mean $x=\frac{1}{\nu}$ and writing $a'=a+\frac{1}{\nu}=\frac{\alpha+1}{\nu}$ we have

$$y=y_0'(a'+X)^{\alpha}e^{-\nu X}\left[1+\sum_{k=4}^{\infty}\frac{S_k'}{k!\,(\alpha+1)_k\,S_0}\,\phi_k'\right] \qquad\qquad (8),$$

where S_k', ϕ_k' are the expressions S_k, ϕ_k with a' substituted for a.

It would be cumbersome to retain all the terms of the series in the reduction. We shall therefore confine ourselves to consideration of the term in S_4 only, when the mode of formation of the terms will become evident, and the succeeding terms in S_5, S_6, etc., can then be written down straight off.

Perhaps the best way to evaluate the S's is to make use of the following simple reduction formulae for the polynomials ϕ_k.

ϕ_k being defined by equation (7), we have

$$\phi_{k+1}=(\phi_1+k)\,\phi_k-k\,!\,\nu\,(a+x)\left\{1+\frac{\phi_1}{1\,!}+\frac{\phi_2}{2\,!}+\frac{\phi_3}{3\,!}+\ldots+\frac{\phi_{k-1}}{(k-1)\,!}\right\},$$

or

$$\phi_{k+1}=(\phi_1+2k)\,\phi_k-k\,(a+k)\,\phi_{k-1} \qquad \ldots\ldots(9).$$

Replacing a by a' and noting that

$$\int_{-a'}^{\infty} y\phi_1'\,dX=\int_{-a'}^{\infty} y\phi_2'\,dX=\int_{-a'}^{\infty} y\phi_3'\,dX=0,$$

while

$$\int_{-a'}^{\infty} y\,dX=S_0,\quad \nu^2\mu_2=a+1,\quad \nu^3\mu_3=2\,(a+1),$$

we get

$$\frac{S_4}{S_0}=\nu^4\mu_4-3\,(a+1)\,(a+3) \qquad \ldots\ldots(10),$$

as we might have expected at the outset, for the fourth moment of the Type III curve (3) is $\frac{3\,(a+1)\,(a+3)}{\nu^4}$, so that the first coefficient of the Romanovsky expansion is proportional to the difference between the Type III fourth moment and the fourth moment of the actual distribution we are fitting. If these fourth moments had been equal, i.e. if the β_1, β_2 point had fallen exactly on the Type III line, we should have had $S_4=0$ along with S_1, S_2 and S_3, and then the first coefficient occurring, S_5, would have been proportional to the difference between the Type III fifth moment and the fifth moment of the distribution. Similarly, if any number of moments of our selected frequency distribution were equal to their Type III values, a corresponding number of terms of the Series (8) would vanish.

To proceed: the value of $\frac{S_4}{S_0}$ is determined from (10) knowing a and ν by finding the fourth moment of the curve (4). Introducing the value of ϕ_4 from (7) we have to this order of approximation, when the right-hand side of (8) as now abridged is expanded

$$y=y_0'\left[(a'+X)^a\,e^{-\nu X}+c_1\,(a'+X)^a\,e^{-\nu X}-c_2\,(a'+X)^{a+1}\,e^{-\nu X}+\ldots+c_5\,(a'+X)^{a+4}\,e^{-\nu X}\right],$$

where

$$c_1=\frac{S_4}{4\,!\,S_0},\quad c_2=\frac{4\nu\,S_4}{4\,!\,(a+1)\,S_0},\quad c_3=\frac{6\nu^2\,S_4}{4\,!\,(a+1)_2\,S_0},\quad c_4=\frac{4\nu^3\,S_4}{4\,!\,(a+1)_3\,S_0},\quad c_5=\frac{\nu^4\,S_4}{4\,!\,(a+1)_4\,S_0}.$$

To get this into a form suitable for integration, let $z=\nu\,(a'+X)$. Then

$$y=y_0''\left[z^a\,e^{-z}+\frac{S_4}{4\,!\,S_0}\left\{z^a\,e^{-z}-\frac{4}{a+1}\,z^{a+1}\,e^{-z}+\frac{6}{(a+1)_2}\,z^{a+2}\,e^{-z}-\frac{4}{(a+1)_3}\,z^{a+3}\,e^{-z}+\frac{1}{(a+1)_4}\,z^{a+4}\,e^{-z}\right\}\right] \qquad \ldots\ldots(11).$$

Now the Probability Integral of Type III is denoted by $I\,(u,\,a)$, where

$$I\,(u,\,a)=\frac{1}{\Gamma\,(a+1)}\int_0^{u\sqrt{a+1}} e^{-z}\,z^a\,dz.$$

Equation (11) is the frequency curve which, by a suitable choice of the constants, can be made to have the same first four moments as any distribution which it is desired to fit. It is of the form

$$y=y_0''\,z^a\,e^{-z}\,(b_0+b_1z+b_2z^2+b_3z^3+b_4z^4).$$

The total area of this curve is

$$\int_0^{\infty} y\,dz=y_0''\,\Gamma\,(a+1)\left[1+\frac{S_4}{4\,!\,S_0}\,(1-4+6-4+1)\right]$$
$$=y_0''\,\Gamma\,(a+1),$$

i.e. it has the same total area as the curve from which it is derived. Hence the Probability Integral of this curve, or the ratio of the area up to any ordinate z to the total area

$$=I\left(\frac{z}{\sqrt{a+1}},\, a\right)+\frac{S_4}{4\,!\,S_0}\times\left(I_1-4I_2+6I_3-4I_4+I_5\right) \quad\text{.....................(12)},$$

where, for simplicity, we have put I_s for $I\left(\frac{z}{\sqrt{a+s}},\, a+s-1\right)$.

The first term is the ordinary Type III Probability Integral, with the same first three moments as the distribution under consideration, and should provide a rough approximation to the Type I Probability Integrals we are seeking. The rest consists of a simple function of five such Probability Integrals, going up by one at a time. They can all be got from the *Tables of the Incomplete Gamma-function*, and so there is nothing but the labour of undertaking the necessary interpolations to stand in the way of a complete determination of the value of (12).

The simple law of formation of the terms in the expression deduced from (8) is now apparent, and without further proof the full value of the Probability Integral of the Romanovsky Generalised Type III curve can now be given. We have:

$$I_R=I_1+\frac{S_4}{4\,!\,S_0}\times(I_1-4I_2+6I_3-4I_4+I_5)+\frac{S_5}{5\,!\,S_0}\times(I_1-5I_2+10I_3-10I_4+5I_5-I_6)+\dots$$
$$+\frac{S_k}{k\,!\,S_0}\times\left(I_1-kI_2+\frac{k(k-1)}{1\,.\,2}I_3-\frac{k(k-1)(k-2)}{1\,.\,2\,.\,3}I_4+\dots+(-1)^k I_{k+1}\right)+\dots \quad\text{......(13)};$$

$\frac{S_4}{S_0}$ has already been given (see equation (10)) and we have by the aid of (9)

$$\frac{S_5}{S_0}=-8(a+1)(5a+17)+20\nu^4\mu_4-\nu^5\mu_5 \quad\text{..(14)},$$

$$\frac{S_6}{S_0}=10(a+1)(3a^2-28a-151)+15(19-a)\nu^4\mu_4-30\nu^5\mu_5+\nu^6\mu_6 \quad\text{...........(15)}.$$

Note: if we substitute the Type III fourth moment for μ_4 in (14) we have

$$\frac{S_5}{S_0}=4(a+1)(5a+11)-\nu^5\mu_5,$$

the actual Type III fifth moment being $\frac{4(a+1)(5a+11)}{\nu^5}$. Again the substitution for μ_4 and μ_5 of their Type III values in (15) gives us

$$\frac{S_6}{S_0}=\nu^6\mu_6-5(a+1)(3a^2+32a+53),$$

the part on the right, when divided by ν^6, being the Type III sixth moment. This bears out what was said earlier.

We shall now apply our result to the example we have chosen. By the aid of the relations given by Professor Pearson in *Phil. Trans.* Vol. 186 A, p. 368, we find

$$\nu=\frac{2\mu_2}{\mu_3}=\frac{583}{9},$$
$$a+1=\nu a'=\frac{4}{\beta_1}=5{\cdot}422\,0393,$$
$$\nu^4\mu_4=116{\cdot}996\,994\,022,$$
$$\nu^5\mu_5=674{\cdot}377\,108\,55,$$
$$\nu^6\mu_6=6150{\cdot}543\,559\,85,$$
$$\frac{S_4}{4\,!\,S_0}=-{\cdot}155\,4489,$$
$$\frac{S_5}{5\,!\,S_0}=-{\cdot}257\,4452,$$
$$\frac{S_6}{6\,!\,S_0}=-{\cdot}301\,4588.$$

We have to remember that the curve (11) does not start at the same point of the x-axis as the curve (4) with which we are comparing it. The two means coincide, but the distance between the start of the two curves is

$$x = \frac{p(p+1)}{(p+q+1)(q-p)} = -\cdot 008\,2305,$$

while the relation between the x of the Type I curve and the u of the generalised Type III is

$$u = \frac{x}{\sigma} = \frac{x}{\cdot 035\,9464}.$$

To find then the values of the probability integrals corresponding to the results obtained for the Type I curve, we require in the first instance, i.e. for I_1, to interpolate for $a = 4\cdot 422\,0393$ and for the following values of u:

$$u_1 = \frac{\cdot 0277\,159 + \cdot 0082\,305}{\cdot 0359\,464} = 1\cdot 0$$

$$u_1' = \frac{\cdot 0754\,717 + \cdot 0082\,305}{\cdot 0359\,464} = 2\cdot 328\,5273.$$

We find $\quad I(u_1, a) = \cdot 0577\,460, \quad I_x(p, q) = \cdot 0558\,876,$

$\qquad\qquad\quad I(u_1', a) = \cdot 5571\,433, \quad I_{x'}(p, q) = \cdot 5588\,886.$

So that fitting our Type I curve with a Type III curve having the same first three moments has produced a fair measure of agreement in the results, an agreement which we may expect to become greater as the ratio of the indices p and q grows larger. Our example was chosen so as to come within that portion of the *Tables of the Incomplete Beta-Function* which had been computed.

To obtain the second, third and fourth approximations to the generalised curve (which is as far as we shall proceed) we have the following systems for u and u':

$$= \frac{z}{\sqrt{a+2}} = \sqrt{\frac{a+1}{a+2}}, \quad u_2' = \frac{z'}{\sqrt{a+2}} = \frac{a+1}{\sqrt{a+2}};$$

$$u_3 = \sqrt{\frac{a+1}{a+3}}, \quad u_3' = \frac{a+1}{\sqrt{a+3}}, \text{ and so on.}$$

We then have on interpolating

$$\begin{aligned}
I_1 &= \cdot 057\,7460, & I_1' &= \cdot 557\,1433,\\
I_2 &= \cdot 019\,6985, & I_2' &= \cdot 388\,4253,\\
I_3 &= \cdot 005\,9030, & I_3' &= \cdot 245\,9791,\\
I_4 &= \cdot 001\,5749, & I_4' &= \cdot 141\,9176,\\
I_5 &= \cdot 000\,3783, & I_5' &= \cdot 074\,9234,\\
I_6 &= \cdot 000\,0826, & I_6' &= \cdot 036\,3708,\\
I_7 &= \cdot 000\,0166, & I_7' &= \cdot 016\,3140.
\end{aligned}$$

The formula used for Interpolation was Casus III Mid-side Bivariate Interpolation Formula. See Introduction, *Tables of the Incomplete Γ-function*, p. xii.

Finally, taking these values in conjunction with those found for S_4, S_5 and S_6, we have denoting the Romanovsky generalised Type III Probability Integral by $I_R(u, p)$,

$$\begin{aligned}
I_R(1\cdot 0,\ 4\cdot 4220393) &= \cdot 057\,7460 + (-\cdot 155\,4489)(\cdot 008\,4487)\\
&\quad + (-\cdot 257\,4452)(\cdot 004\,3434) + (-\cdot 301\,4588)(\cdot 001\,7975)\\
&= \cdot 057\,7460 - \cdot 001\,3133 - \cdot 001\,1182 - \cdot 000\,5419,
\end{aligned}$$

so that the successive approximations are

$$\begin{aligned}
&\cdot 057\,7460,\\
&\cdot 056\,4327,\\
&\cdot 055\,3145,\\
&\cdot 054\,7726,
\end{aligned}$$

while the true value $\qquad I_x(4, 49) = \cdot 055\,8876.$

On the other side of the mode we have

$$I_R\,(2{\cdot}328\,5273,\ 4{\cdot}422\,0393) = {\cdot}557\,1433 + (-{\cdot}155\,4489)\,(-{\cdot}013\,4303)$$
$$+ (-{\cdot}257\,4452)\,(-{\cdot}006\,1220) + (-{\cdot}301\,4588)\,(-{\cdot}000\,1338)$$
$$= {\cdot}557\,1433 + {\cdot}002\,0877 + {\cdot}001\,5761 + {\cdot}000\,0403,$$

the successive results being

$$\begin{aligned} &{\cdot}557\,1433,\\ &{\cdot}559\,2310,\\ &{\cdot}560\,8071,\\ &{\cdot}560\,8474, \end{aligned}$$

while the true value $\qquad I_{x'}\,(4,\ 49) = {\cdot}558\,8886.$

If we compare the differences of the above results, i.e. the ratio of the area between the ordinates corresponding to x and x' to the total area, we find

I_R 1st approx.		$= {\cdot}499\,3973$,
2nd	,,	$= {\cdot}502\,7983$,
3rd	,,	$= {\cdot}505\,4926$,
4th	,,	$= {\cdot}506\,0748$,
while the exact value		$= {\cdot}503\,0010$.

The second approximation, going up to the fourth moment, appears to give the best agreement with the Type I result; the succeeding approximations do not better the fit. On the contrary, since the terms in S_4, S_5 and S_6 are positive on one side of the mode, and negative on the other, the closer we approximate to the Romanovsky curve, the larger does the area between the two ordinates become. Whether the extra terms would change their sign eventually if we went far enough, we are not in a position to say, as we have not made the necessary computation. There seems some sign in the last approximation of an approach to an upper limit for the area measured, and we may infer thereafter a decrease. But if we did not know beforehand what our result was, there is absolutely nothing to show us at what point to stop so as to get the best result. The labour involved in producing even the results shown has been fairly strenuous, involving as it does the working out of fourteen bi-variate interpolations and the rather troublesome calculation of the S's. Enough has been done, we hope, to show that one ought to hesitate before accepting any one approximation as better than any other. The expansion is an asymptotic one and must ultimately diverge. For while the expression

$$I_1 - kI_2 + \frac{k(k-1)}{1\,.\,2} I_3 - \frac{k(k-1)(k-2)}{1\,.\,2\,.\,3} I_4 + \ldots + (-1)^k I_{k+1}$$

seems to decrease steadily as k increases, the quantities S_4, S_5, S_6 exhibit no such corresponding decrease. On the contrary they may become large, being proportional, as we have shown, to the differences between the theoretical Type III moments and the moments of the fitted distribution. For the higher moments we may expect such differences to become larger and larger.

Now the curve which we have fitted to our data is of the form

$$y = y_0'' z^a e^{-z} (b_0 + b_1 z + b_2 z^2 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

where z may go from 0 to ∞. If we stop at the term in z^6 we have a curve which has the same first six moments as our Type I curve

$$y = y_0 x^3 (1-x)^{48} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Theoretically, then, within the range $x = 0$ to 1 or $z = 0$ to $\frac{58}{3}$ the curve (16) ought to fit (17) very much better than the ordinary curve of Type III

$$y = y_0'' z^a e^{-z},$$

which has only three moments the same, unless, and this is an important point, the curve (16) has negative frequencies at any point. The polynomial in z has the form

$$1 + \frac{S_4}{4!\,S_0}\left(1 - \frac{4z}{a+1} + \frac{6z^2}{(a+1)_2} - \frac{4z^3}{(a+1)_3} + \frac{z^4}{(a+1)_4}\right)$$
$$+ \frac{S_5}{5!\,S_0}\left(1 - \frac{5z}{a+1} + \frac{10z^2}{(a+1)_2} - \frac{10z^3}{(a+1)_3} + \frac{5z^4}{(a+1)_4} - \frac{z^5}{(a+1)_5}\right) + \dots$$

If we go up to the term in S_6 and substitute the values of our example we have for the polynomial the value

$$f(z) = \cdot 285\,647 + \cdot 685\,678\,z - \cdot 230\,583\,z^2 + \cdot 035\,697\,z^3 - \cdot 002\,740\,z^4 + \cdot 000\,101\,z^5 - \cdot 000\,001\,z^6.$$

Inspection shows that there is a root of this equation between 0 and -1 and another between 65 and 70. Now to cover the range of the Type I curve z should go from 0·533 to $\nu = 58{\cdot}2$. Within this range, it is true, $f(z)$ is positive and therefore

$$y = y_0''\, z^a e^{-z} f(z)$$

is also positive, but just beyond it, although the ordinates are exceedingly small, they are nevertheless negative. Thus the negative frequencies when multiplied by powers of the large distances from the mean ($z = 5{\cdot}4$) are going to affect the moments, and it becomes clear why this curve can exist, having the same first six moments as the Type I curve, and yet giving a very bad fit between two ordinates adjacent to the mode.

This would appear to render the method quite useless as a means of describing and fitting frequency distributions better than can be done at present with the aid of Professor Pearson's system of curves. The first term of the generalised expansion (13) gives a rough approximation to what we are seeking, but that is no more than we knew already, namely, that when a Type I distribution approaches, on the $\beta_1\beta_2$ diagram, the Type III line, we may get a rough idea of the frequencies by calling it Type III and fitting by means of the first three moments, and ignoring the fourth. To go even one step further, involving the calculation of five Incomplete Γ-functions, does not seem, on our experience, to give a substantially better result, commensurate with the labour involved in obtaining it. If the Romanovsky Generalised Curves are to be of value to the statistician, they must be shown to be capable of giving (a) a better fit than any other curves to statistical data, or (b) as good a fit by a speedier method, or a method susceptible of computation where existing methods are not. Now if a distribution has a β_1 and β_2 which satisfy the relation $2\beta_2 - 3\beta_1 - 6 = 0$ within the probable errors of the β's, the term in S_4 will be insignificant, from (10), and so an application of the generalised curve will only differ from the ordinary Type III by terms involving fifth, sixth, etc., moments which are subject to such large probable errors that the retention of terms involving them is of doubtful expediency. So we should anticipate that a Type III fit would not be bettered. On the other hand, as soon as we stray from the Type III line, the convergence of the series begins to be endangered. Our example may be considered a very favourable one, coming, as it does, so near the Type III line, and yet our results do not lead us to hope for anything as good, even, as would be obtained from fitting a Type I curve. The Probability Integrals of this Type will be obtained directly as soon as the *Tables of the Incomplete Beta-function* are published, while should the constants of the fitted curve lie outside the scope of these tables, there are various approximate formulae which we may apply in less time than it would take to work out the Romanovsky method, even up to four moments. For example, a comparatively few terms of a parts-reduction formula* would soon give us a better result, and would have this advantage, that an examination of the first term neglected would provide at once an idea of the accuracy of our result. Then again an application of the approximate method, given by Soper*, of fitting Incomplete Γ-functions at the start of the Type I curve, would give us a more dependable result: both these methods give perfectly satisfactory results, if the series be carried on long enough. This takes time, and as

* See *Tracts for Computers*, No. VII. pp. 19—20 and 40—43.

what we set out to seek was a formula speedier of computation than either, we are still as far from our goal as ever.

Quite apart, however, from this attempted application of Romanovsky's method, to which end, indeed, this investigation was carried out, his curves do not appear to better the existing types, owing to the expansion in terms of functions which are not suited to the purpose. For example, except for a small region near the point $\beta_1=0$, $\beta_2=3$ (for the Thiele-Charlier curves) and for a small area adjacent to the line $2\beta_2-3\beta_1-6=0$ (Generalised Type III) the series resulting from these expansions are not convergent, and hence cannot give us really satisfactory fits. The curve we obtained was shown to have negative frequencies over part of its range, and the same is doubtless true of similar curves obtained in analogous ways, all of which gives point to Professor Pearson's assertion* that fitting with the (Pearson) curve with the true fourth moment will give a theoretical distribution in a much more manageable form, and a better fit to the observations, than Romanovsky's curve will provide. To go beyond the fourth, and utilise higher moments, is to render the result problematical, owing to the high probable errors involved.

I am indebted to Mr J. O. Irwin for aid in checking the rather laborious arithmetic of this paper.

II. On The Linear Correlation Ratio in the case of Certain Symmetrical Frequency Distributions.

By J. R. MUSSELMAN, Ph.D., Johns Hopkins University.

Professor Pearson has developed a linear correlation ratio†, on the assumption of a normal frequency surface, which is of use when one of the characters is in broad categories. It is the object of this article to extend his linear correlation ratio to the frequency surface, with linear regression, for which the β_1's in the case of both marginal totals are zero, and the β_2's are equal but arbitrary.

The equation of this surface‡ can be written as

$$Z=\frac{N}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-r^2}}\frac{n-1}{n-2}\frac{1}{\left\{1+\frac{1}{2(n-2)}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}-2r\frac{xy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right\}^n},$$

where $\beta_2>3$; and the marginal totals are given by

$$\phi_1(x)=\frac{N}{\sqrt{2\pi}\,\sigma_1}\frac{1}{\sqrt{n-2}}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)}\frac{1}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}},$$

where
$$n=\frac{3(\beta_2-2)}{\beta_2-3}.$$

The linear correlation ratio is

$$\zeta_{y.x}=\sum_{x>\bar{x}}{}_{+}\frac{n_x(\bar{y}_x-\bar{y})}{N\sigma_2}+\sum_{x<\bar{x}}{}_{+}\frac{n_x(\bar{y}-\bar{y}_x)}{N\sigma_2}.$$

Calling the first half of this expression $\zeta'_{y.x}$, and the second half $\zeta''_{y.x}$, we have

$$\left.\begin{aligned}\zeta'_{y.x}&=\sum_{x>\bar{x}}{}_{+}\frac{n_x(\bar{y}_x-\bar{y})}{N\sigma_2}=\frac{\int_{\bar{x}}^{\infty}(\bar{y}_x-\bar{y})\,\phi_1(x)\,dx}{N\sigma_2}\\ \text{and}\quad \zeta''_{y.x}&=\sum_{x<\bar{x}}{}_{+}\frac{n_x(\bar{y}-\bar{y}_x)}{N\sigma_2}=\frac{\int_{-\infty}^{\bar{x}}(\bar{y}-\bar{y}_x)\,\phi_1(x)\,dx}{N\sigma_2}.\end{aligned}\right\}\quad\text{.........................(1)}$$

* See Note on Prof. Romanovsky's paper, *Biometrika*, Vol. XVI. p. 116.

† Pearson, *Biometrika*, Vol. XVII. pp. 459—461.

‡ Pearson, *Biometrika*, Vol. XV. p. 234 *et seq.*

Since this surface has linear regression and is referred to its mean as origin

$$\bar{y}_x - \bar{y} = r\frac{\sigma_2}{\sigma_1}x.$$

Substituting this in (1) we have

$$\begin{aligned}\zeta'_{y,x} &= \frac{r}{N\sigma_1}\int_0^\infty x\phi_1(x)\,dx \\ &= \frac{r\,\Gamma(n-\frac{1}{2})}{\sqrt{2\pi}\,\sigma_1{}^2(n-2)^{\frac{1}{2}}\,\Gamma(n-1)}\int_0^\infty \frac{x\,dx}{\left(1+\dfrac{x^2}{2(n-2)\sigma_1{}^2}\right)^{n-\frac{1}{2}}} \qquad (2). \\ &= \frac{r(n-2)^{\frac{1}{2}}}{\sqrt{2\pi}\,(n-\frac{3}{2})}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)}\end{aligned}$$

It is readily seen that $\zeta''_{y,x}=\zeta'_{y,x}$ and hence, solving for r,

$$\zeta_{y,x} = \sqrt{\frac{\pi}{2}}\,\frac{n-\frac{3}{2}}{(n-2)^{\frac{1}{2}}}\,\frac{\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}\left\{\Sigma_{+\atop x>\bar{x}}\frac{n_x(\bar{y}_x-\bar{y})}{N\sigma_2}+\Sigma_{+\atop x<\bar{x}}\frac{n_x(\bar{y}-\bar{y}_x)}{N\sigma_2}\right\} \qquad (3).$$

In the use of this formula $\bar{x}$ will fall somewhere, generally, inside one group of the broad categories. As the formula is valid only when $x>\bar{x}$ and $x<\bar{x}$ we must divide up this group. Calling the frequencies on each side of the mean in this group n_1, n_2; with values x_1 and x_2 at the ends of the group interval and $\bar{y}_1$, $\bar{y}_2$ the y means and $\bar{x}_1$, $\bar{x}_2$ their abscissae, we have, since the regression is linear,

$$\frac{\bar{y}_2-\bar{y}}{\sigma_2}\frac{n_2}{N}=\frac{r(\bar{x}_2-\bar{x})}{\sigma_1}\frac{n_2}{N}=r(t_0-t_2),$$

$$\frac{\bar{y}-\bar{y}_1}{\sigma_2}\frac{n_1}{N}=\frac{r(\bar{x}-\bar{x}_1)}{\sigma_1}\frac{n_1}{N}=r(t_0-t_1),$$

where

$$\left.\begin{aligned}t_0&=\frac{1}{\sqrt{2\pi}}\frac{(n-2)^{\frac{1}{2}}}{n-\frac{3}{2}}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)}\\ t_1&=\frac{1}{\sqrt{2\pi}}\frac{(n-2)^{\frac{1}{2}}}{n-\frac{3}{2}}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)}\frac{1}{\left\{1+\dfrac{x_1{}^2}{2(n-2)\sigma_1{}^2}\right\}^{n-\frac{3}{2}}}\\ t_2&=\frac{1}{\sqrt{2\pi}}\frac{(n-2)^{\frac{1}{2}}}{n-\frac{3}{2}}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)}\frac{1}{\left\{1+\dfrac{x_2{}^2}{2(n-2)\sigma_1{}^2}\right\}^{n-\frac{3}{2}}}\end{aligned}\right\} \qquad (4).$$

Hence writing $\zeta_{y,x}=r$:

$$r=\sqrt{\frac{\pi}{2}}\,\frac{n-\frac{3}{2}}{(n-2)^{\frac{1}{2}}}\,\frac{\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}\left[\left\{\Sigma_{+\atop x>\bar{x}}\frac{n_x(\bar{y}_x-\bar{y})}{N\sigma_2}+\Sigma_{+\atop x<\bar{x}}\frac{n_x(\bar{y}-\bar{y}_x)}{N\sigma_2}\right\}+r(2t_0-t_1-t_2)\right] \qquad (5),$$

omitting median array, or

$$r=\left\{\Sigma_{+\atop x>\bar{x}}\frac{n_x(\bar{y}_x-\bar{y})}{N\sigma_2}+\Sigma_{+\atop x<\bar{x}}\frac{n_x(\bar{y}-\bar{y}_x)}{N\sigma_2}\right\}\left(\frac{1}{t_1+t_2}\right) \qquad (6),$$

omitting median array.

Formula (6) then gives the exact value of r, if the distribution of the character in broad categories follows the type of curve $\phi_1(x)$.

To use it, the constants of the curve $\phi_1(x)$ must be calculated from the distribution, whence t_1 and t_2 are readily found and the formula (6) can be used. This involves more computation than in the case of the normal frequency surface for which tables of the ordinates are available. Let us see how much error will be caused by using the normal z_1 and z_2 in the formula (6) instead of t_1 and t_2. The following table is calculated for the curve $\phi_1(x)$ at $\frac{x}{\sigma}=\cdot 5$ for various values of β_2.

We must then calculate the area from the mean to $\frac{x}{\sigma} = \cdot 5$ for each curve and ascertain the value of the ordinate of the normal curve which has the same area.

β_2	n	t_2	Area*	z_2	z_2/t_2
3·03+	100	·35142	·19208	·35176	1·0010
3·06+	50	·35075+	·19272	·35143	1·0019
3·1	33	·35003	·19340	·35109	1·0030
3·2	18	·34814	·19519	·35018	1·0059
3·3	13	·34637	·19685+	·34933	1·0085+
3·4	10·5	·34472	·19840	·34853	1·0110
3·5	9	·34317	·19983	·34778	1·0134
3·6	8	·34172	·20112	·34707	1·0157
3·8	6·75	·33906	·20362	·34578	1·0198
4·0	6	·33669	·20578	·34461	1·0235+
4·5	5	·33178	·21021	·34218	1·0313

When $\beta_2 < 3$ the equation of the surface† is

$$Z = \frac{N}{2\pi\sigma_1\sigma_2} \frac{1}{\sqrt{1-r^2}} \frac{n+1}{n+2} \left\{1 - \frac{1}{2(n+2)} \frac{1}{1-r^2} \left(\frac{x^2}{\sigma_1^2} - 2r\frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right\}^n,$$

and
$$\phi_1(x) = \frac{N}{\sqrt{2\pi}\,\sigma_1} \frac{1}{(n+2)^{\frac{1}{2}}} \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} \left(1 - \frac{x^2}{2(n+2)\sigma_1^2}\right)^{n+\frac{1}{2}},$$

where
$$n = \frac{3(\beta_2 - 2)}{3 - \beta_2}.$$

Proceeding as before,

$$\zeta'_{y,x} = \int \frac{(\bar{y}_x - \bar{y})}{N\sigma_2} \phi_1(x)\,dx = \frac{r}{N\sigma_1} \int_0^a x\phi_1(x)\,dx$$
$$= \frac{r}{\sqrt{2\pi}\,\sigma_1^2} \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} \frac{1}{(n+2)^{\frac{1}{2}}} \int_0^a x\left(1 - \frac{x^2}{2(n+2)\sigma_1^2}\right)^{n+\frac{1}{2}} dx$$
$$= \frac{r(n+2)^{\frac{1}{2}}}{\sqrt{2\pi}\,(n+\frac{3}{2})} \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})}.$$

As $\zeta''_{y,x} = \zeta'_{y,x}$, then if we put $\zeta_{y,x} = r$, we have

$$r = \sqrt{\frac{\pi}{2}} \frac{n+\frac{3}{2}}{(n+2)^{\frac{1}{2}}} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+2)} \left\{\sum_{x>\bar{x}}^{+} \frac{n_x(\bar{y}_x - \bar{y})}{N\sigma_2} + \sum_{x<\bar{x}}^{+} \frac{n_x(\bar{y} - \bar{y}_x)}{N\sigma_2}\right\}.$$

Similarly we can derive that

$$r = \left\{\sum_{x>\bar{x}}^{+} \frac{n_x(\bar{y}_x - \bar{y})}{N\sigma_2} + \sum_{x<\bar{x}}^{+} \frac{n_x(\bar{y} - \bar{y}_x)}{N\sigma_2}\right\} \left(\frac{1}{t_1 + t_2}\right) \quad \text{.......................(7)},$$

omitting median array,

where
$$t_1 = \frac{1}{\sqrt{2\pi}} \frac{(n+2)^{\frac{1}{2}}}{(n+\frac{3}{2})} \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} \left(1 - \frac{x_1^2}{2(n+2)\sigma_1^2}\right)^{n+\frac{3}{2}}$$

and
$$t_2 = \frac{1}{\sqrt{2\pi}} \frac{(n+2)^{\frac{1}{2}}}{(n+\frac{3}{2})} \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} \left(1 - \frac{x_2^2}{2(n+2)\sigma_1^2}\right)^{n+\frac{3}{2}}.$$

* The areas were computed to five place accuracy by use of manuscript tables in the Biometric Laboratory of the incomplete Beta-function where available; in other cases by the method of Wishart, *Biometrika*, Vol. XVII. p. 469.

† Pearson, *Biometrika*, Vol. xv. p. 240.

If we substitute z_1 and z_2 the ordinates of the normal curve in formula (7) for t_1 and t_2 we shall introduce a certain amount of error. The following table is calculated for the above curve $\phi_1(x)$ at $\frac{x}{\sigma} = \cdot 5$ for the given values of β_2.

β_2	n	t_2	Area	z_2	z_2/t_2
2·94	47	·35333	·19022	·35268	·99816
2·9	27	·35425+	·18936	·35311	·99678
2·8	12	·35662	·18674	·35439	·99375−
2·7	7	·35918	·18457	·35545	·98962
2·6	4·5	·36195+	·18172	·35680	·98577
2·5	3	·36498	·17879	·35818	·98137
2·4	2	·36828	·17540	·35973	·97678
2·2	0·75	·37591	·16753	·36323	·96627
2·0	0	·38525+	·15748	·36744	·95377
1·8	−0·5	·39693	·14434	·37253	·93853
1·5	−1	·42108	·11503	·38224	·90776

Conclusion. If the distribution in broad categories follows the curve for which $\beta_1 = 0$, equations (6) and (7) will give the exact values of the linear correlation ratio. If instead of using them, we use the assumption of a normal curve, the two tables give the error at $\frac{x}{\sigma} = \cdot 5$, which generally covers the middle group in the distribution. For β_2 lying between 2·4 and 4·0 the error is approximately two per cent., and this is within the usual range of statistical error. Hence it seems justifiable to use the assumption of normal distribution, if our marginal total is such that $\beta_1 = 0$ and $2 \cdot 4 < \beta_2 < 4$.

III. On the Avuncular Relationship.

By KARL PEARSON.

If the correlation between pairs of relations be linear, and if the individual ancestral correlations decrease in geometrical progression—hypotheses satisfied by the simple Mendelian theory whether of gametic or somatic constitutions—then the multiple regression equation ("Law of Ancestral Heredity"), and all the collateral correlations can be deduced from a knowledge of three constants, namely:

(i) the coefficient, ϵ, of assortative mating supposed constant for all generations,

(ii) the mean correlation ρ_1 of parent and offspring, and

(iii) the mean fraternal correlation r, which is equal to R^2, where R is the multiple correlation of offspring on all ancestry*.

Using the notation of the paper just referred to, in particular, $c_p = \gamma\eta^p = p$th coefficient in multiple regression formula; r_{p-q} = correlation coefficient of pth and qth midparents; ρ_p = mean correlation coefficient of individual pth ancestor and offspring, we find for r_{cc} the correlation of cousins and for r_{un} the correlation of uncle (or aunt) with nephew (or niece) the following values†:

$$r_{cc} = \frac{1}{2} r - \frac{1}{2} \gamma^2 \eta^2 \left(1 - \frac{r}{1+\epsilon}\right.$$

$$r_{un} = \frac{1}{\sqrt{2}} \frac{\gamma\eta}{1-\eta^2} (\gamma\eta^2 + (1+\eta^2) r).$$

Taking stature for males we find

$$\rho_1 = \cdot 5040, \quad \epsilon = \cdot 2804, \quad \rho_1' = \cdot 393{,}627, \quad \sqrt{2}\rho_1' = \cdot 556{,}673, \quad r = \cdot 5196.$$

* *Biometrika*, Vol. XVII. p. 131. See also p. 134.

† Algebraical proofs to be published later.

Hence, since $$\beta = \sqrt{2}\rho_1' + \sqrt{(1-r)\left(1 - \frac{2\rho_1'^2}{r}\right)}$$
$$= \cdot 997{,}006.$$

Again $$\eta = \frac{\beta - \sqrt{2}\rho_1'}{1-r} = \cdot 916{,}596,$$
$$\gamma\eta = \beta - \eta = \cdot 080{,}410.$$

Thus we find $$r_{cc} = \cdot 2579, \quad r_{un} = \cdot 3686.$$

These would be the values of cousin and uncle correlations for stature.

If we take the extreme case for $\beta = 1$, $\gamma\eta = 1 - \eta$, and accordingly

$$r_{cc} = \frac{1}{2}r - \frac{1}{2}(1-\eta^2)\left(1 - \frac{r}{1+\epsilon}\right),$$

$$r_{un} = \frac{1}{\sqrt{2}}\frac{1}{1+\eta}(\eta(1-\eta) + (1+\eta^2)r),$$

then the quadratic for η shows that as β approaches unity so does η, or we have

$$r_{cc} = \frac{1}{2}r, \quad r_{un} = \frac{1}{\sqrt{2}}r.$$

For twelve measured characters, $r = \cdot 5250$ in the mean, hence

$$r_{cc} = \cdot 2625, \quad r_{un} = \cdot 3712,$$

no great difference from the previous values for stature.

It is clear from these results that we might anticipate that cousins would be about half as closely related as brothers and uncle and nephew about 40 °/ₒ more closely than cousins.

In 1907 Miss E. M. Elderton prepared 70 correlation tables for quantitative and qualitative characters in cousins and found an average value of ·265, a value in accordance with the above theory. But in 1907 she also worked out 56 tables for qualitative characters for uncles, aunts, nephews and nieces. The result was a mean correlation of ·2384 slightly less than for cousins, but explicable if allowance be made for difference of generation. The data were chiefly taken from my schedules of family record.

Dr Snow working by Mendelian theory* found a near equality between the correlations for the cousinal and avuncular relations. Notwithstanding Dr Snow's apparent theoretical corroboration of Miss Elderton's observed values, the point has always seemed to me an obscure one, and one demanding further investigation. Sir Francis Galton would not hear of the avuncular and cousinal correlations being equal. He wrote to me on Oct. 30, 1907:

"There is one serious difficulty in the conclusion, namely that the relation between a person and his uncle is the same as that between that person and his uncle's child. This cannot be the case unless the wife of the uncle is a counterpart of himself."

Dr Snow's result for Mendelism seemed to indicate that the equality of r_{cc} and r_{un} might be possible, but Sir Francis was very strong on the point, and the avuncular correlation tables have remained nearly twenty years unpublished. Recently I reached in lecture the above results for r_{cc} and r_{un} confirming Galton's point that the avuncular relationship was closer than that of cousins. Hence either there is some error in my recent work, or Galton is right and there is a slip in Dr Snow's work, and further my family records which give reasonable values for cousinships are not to be depended on for the avuncular relationships. The determination for a measurable character of the correlation of uncle and nephew would be of very great value.

* *R. S. Proc.* Vol. LXXXIII. B. pp. 48—53, 1911.

Alexander Tchouproff

Alexander Tchouproff, 1874–1926. "Grundbegriffe und Grundprobleme der Korrelationstheorie," 1925.

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ON THE SKULL AND PORTRAITS OF GEORGE BUCHANAN.

BY KARL PEARSON, F.R.S.*

I EXPECT that many of those present will consider it an act of great rashness that a southerner should venture into the metropolis of northern Britain and dare, a mere scientist, to talk about a man whom many Scots look upon as Scotland's greatest scholar, leading historian and the first writer in Scotland to propound a political philosophy.

But I would remind you that for good or ill we cannot separate the history of our two countries. Buchanan played a large part—a larger part than some of us recognise—in those still obscure proceedings which, starting at Langside, were continued at York and Hampton Court, and culminated only in the final scene at Fotheringay. Mary Stewart's history is as much English as Scottish history, and the story of both countries would not only have been different, but differently written, had the Tudor blood been as dominant in Mary as in Elizabeth; while the psychological bearing of the extent to which Mary did inherit Tudor characteristics has been often overlooked by the historians of both countries.

The time has come when we are justified in taking, not a heptarchical view of the history of Britain, but a British view, which would recognise that local histories can be interwoven into a wider history of our united countries. Queen Mary, George Buchanan, Darnley, Moray, Bothwell, and the rest of the Scottish actors of those days really played their parts on a wider stage than local history provided; that is not only the source of their perennial interest, but the excuse for my appearance and choice of a subject here this afternoon.

Previous lecturers for the Henderson Trust have told us that while much of the old phrenology is no longer acceptable in the light of modern science, yet increasing acquaintance with the function of brain centres may lead us, not so much from observations on the external surface of the skull as from the markings on the endocranial cast, to suggestions—not to use too emphatic a word—as to character and abilities. I should like at once to state that this is not to be the line of my inquiry this afternoon. I think an endocranial cast of George Buchanan's skull would be of interest, but I have not ventured to take it nor to ask for it to be taken.

I want to put before you something much less abstruse, and more direct in its appeal to your imaginations.

In the rough and tumble of everyday experience, in trusting to our fellow men, in the appointment of colleagues, assistants and servants, and even in the

* Being a lecture delivered on June 4 in the Anatomy Theatre of the University of Edinburgh for the William Ramsay Henderson Trust.

choice of acquaintances and friends, we are largely influenced by our reading of character from physiognomy. We may admit it or we may not, but it is a fact that facial expression draws our confidence or serves as a warning mark. The reading of physiognomy is not a science with its ascertained rules; it is not wholly, I believe, a result of past experience, but rather an instinct modified by experience. We are reduced to the sentiment of Martial, as englished by Tom Brown:

> "I do not love thee, Doctor Fell,
> The reason why I cannot tell;
> But this alone I know full well,
> I do not love thee, Doctor Fell."

Even a Christchurch undergraduate is susceptible to facial expression!

Probably in judging half-unconsciously by physiognomy we are right in nine cases out of ten, and badly wrong on the tenth occasion. We should do better, if we judged by the dynamic expression as given by the film of an individual rather than by the static expression conveyed by a bust or portrait.

The reconstruction from the skull of the head is a branch of science, which has of recent years made some progress, but while the facial expression does to some extent depend on the form of the skull, and the strength of the musculature which can be partially judged from the skull markings, we are left wholly in the dark as to those facial lines which leave no permanent traces on the skull itself, and which have so much to do with our judgment of character.

Why is it that we desire to see and to acquire portraits of great men, or of the individual men we treat as our personal heroes or saints? I think it undoubtedly arises from the fact that a portrait, at any rate a good portrait, helps us to a truer characterisation, a truer appreciation, than many pages of verbal description. We are thrown back on our instinctive judgment of physiognomy. I believe that at present one of the chief services which can be done by aid of the skull of a great man is to find out from it which portrait, if portraits exist, truly represents him. With all due reverence I would exhume the skull of Shakespeare, of Newton or of Milton, and then we should be able to standardise their portraiture, and say: Here is the true man, and what would also be of interest: There is the artistic liar!

Something of this kind has been done in my Laboratory for two historic skulls—that of Sir Thomas Browne and that of Henry Stewart, Lord Darnley; and I may be permitted to indicate here how the method serves to differentiate good from bad portraiture.

On Plate I we have a portrait of Jeremy Bentham fitted with the significant lines of his "Auto-Icon." The mummified head, when the drooping mandible is brought into touch with the upper jaw, gives a quite passable fit, showing that the artist was reasonably truthful. On Plate II we have a contrast between two portraits of Sir Thomas Browne fitted with the outline of his skull adjusted as closely as possible to the two representations. The earlier or Allix portrait agrees well with the skull, while the later portrait, almost certainly painted after his

death for the Royal College of Physicians, exaggerates his low frontal into "high-brow" intellectuality. The artist was quite clearly an incorrigible idealiser.

Can we apply the same method to obtain further light on George Buchanan as judged from his portraits? To attempt this will be the purpose of my lecture this afternoon. But before we proceed further with the portraits, it is wise to inquire what evidence we have for the authenticity of the skull.

Buchanan died on September 28, 1582, aged 76 years.

There is nothing in the skull to contradict this age. It is the calvarium of an old man; there is no mandible, which would have told us something additional, but the teeth in the upper jaw must have been present at death, although the alveolar border has extensively receded.

The bony walls of the skull have been reduced in parts to the thinness almost of a railway ticket in a manner that is very characteristic of extreme age, and familiar to those who have handled large numbers of crania. The zygomatic arches, at least to judge from the one that is complete, could never have been massive and have dwindled with age almost to the thickness of a quill. The owner of the skull could hardly have had prominent cheek bones, except by extreme thinness of the flesh.

Coming to the history of the skull itself, we naturally find its first purloining—as in the case of Sir Thomas Browne's—involved in obscurity. It is the more so, perhaps, in this case, as the purloiner was a man of position in Edinburgh—no other than the Rev. John Adamson, Principal Regent of the University from 1623 to his death in 1653. Adamson was a great admirer of Buchanan, but he was hardly likely to state, during his official life, how and when he had despoiled Buchanan's grave. Puissant as we all recognise the principal of a university to be, humbly as we obey his dictates, yet we perceive that his autocracy—wide as it is—does not include body-snatching as a prerogative. There are some things which even the principal of a university cannot do publicly, but we may be fairly sure that when he does them *sub rosa*, he will carry them through successfully, and obtain exactly what he is seeking. I think we need not doubt that Adamson, having set his heart on Buchanan's skull, was not likely to fail in getting the actual article.

Adamson was educated at St Andrews, being born in the year 1576; he held I think a professorship of philosophy there, but in 1598 became a regent in Edinburgh, holding his chair there till 1604, when he accepted a ministerial call to North Berwick, being afterwards translated to Libberton and returning to Edinburgh as Principal in 1623.

In 1598, George Buchanan had been buried for 16 years, in 1623 for 41 years. We must further note that Buchanan had died in poverty, directing the small sum of money he had left to be given to the poor, and telling his friends to let him lie where he was till the City of Edinburgh buried him. He appears to have been interred without a tombstone. Adamson in his epitaph or eulogy on Buchanan draws attention to this fact, saying that only those need monuments who have

nothing else to be honoured by. A grave without a tombstone is very likely to be encroached upon for later interments, and Adamson was undoubtedly familiar with the situation of Buchanan's grave. Probably he obtained the skull from the grave-diggers at some later interment at or near Buchanan's tomb. Sir Robert Sibbald, born twelve years before Adamson's death, writing of Adamson within thirty years of the latter's death, tells us, somewhat naively, that by his, Adamson's, care the skull of Buchanan, dragged from its grave, has been preserved in the Library of the University of Edinburgh. There it remained, accompanied by a copy of Adamson's Latin verses, to which I have already referred, till 1817, when it was transferred to the Museum of Anatomy. About 100 years after Buchanan's death Mr Robert Henderson, University Librarian, refers to the skull, and says of Adamson: "It 's storied of him that being at Buchanan's burial, he bargained for the skull; which having got from the graveman within the year or so, he carefully kept in his life-time; and being found in his study so inscribed after his removal was handed down to us." A child of six might be, if improbably, at the burial of Buchanan, but he certainly would not have bargained with the graveman for Buchanan's skull on that occasion. However the main point of Henderson's account, that the skull was found in Adamson's study after his death, inscribed with the name of Buchanan, doubtless corresponds to the truth. And if so, Adamson would hardly have obtained the skull without the assistance of the sexton, or kept it without a fair amount of certainty that it came from George Buchanan's tomb.

It is noteworthy that while John Adamson says there was no tombstone to Buchanan's grave, we read in 1701 of the existence of such a stone.

"At Edinburgh, the 3rd day of December 1701, the same day the council being informed, that the through stone [tombstone] of the deceast George Buchanan lyes sunk under the ground of the Greyfriars: therefore they appoint the chamberlain to raise the same, and clear the inscription thereon; so that the same may be legible."

If we could ascertain at what date that stone was erected, we should have an upper limit to the date when the Principal of the University robbed the Greyfriars, for there was clearly no stone when he wrote his verses.

I have not succeeded in finding any real work done on the skull in relation to the portraits.

Mr William Carruthers in the *Scottish Review* for October 25, 1906, dismissed the skull, stating that it could not be that of George Buchanan, because it was not like the portrait in the National Portrait Gallery in London. The question seems to be: Is it like any other portrait? if it is, it would only show that the National Portrait Gallery picture is not a truthful representation of Buchanan.

Sir William Hamilton is said to have tested Buchanan's portraits against the skull about 1835 by measurement and found that *only* the Chouet woodcut (see Plate III) fulfilled his tests. In other words he appears to have rejected the two portraits in the Senate Hall of his own University, which are of the National Gallery type. Speaking of the Chouet wood engraving and its reproduction by

Freiherus, Hamilton writes: "The head in both is thoroughly Scottish in character with a long and well-formed nose, well-defined cheek bones and a long upper lip as in the skull." Now these three characters are all present in the National Portrait Gallery type (see Plate IV), but appear to be quite absent in the skull.

I am not clear what a head "Scottish in character" may be. It is not possible craniometrically to distinguish really significant differences between the Lowland Scottish measured by Turner and the 17th century Londoners measured in my own Laboratory. Recently some 400 crania from a graveyard in London dating round the plague year 1665 have been measured by us and we have obtained the type contours on the average of some 140 male skulls. The upper lip and the nose of this typical skull from the pauper graveyard are identical in length with those of the Buchanan skull, which like these 17th century Londoners, had no specially marked zygomatic arches*. The main difference between the Buchanan skull and that of the 17th century Londoner lies in the very feeble development in the former of the occipital and cerebellar regions.

I do not think we can place any weight on Sir William Hamilton's rejection of all but the Boissard-Chouet engraving, especially as that engraving is in my opinion undoubtedly a copy of the Edinburgh or Bronckhorst type of portrait, a point which Sir William Hamilton seems to have overlooked.

Lastly I may refer to Professor Hume Brown's *Life of George Buchanan*, wherein he writes: "It may be added that what on good authority is supposed to be Buchanan's skull is preserved in the Anatomical Museum of the University of Edinburgh. Its general outline is exactly like that of the best portraits, and by its dome-like shape and extreme tenuity it has all the marks of a high cerebral development."

The skull has certainly extreme tenuity, but why this should be a mark of "high cerebral development" instead of as usual the concomitant of old age, I do not understand. As to its dome-like character, its auricular height is just the average of our Londoners †, and the "dome," such as it is, is produced in appearance by the marked absence of occipital development. Professor Hume Brown does not tell us the locus of the best portraits which the general outline of the skull is exactly like. The fact that Sir William Hamilton and Professor Hume Brown appear to reach opposite conclusions may, perhaps, justify a biometrician in endeavouring to advance further by more elaborate measurements.

I have placed in the accompanying Table in parallel columns the usual series of cranial measurements taken in the Biometric Laboratory, for (i) the average male Londoner of the 17th century, (ii) the late 16th century George Buchanan, (iii) the 17th century author of the *Religio Medici*, and (iv) the early 19th century

* Zygomatic Breadth: Buchanan's Skull, 130 mm.; Farringdon Street Graveyard, 130·1 mm. Facial Breadth: Buchanan's Skull, 92·5 mm. (?); Farringdon Street Graveyard, 91·4 mm. (Moorfields Plague Pit, 93·9 mm.).

† Males: 17th century Londoners. Farringdon Street Poor Graveyard 110·0, Whitechapel Plague Pit 112·1, Moorfields Plague Pit 111·8. These are *mean* values, hence the Buchanan skull, 110·0, is simply mediocre.

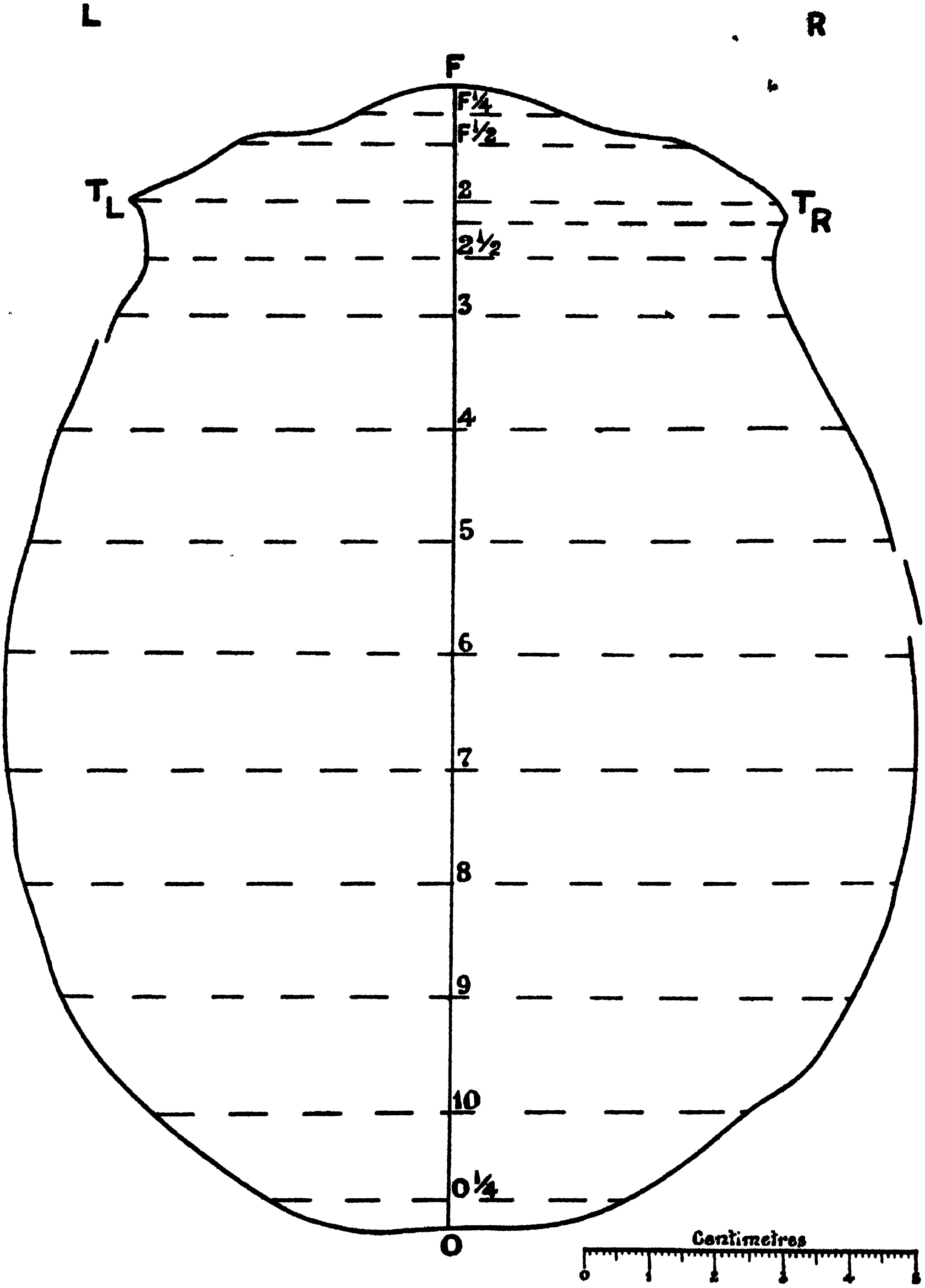
Comparison of the Cranial Measurements of George Buchanan with those of other Skulls.

Cranial Character	Average London Poor 17th Century ♂s	Skull of George Buchanan	Skull of Sir Thomas Browne	Skull of Jeremy Bentham	Deviation of Buchanan skull from 17th Century ♂s. Difference / S.D.
Capacity	1481	1360 (??)	1509	1475	− ·93
Length (Flowers)	186·1	171·3	190·0	183·5	−2·75
Length (Maximum)	188·8	174·0	194·6	186·0	−2·29
Breadth (Parietal)	142·4	140·0	143·7	147·0	− ·41
Breadth (Minimum frontal)	96·8	98·8	87·4	99·0	+ ·34
Height (Basio-vertical)	130·4	130·6 (?)	122·0	—	+ ·04
Height (Basio-bregmatic)	129·7	129·9 (?)	121·9	132·0	+ ·04
Height (Auricular)	110·0	111·0	102·6	116·0	·00
Skull Base	100·1	100·9 (?)	102·7	102·0	+ ·18
Arc, Transverse (Apex)	309·0	304·0	297·0	325·0 (?)	− ·39
Arc, Sagittal	378·8	355·4	380·0	365·0 (?)	−1·65
Arc (Nasion to Bregma)	129·3	119·5	129·0	130·0 (?)	−1·51
Arc (Bregma to Lambda)	128·1	124·0	123·0	120·0 (?)	− ·51
Arc (Lambda to Opisthion)	120·6	111·9 (? Reconst.)	128·0	115·0 (??)	−1·11
Chord (Lambda to Opisthion)	97·3	91·3 (? Reconst.)	99·8	97·1 (?)	−1·17
Horizontal Circumference	530·0	503·0	549·0	540·0 (?)	−1·70
Alveolar Point to Nasal Spine	19·2	21·5	21·7 (?)	22·0 (??)	+ ·82
Facial Height	70·5	69·6	73·1 (?)	73·0 (?)	− ·20
Facial Breadth	91·4	92·5 (?)	90·9	97·0 (??)	+ ·18
Bizygomatic Breadth	131·0	130·0	132·3	127·0 (?)	− ·21
Nasal Height, R.	51·8	51·0	52·2	52·8	− ·28
Nasal Height, L.	51·7	51·7	52·2	52·8	·00
Nasal Breadth	24·6	25·6 (?)	22·6	26·7	+ ·40
Dacryal Subtense	12·8	12·0	12·8	11·9 (?)	− ·45
Dacryal Chord	22·2	22·2	20·9	22·2 (?)	·00
Dacryal Arc	36·1	32·0	32·7	33·0 (?)	−1·15
Simotic Subtense	4·6	4·5	5·0	4·3	− ·09
Simotic Chord	9·2	10·4	8·7	9·5	+ ·63
Breadth, R. Orbit	42·3	45·8	43·7	45·2 (?)	+2·24
Breadth, L. Orbit	42·4	44·8	42·8	46·0 (?)	+1·62
Height, R. Orbit	34·3	35·9	36·3	36·0 (?)	+ ·68
Height, L. Orbit	34·3	36·8	36·7	36·0 (?)	+1·09
Palate Height	11·1	10·5	16·5	12·0 (?)	− ·42
Palate Breadth	39·3	36·8	37·5 (??)	35·0 (?)	− ·82
Palate Length to base of Spine	46·0	38·7 (??)	49·0 (??)	47·0 (?)	−2·75
Profile Length	94·4	89·5 (?)	98·3 (?)	89·0 (?)	− ·92
Length, Foramen Magnum	36·8	32·6 (?? Reconst.)	34·6	36·2 (?)	−1·40
Breadth, Foramen Magnum	30·6	—	30·1	31·0 (?)	—
Profile Angle	85°·9	87°·8	84°·7 (?)	93°·0 (?)	+ ·60
Angle at Nasion	64°·2	60°·5 (?)	65°·4	58°·0	−1·01
Angle at Basion	42°·5	42°·3 (?)	42°·7 (?)	44°·5	− ·05
Angle at Alveolar Point	73°·3	77°·2 (?)	71°·9 (?)	77°·5	+1·07
1st Cephalic Index (100 B/L)	75·4	80·5	73·8	79·0	+1·47
2nd Cephalic Index (100 H/L)	69·3	75·1 (?)	62·7	70·9	+1·80
3rd Cephalic Index (100 B/H)	109·1	107·2 (?)	117·8	111·4	− ·40
Compound Cephalic Index (100 $(B-H)/L$)	6·7	5·4 (?)	11·2	8·1	− ·36
Facial Index	77·1	75·2 (?)	80·4	75·3 (?)	− ·30
Nasal Index, R.	47·5	50·2	43·3	50·6	+ ·63
Nasal Index, L.	47·5	49·5	43·3	50·6	+ ·46
Foraminal Index	83·4	—	85·7	85·6	—
Dacryal Index	58·1	54·1	61·2	53·6 (?)	+ ·43
Simotic Index	50·7	43·3	57·5	45·3 (?)	+ ·57
1st Palate Index (Breadth/Length) ...	85·4	95·1 (??)	76·5 (??)	74·5	+1·75
2nd Palate Index (Height/Breadth) ...	28·5	28·5	33·7 (??)	34·3 (?)	·00
Occipital Index	58·0	58·3 (?)	55·8	61·3	+ ·11
Orbital Index, R.	81·0	78·4	83·1	79·6 (?)	− ·43
Orbital Index, L.	80·9	82·1	85·7	78·3 (?)	+ ·23
External Orbital Width	98·1	98·5	98·0	100·4 (?)	+ ·10

Jeremy Bentham. None of these skulls is in anything like perfect condition, and that of Jeremy Bentham is in the mummified head, so that many measurements are difficult to take upon it, and are marked in the table with queries and double queries. Sir Thomas Browne's skull is in a more complete condition than Buchanan's, but still owing to breakages certain measurements are doubtful. In the case of Buchanan, a portion of the left orbit, a considerable portion of the occipital bone and the whole of the foramen magnum have disappeared. There is no mandible, and accordingly I have not given the mandibular measurements for the other crania. In the case of Buchanan I have given measurements obtained by a reconstruction in plasticene of the base of the skull. I do not propose to enter fully into the methods adopted in reconstructing parts of a skull. They can be carried out with more or less exactness, when we know the correlations of its component bones. As an illustration I may take the reconstruction of the basion or anterior point of the foramen magnum. We have first to measure as far as is possible the capacity of the skull. After closing the orifices this was found by mustard seed packing to be 1360 $cms.^3$, very small as compared with the average English, or either Sir Thomas Browne's or Bentham's skull. Now we have a formula for finding the capacity from maximum length, parietal breadth and auricular height. This formula gave a capacity of 1375 $cms.^3$, quite as close to the capacity obtained by mustard seed as could be hoped for. Assuming Buchanan's capacity as known, we can compute the height from the bregma to the basion because we have a second formula giving the capacity in terms of maximum length, parietal breadth and basio-bregmatic height. Proceeding in this way one is able to determine reasonably accurate positions for points on the missing part of the base of the skull, and fill in the portion broken away with plasticene. That is what is meant by the word "Reconstruction" against certain measurements in the table.

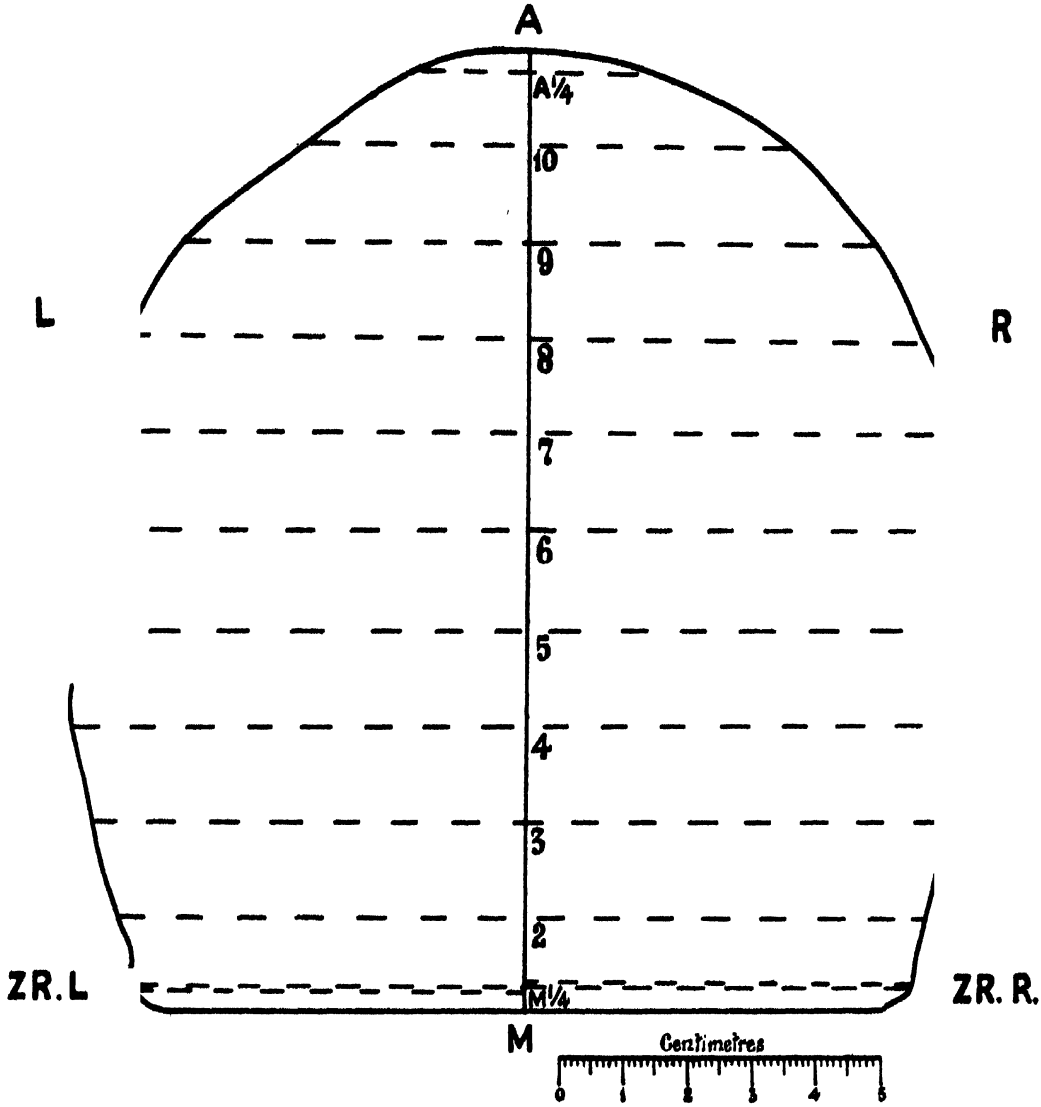
We have seen that Buchanan's skull had a small capacity, and it is at once clear why it had. The height of the skull and its breadth are not small, they are simply mediocre, but the length is remarkably small. Sir Thomas Browne compensated his low frontal and small auricular height by a great length and an increased parietal breadth. But nothing of this kind occurs in Buchanan's skull. It is as if the occipital region and the cerebellum had been simply cut away. This can be shown at once by superposing the horizontal contour of Buchanan's skull on the type or average contour of 140 male English skulls*. You will remark at once nearly complete correspondence in the frontal region, but a total absence of occipital development in Buchanan's skull. You may see the same point in the other two main contours. In the transverse or vertico-auricular section Buchanan is very nearly simple British, he might easily have been a 17th century Londoner; in the sagittal section he is also mediocre British until we come to the occipital, and here comparison is not feasible, because in his skull that region may be said to fail entirely.

* Tissues of the average Farringdon Street contours accompany Miss Beatrix Hooke's paper in the preceding part of this journal and are reproduced with this part.



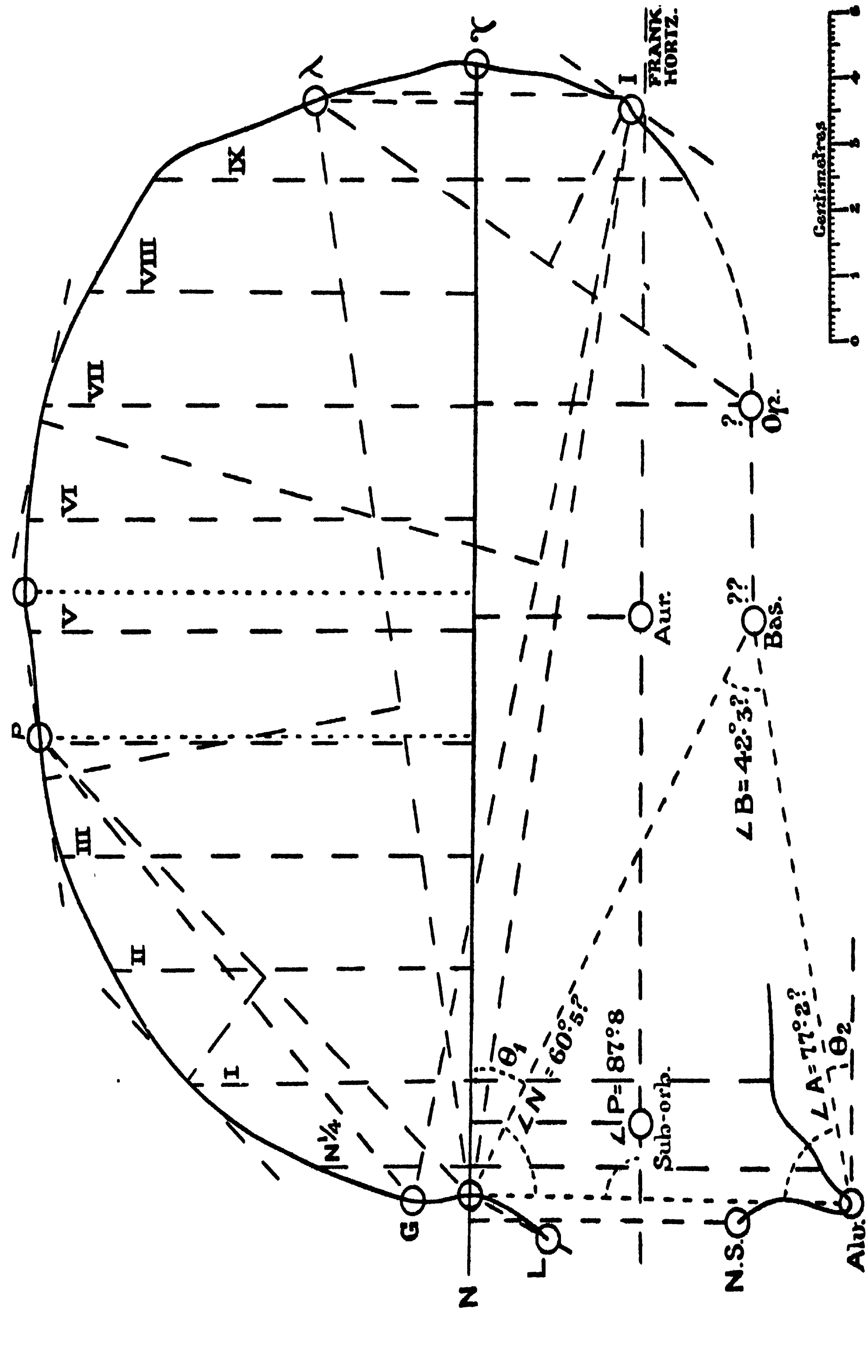
George Buchanan's Skull, Horizontal Contour.

Fig. I.



George Buchanan's Skull, Transverse Contour.

Fig. II.



George Buchanan's Skull, Sagittal Contour.

Now let us examine the characteristics of Buchanan's skull by another method, namely by taking the difference between the character in Buchanan's case and the corresponding average value in English crania, and dividing this difference by the average variability of the character in the English population. When this ratio is:

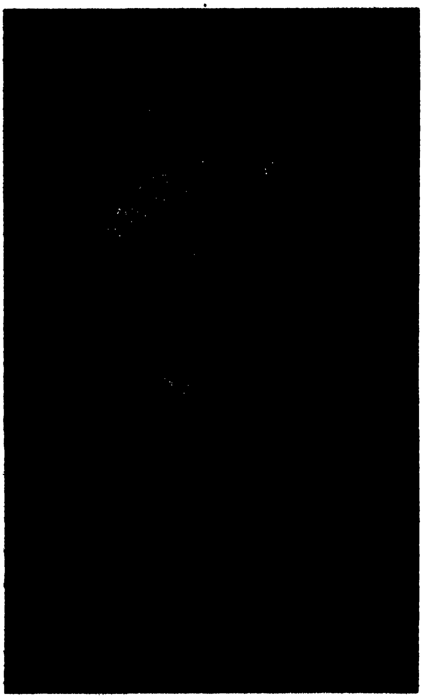
1,	1 person in	6	has the characteristic more emphasised.			
1·5,	1 „ „	14 to 15	„	„	„	
2,	only 1 „ „	50	„	„	„	
2·5,	only 1 „ „	100	„	„	„	

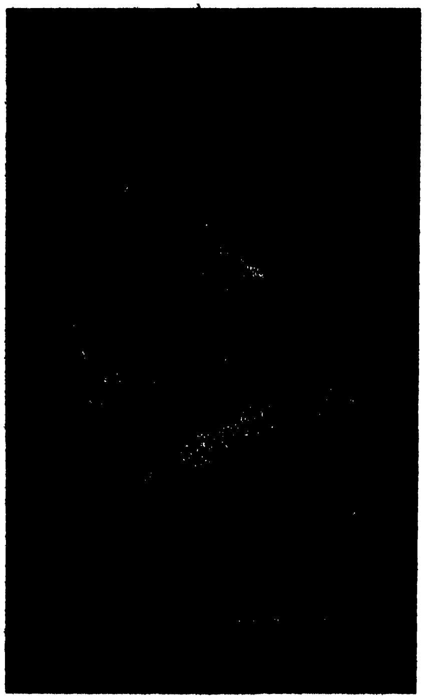
I think a characteristic can hardly be spoken of as individually remarkable until it occurs only once in at least 50 or more individuals. Values of 1 to 2 may be said to contribute somewhat to individuality, but not to be markedly personal features. Now let us look down the fifth column of our Table on p. 238 and see the points in which the owner of this skull possessed individuality. We may examine first certain points in which he certainly had no individuality. I refer to those mentioned by Sir William Hamilton. George Buchanan had certainly not a long nose, it was just the nose of the average man we meet in the street; he had not a long upper lip, one man at least in every five we see has a longer lip than George Buchanan had; again he had not well-defined cheek bones, for his facial breadth and bizygomatic breadth are just normal. In addition he had not a high forehead, his cranial height, whatever estimate we take of it, is simply mediocre, and his frontal bone has precisely the shape of the average British frontal bone. The skull in no particular corresponds with that of a man of high brow, long nose, long upper lip and well-defined cheek bones.

Now let us see the points in which the owner of the skull was individual. First we see from the $-2·75$ and $-2·29$ that he was individual in the extreme shortness of his head. Next we see that the sagittal arc and its components were short, and that the horizontal circumference was small; this of course follows from the shortness of the length, but the individuality is by no means so marked. In other words the arcs have made up for some of the deficiency of the occipital region. We may express this by saying that the skull, if small, is "well-filled." The idea indicated by "well-filled" is the following: Suppose we made a box of the exact length, height and breadth of the skull, and placing in it a bladder, blew air into the bladder until it *just* touched all six sides of the box. That would *not* be a "well-filled" bladder. Now continue to blow air into it; its three principal diameters would be forced to remain the same, but the sagittal, transverse and horizontal arcs would continue to increase and compensate for the smallness of the principal diameters; such a bladder would then be "well-filled." This is the best interpretation I can place on the term a "well-filled" skull*.

* It must be noted, however, that if the capacity be reconstructed from the three principal arcs (see *Biometrika*, Vol. III. p. 370) its value is found to be 1342·5, actually *less* than that reconstructed from the diameters (L, B, OH), i.e. 1372·7. The mean of these two values (1358) is very close to the value obtained by packing with mustard seed.

The next value in excess of unity is that for the dacryal arc. Buchanan's dacryal chord is just the average, but the dacryal arc is short. He must therefore have had a flat bridge to his nose. You will remember this type of nose occurs in the busts of Socrates. I think we must not on this account rush to the conclusion that Buchanan shared the wisdom of Socrates. Indeed modern medicine attributes such noses to courses of conduct hardly wise. But the point I would have you bear in mind is that a picture which has a narrow nasal bridge cannot be a true portrait of the owner of this skull.





Passing down our last column (p. 238) we see that the next value which attracts our attention is the size of the orbits; these, notably the breadth of the right orbit, are marked characteristics, especially when we consider that the skull is small. We must look out in the portraits for large orbits, especially the right one.

The next character which attracts our attention is the markedly short palate; but I think we must proceed here with great caution. I have marked the measurement with a double query, the palate is too much broken to have a good reconstruction, and the skull being short we might assume it to have a short palate. I am not certain of this however, for the auricular point is much in the average position, and accordingly the position of the basion as reconstructed cannot be far out, so that the skull base must be near the normal length, and the palate would not necessarily have been so short. The fact is that the chief peculiarity of Buchanan's skull lies entirely in the occipital region, and as there is no profile

portrait this peculiarity, and naturally that of the palate, cannot be of service for the portraiture.

We now pass to the length of the foramen magnum. This is again a doubly-queried measurement, but I think the foramen was very small; as it is of no service for portraiture we may anyhow pass it by. In going further down our column we notice some slight individuality in the fundamental triangle, but it is only a "one-man-in-six" individuality and may be entirely due to a small error in the reconstruction of the basion. We again need not stay over this point.

Next we reach the two cephalic indices, which are naturally in excess, since the height and breadth are normal, while the length is very short. A cephalic index of 80·5 would be about 83 on the living individual. Out of 3377 living British persons I found 249 or 7·4 % with a cephalic index of 83·5 or upwards. This is about 1 in 14 to 15, precisely what I have said before was indicated by a figure of 1·5 in the last column of our table. Now a cephalic index of 83 for the living head is by no means uncommon in Scotland. Buchanan is said to have sprung from Killearn in Stirling, which according to Dr Tocher's measurements of Scottish soldiers is the most long-headed district of Scotland (mean cephalic index 77·3). His name, however, if we discard the mythical Irishman, suggests a north-eastern rather than mid-western descent. But the best of Aberdonians cannot claim an average cephalic index of more than 78·4, in practical accordance with that of Britishers south of the Tweed. Whence George Buchanan's round-headedness came we cannot say, so ignorant are we of many of the lines of his ancestry. It is possible, but by no means needful, to suppose foreign blood*. Unfortunately this very marked feature of the skull is of no avail for determining portraiture. As far as portraiture is concerned we are left with facial characters only. Most of these characters possess no marked individuality, they are close to those of the average British skull. But there are certain characters which are individual. We should not expect to find a long nose with a narrow bridge, but an average length of nose with a flattened bridge. We should expect not a long upper lip but one of moderate dimensions. We should not expect emphasised cheek bones, but rather a rotundity of face, and this would be surmounted by a hemispherical skull cap suggesting no great height of the forehead. There is nothing in the skull whatever to suggest a long oval face. George Buchanan was certainly *not* a "high brow."

The skull itself will give you the best idea of these points (see Plates V to IX). It is difficult now-a-days to speak of a bullet-headed man, for the shape of a bullet

* Professor Hume Brown says Buchanan came of the happy mixture of Celtic and Teutonic blood. If Celtic origin for Buchanan be assumed because one of his distant ancestors is said to have come from Ireland, and he himself could speak Gaelic, we must anthropologically assert that there is no such thing as a "Celtic race," only a mass of men of very different head shapes, whose ancestors spoke the same family of languages. As for "Teutonic," it is again a linguistic and not a racial term. Of a brachycephalic German ancestor—to account for Buchanan's round-headedness and absence of occipital—I can find no trace in such pedigrees of the Heriots as exist. If by "Teutonic," we are merely to understand Lowland Scottish, these are identical with our Londoners and, so far from being essentially Anglo-Saxon, have for their nearest congeners the men of the Iron Age in Britain.

has changed, but we may I think safely say the owner of this skull had an apple-shaped rather than a pear-shaped head.

With this preliminary consideration of the skull we now turn to the portraiture. As in the case of many men of distinction there exist several incompatible types of portraiture, so for Buchanan there are divergent pictures which undoubtedly cannot all represent him. Those who start with the idea that a particular portrait is certainly that of Buchanan, citing written evidence for it of a certain grade of probability, but then find this portrait does not fit the skull, ought, I hold, to do one of two things:

(i) to consider whether the evidence for the authenticity of the picture is more or less reliable than the evidence for the authenticity of the skull;

(ii) to consider whether the skull is more appropriate to other portraits which have, perhaps, less evidence of their authenticity, but which may after all represent the Simon Pure.

What they certainly must not do is to disregard the skull straight away because it does not accord with the portrait they have selected as giving their own ideal of George Buchanan. That I take it was the line adopted by Mr William Carruthers, who simply dismissed the skull because it was not like one type of presumed portrait. Nor must anyone on the other hand, overlooking the fact that craniometry is a definite branch of science, assert that the skull possesses attributes which actually do not belong to it. A good illustration of this occurs in Dr Aeneas Mackay's article on George Buchanan in the *Dictionary of National Biography*. He writes:

"Tradition dating from a short period after his [Buchanan's] death ascribed to him the skull preserved in the Anatomy Museum of the University, of which there is a print in Irving's life, and which certainly resembles the best authenticated portraits of him which have been preserved, that by Boinard, engraved in Beza's 'Irones' and of which a copy is in the University of Edinburgh."

It is difficult to imagine more errors than are crammed into the last three lines of this statement, overlooking the grammatical blunder, by which "the best authenticated portraits" is suddenly reduced to the singular as "that by Boinard."

Now let us first be quite sure of our dates. George Buchanan died in 1582, aged 76. Beza's volume of *Icones* [not "Irones"!] was published in 1580; it contains no engraving of Buchanan, and no biography of Buchanan. In 1581 Beza's *Icones* was translated into French by Simon Goulard; there was no portrait of Buchanan in it.

Only in 1673, more than ninety years after Buchanan's death, Pierre Chouet published an anonymous volume at Geneva entitled: *Les Portraits des Hommes Illustres, qui ont le plus contribué au Restablissement des belles lettres et de la vraye Religion.*

There is no text to this work, beyond the list of portraits, and no explanation of its origin. All the blocks of Beza and Goulard are used, and *many more besides.* Among the latter is one of GEORGE BVCANAN (see Plate III). It has been too

hastily assumed that Beza had a block cut which neither he nor Goulard used, and that this was preserved for 90 years, to be published later in the following century by Pierre Chouet, also at Geneva.

That portraits of John Knox and George Buchanan were sent to Beza is well known. They were sent by Peter Young, who in a postscript to a letter of Nov. 13, 1579, addressed to Beza, wrote "Just as I am signing this letter the painter has opportunely come in, and brought in a box the likenesses of Buchanan and Knox*."

Further, we find in the Scottish Treasury Accounts: "To Adrian Vaensome, Fleming, painter, for twa picturis, and send to Theodorus Besa, conforme to ane precept, as the samin producit upon compt beris £8. 10*s*."

These pictures must have been painted when Buchanan was 73 years of age. The Beza portrait was not therefore, as Dr Mackay states, painted by Boinard (*sic*), but by Vansom†; it did not appear in Beza's *Icones*, and there is no reason to suppose a copy of that particular portrait is now in the University of Edinburgh.

But let us go a stage further. In 1598 J. J. Boissard (not Boinard, be it noted!), fifteen years after Buchanan's death, issued his *Icones Quinquaginta Virorum illustrium Doctrina et Eruditione*. This is a work independent of Beza's *Icones*, and in this is a portrait of George Buchanan (see Plate III)—"Georgivs Bvchananvs Æta suæ 76." This engraving is by Granthomme; Boissard was the author of the book, not the engraver. Now this engraving in Boissard's *Icones* opens a new field of inquiry. For while the crude engraving in Chouet of 1673 does appear to be the same man as in Boissard, it is not possible to believe that the Boissard copied the Chouet. In the first place there is no evidence that a man working in *Frankfurt* in 1598 would have access to an unpublished engraved block of Buchanan in Geneva, even supposing the block was at that time in existence. Further, and what is more important, the Boissard portrait is clearly a copy of the Bronckhorst type of portrait, and it bears the same age, 76‡, as the picture in the National Portrait Gallery in London. It is not only the age which associates the 1598 engraving with the London picture, but the lobe and arrangements of the left ear, cap and locks of hair are alike, and differ from that of the St Andrews portrait of this type, which indeed bears the age 75 and not 76 (see Plate X). Other pictures of the Bronckhorst type, such as those in the Edinburgh University Senate Hall (with scroll in right hand) and at Glasgow University need not be considered from this standpoint. One item may, however, be referred to. The Granthomme engraving in Boissard has a large fur collar. Such a collar it is difficult to discover in the London or St Andrews pictures, but it does appear in the second University of Edinburgh portrait—that with a book and hands resting on a balustrade. The lobe of the left ear and the hair arrangement are however

* Hume Brown: *John Knox*, 1895, Vol. II. pp. 322—324.

† No trust can be put in the Scottish spelling of foreign names in those days. This name may be identical with Vansome, in which form it also occurs, or with that of Vansomer. The best known painter of the latter name, as Mr H. M. Hake informs me, was active in the seventeenth century, but had progenitors of less importance in the sixteenth century.

‡ Not the age 73 of the Vansom portrait be it noted.

very different (cf. Plates III, IV, X, XI and XIII). It is very difficult to believe that the Edinburgh balustrade portrait is an independent portrait of Buchanan made two or three years before the Bronckhorst—the attitude is so completely the same, and the motto upon it is the same*. A copyist might omit the hanging lobe of the ear, but he would be extremely unlikely to introduce it *de novo*. The Edinburgh scroll picture, the Glasgow picture—which is almost certainly an early copy—and the St Andrews portrait of this type, which is dated a year before the London, leave the left ear in obscurity. On this account I am inclined to give precedence to the London portrait over all these with its marked individuality of left ear.

Another "balustrade" type of portraiture has come to my knowledge since my lecture; it is in the possession of Mr T. D. Findlay of Glasgow. It is difficult to say whether the Edinburgh or Glasgow portrait must be considered the original. The Glasgow portrait bears the same words "*Sic Buchananus ora*, etc.," but has no statement of Buchanan's age. The fact that the book and hands are thrown more into shade than in the Senate Hall picture, and a general impression of more life and less woodenness, are in favour of Mr Findlay's picture; but on the other hand the details of the hair over the left ear and of the fur collar are much more highly elaborated in the Edinburgh picture. I must leave my readers to judge for themselves which picture is the earlier. See Plates XI and XI *bis*.

Now it appears to be universally agreed by those who have written on the portraits that the London and the St Andrews pictures are by Bronckhorst, and that the Chouet engraving follows the missing portrait by Vansom sent to Beza, but if this be so, how does it come about that the Chouet engraving has for its *right* ear the peculiar lobe, the cap covering a part of the upper ear and the hair arrangement of the Bronckhorst *left* ear? We may suppose one to be a reversal of the other, made by the artist drawing directly onto the plate. But why should the cap be just covering the upper part of the ear and the curl be practically the same in two independent portraits, taken by different artists at an interval of two or three years? Further the London picture shows Buchanan with a black velvet cap, the Boissard-Granthomme engraving has this velvet, and the crumpling of the velvet is indicated by a series of "light" lines or patches descending to the forehead. These light effects are surely misinterpreted in the Chouet, and instead of a black velvet cap, when the wood-engraver sets to work, we have ruled lines, indicating a pattern on the cap, making it look like a printed cotton night-cap rather than the usual scholar's head-dress. No such cotton night-cap appears anywhere else in Beza's *Icones* or Chouet's engravings (see Plate III). I suggest that there is strong evidence that the Vansom painting of Buchanan never reached Beza or, if it did, he never had it engraved, but that Chouet, when 90 years later

* Mr William Carruthers (George Buchanan, *Glasgow Quatercentenary Studies*, 1906, Maclehose and Sons, 1907, p. 359) associated this portrait closely with the London one, without saying that it is a copy. He says that on the upper right corner is written "Ann. Aet. 76," i.e. the same date as the London picture, but my photograph of it has quite clearly, not on the upper right, but the upper left corner, "Aetatis 73," which if we could trust it, would make this picture three years older than the London picture and the subject's age the same as that of the Vansom picture.

Jeremy Bentham,
with contour lines from a photograph of the "Auto-Icon."

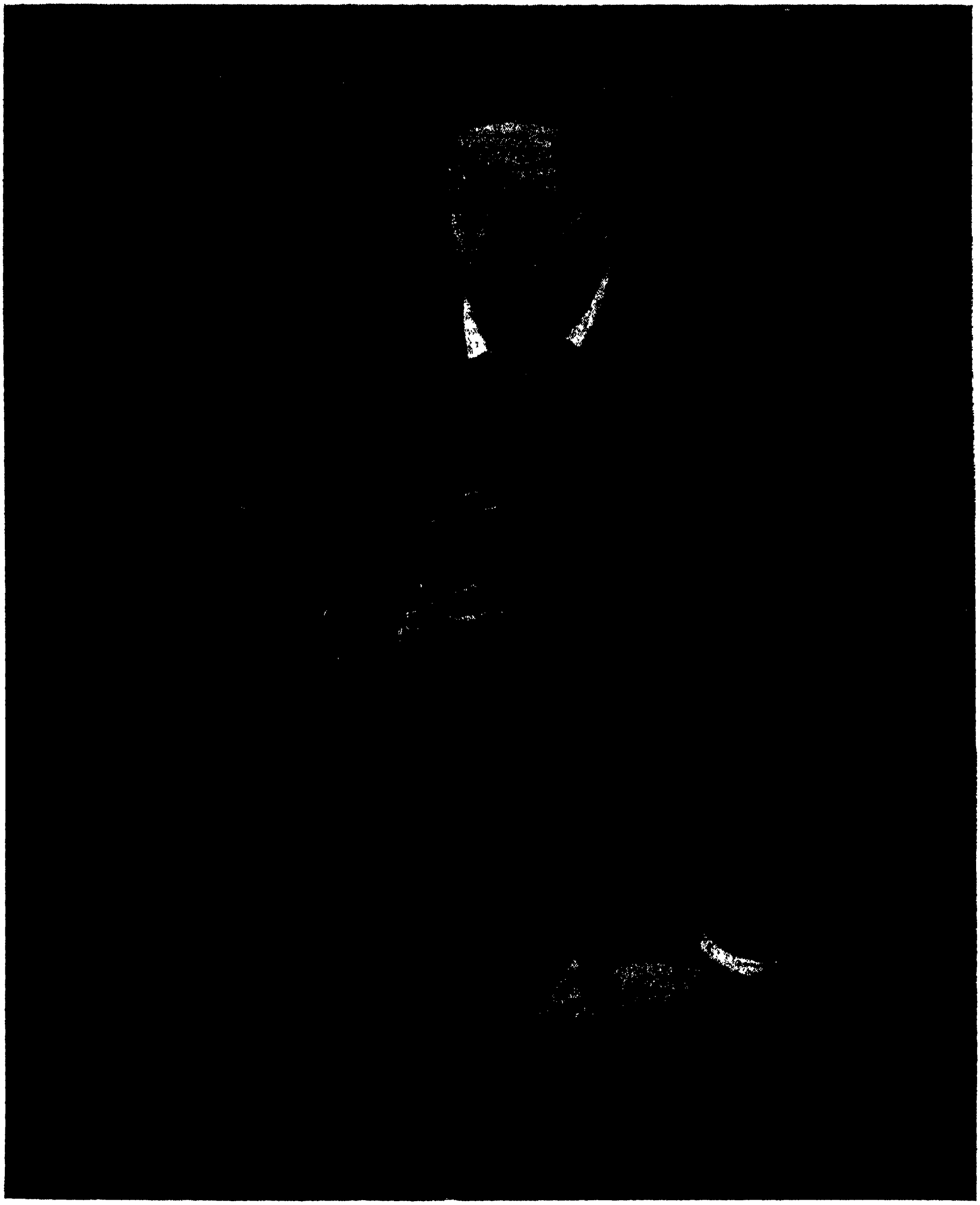
Sir Thomas Browne
NATIONAL PORTRAIT GALLERY

Allix Portrait of Sir Thomas Browne (now in the National Portrait Gallery). The fit of the skull shows a passable portrait.

Sir Thomas Browne
ROYAL COLLEGE OF PHYSICIANS

Portrait of Sir Thomas Browne (in the Royal College of Physic
The artist has idealised absurdly the forehead to empl
intellectuality.

GEORGE BVCANAN.

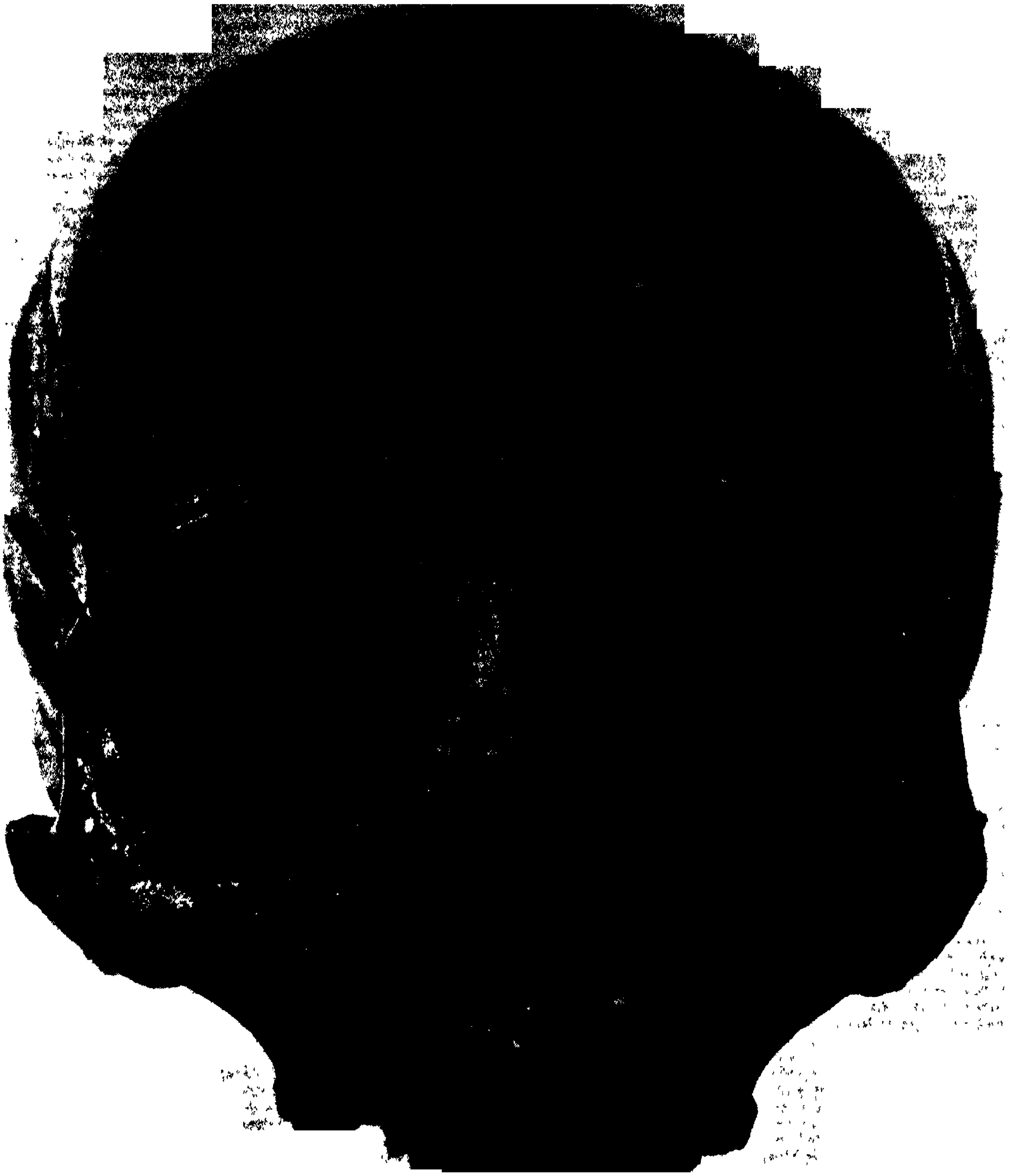


Bronckhorst type, from the National Portrait Gallery, London. Asserted date, 1581.

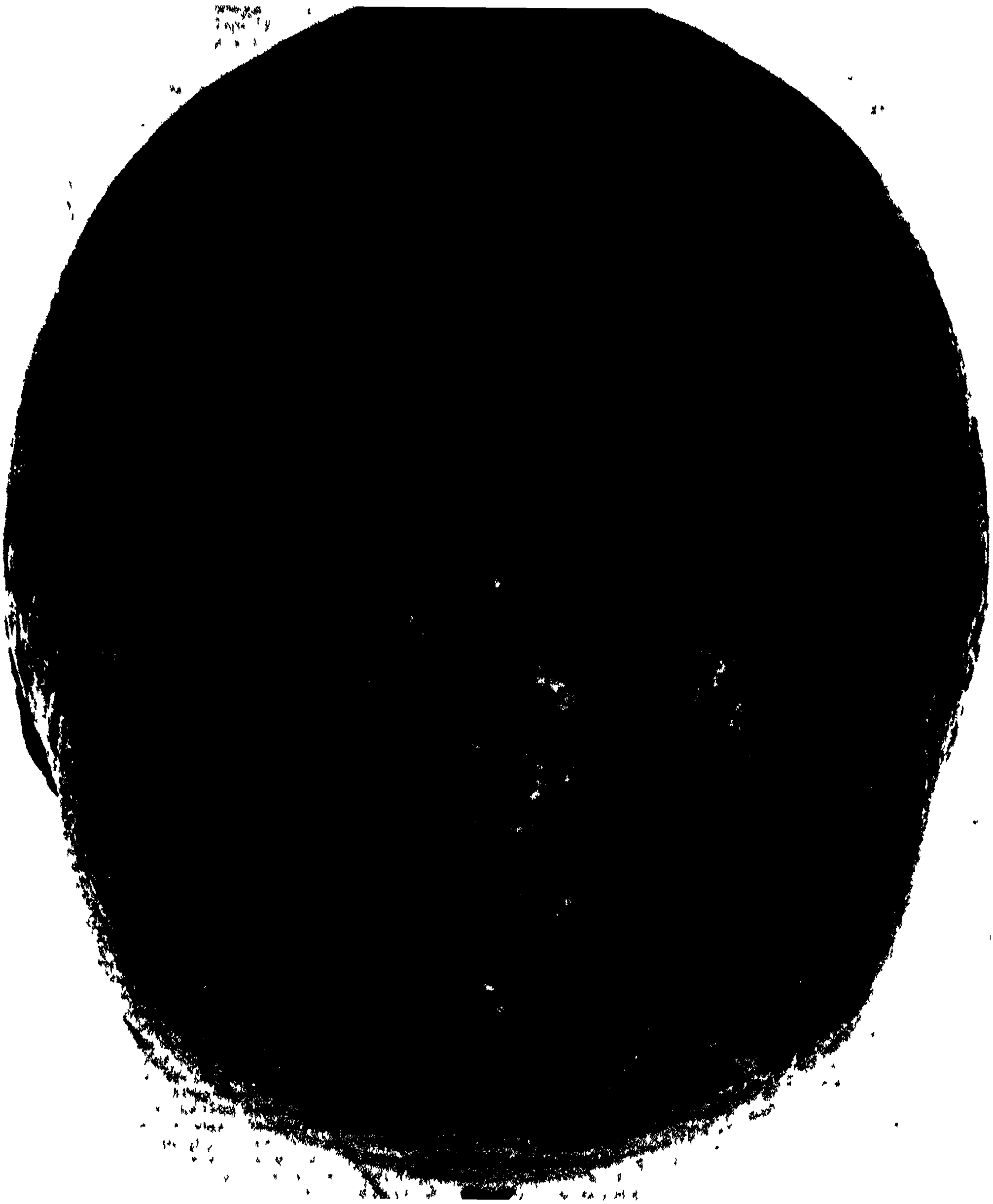
[*Reproduced by permission.*]

Bronckhorst type, from the Landesbibliothek, Wolfenbüttel. A copy made 14 years after Buchanan's death either from the National Portrait Gallery (London) painting, or more probably from a missing picture from which the latter also was copied. [*By kind permission.*]

Plate V

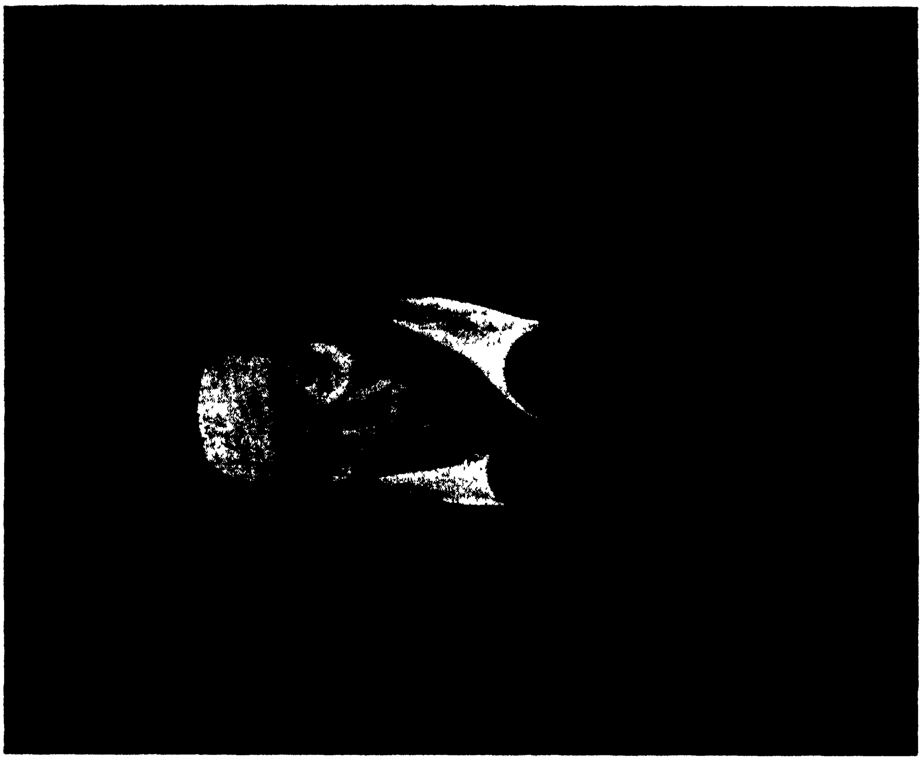


Buchanan's Skull; *Norma Facialis.*

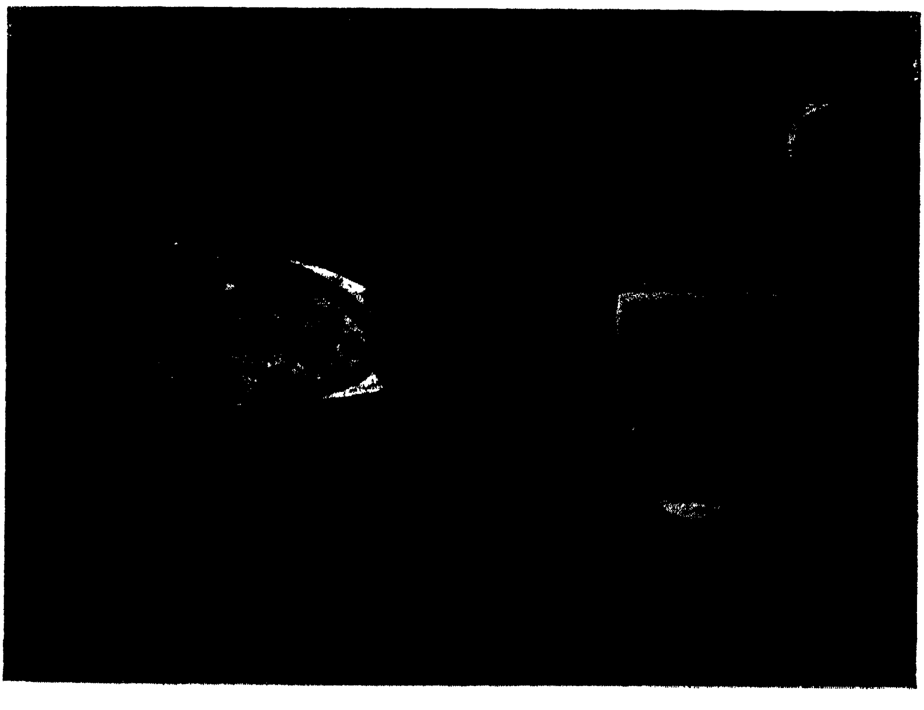


Buchanan's Skull; *Norma Verticalis.*

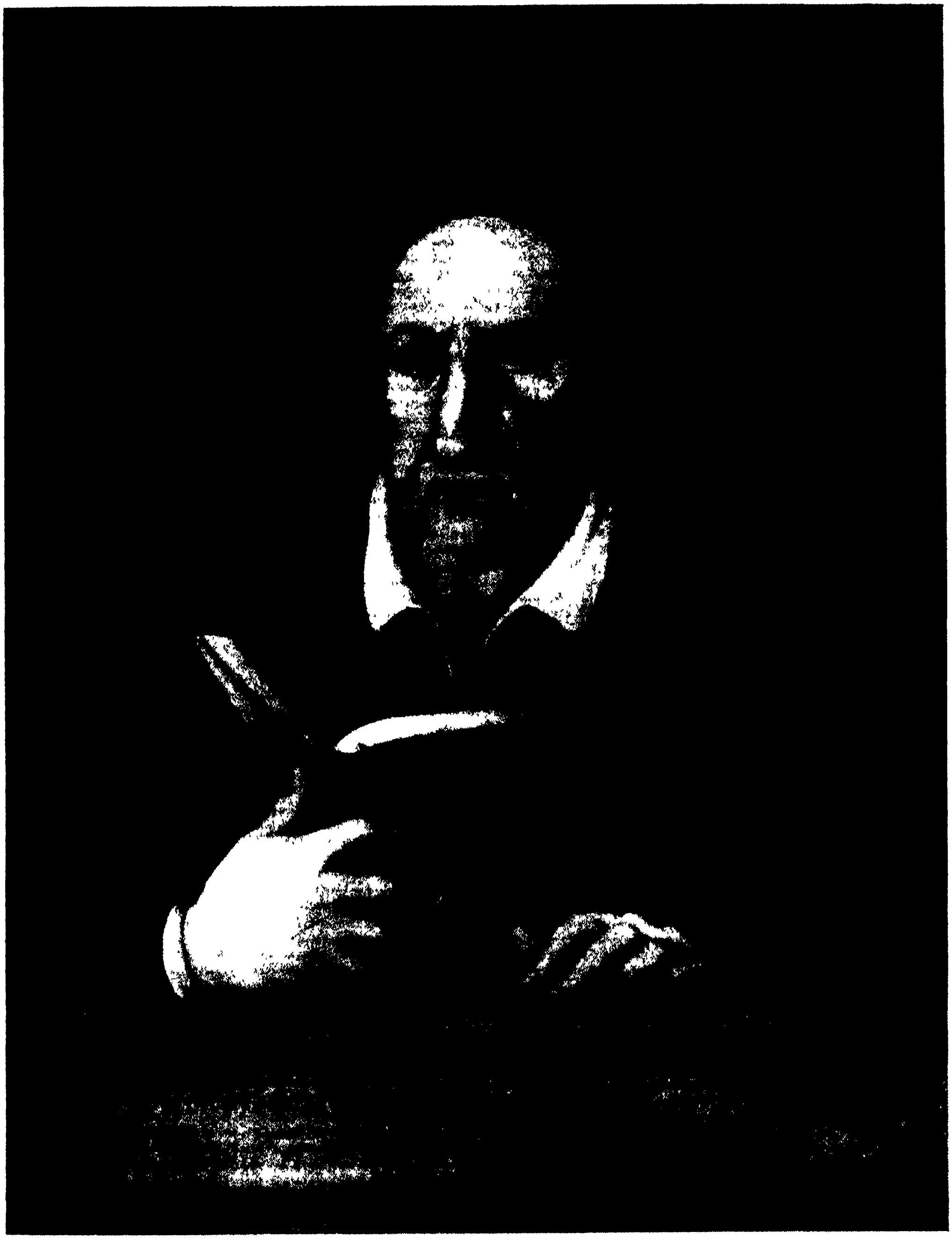
Buchanan's Skull; *Norma Occipitalis.*



Bronckhorst type, from Glasgow University. A copy of one of the earlier pictures. [*By kind permission.*]



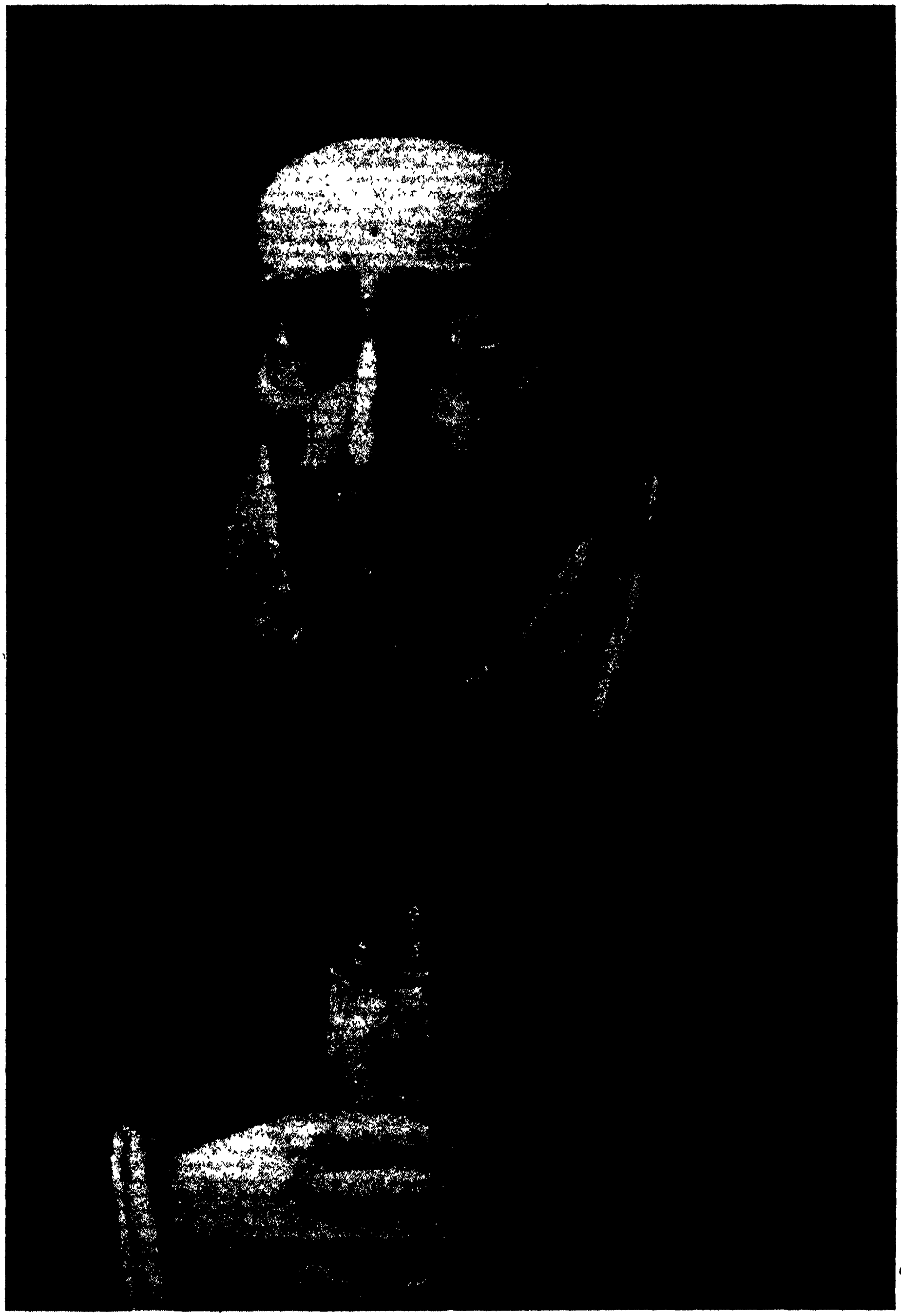
Bronckhorst type, from St Andrews University. Asserted date, 1580. [*By kind permission.*]



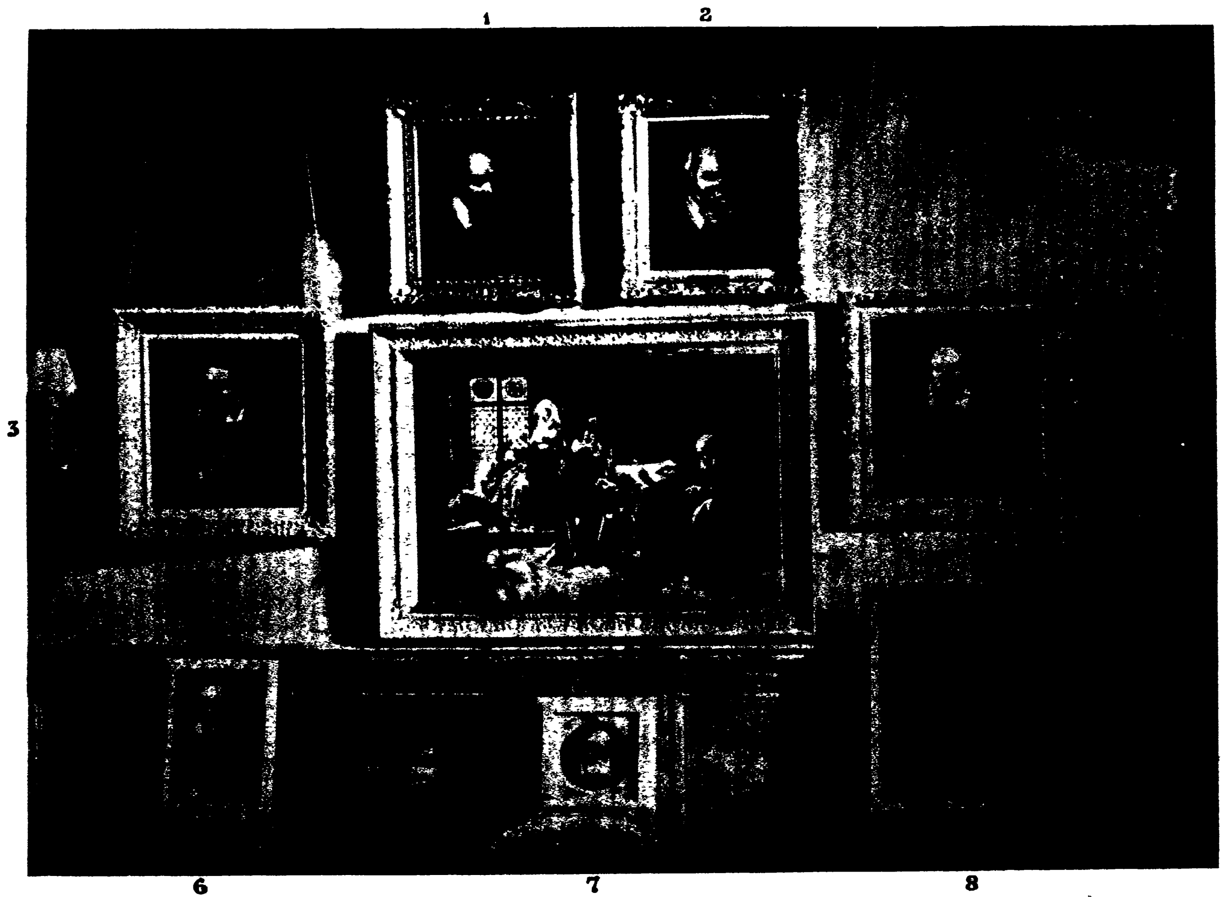
"Balustrade" Portrait, from the Senate Hall, Edinburgh University. Asserted date, 1579.
Clearly Bronckhorst type. [*By kind permission.*]

Portrait of George Buchanan of the Edinburgh "Balustrade" type in the possession of Mr T. D. Findlay of Glasgow. [*By kind permission.*]

Engravings clearly made from the Edinburgh "Balustrade" type of portrait. From originals in the Department of Prints, British Museum. Probably used as title-pages for hitherto undetermined editions of some work or works of Buchanan. [See p. 250 of the text.] Probable first to second quarter of 17th century.



Highly idealised portrait of the Bronckhorst type, in the Senate Hall, Edinburgh University. Note the disappearance of the lobe of the left ear. Probably a rather late copy. [*By kind permission.*]

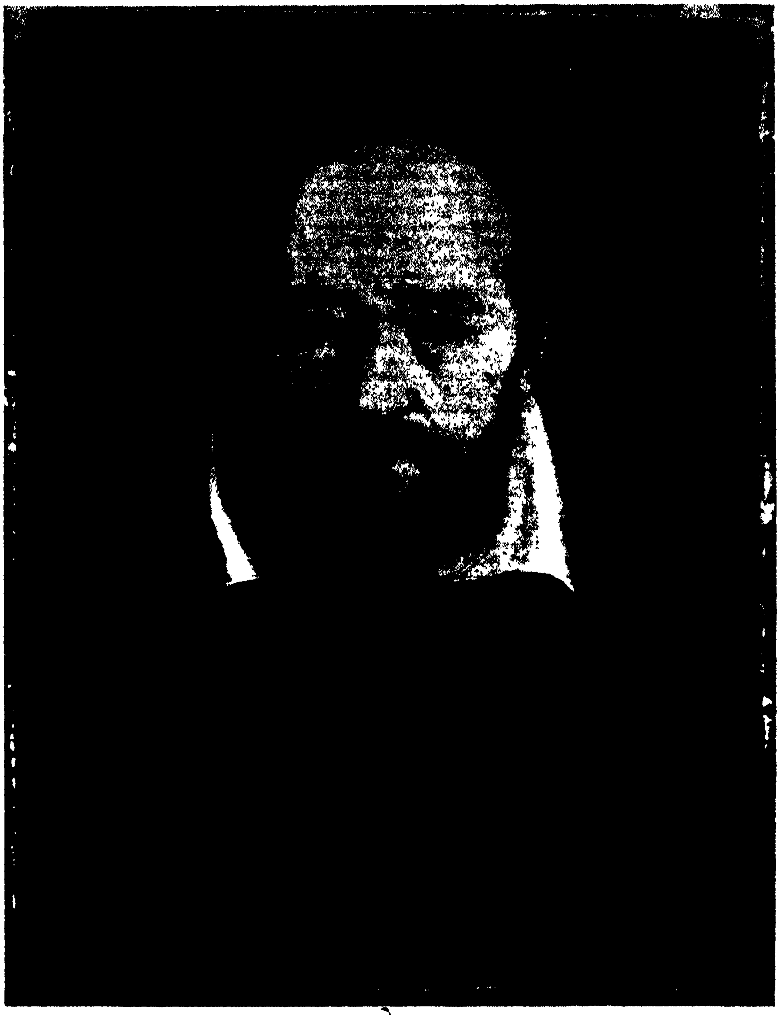


(1) Copy of 2 belonging to Buchan Society. (2) So called "Titian" portrait at St Andrews.
(3 & 4) The Aberdeen portraits, copies of the Edinburgh. (5) Mrs Buchanan's modern picture.
(6) Edinburgh. (7) Duke of Sutherland's. (8) Edinburgh.

General indication of the various types of Buchanan portraiture, taken by kind permission of Mr Maitland Anderson from his *The Writings and Portraits of George Buchanan*, 1906.

The so-called "Titian" portrait at St Andrews University, probably originally in the possession of the Earl of Buchan.
[*By kind permission.*]

Photograph of portrait of President Jeannin of Dijon, from the Musée of that city, showing that Titian's "Buchanan" is a faked portrait. [Photographed for this work.]



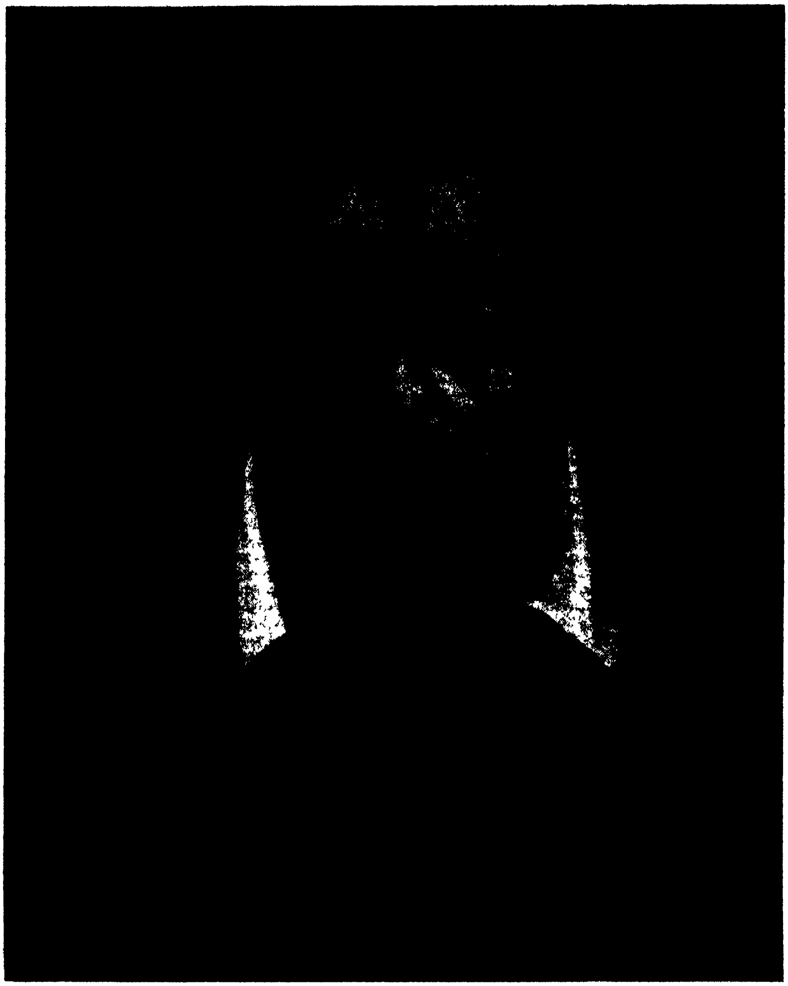
The Royal Society or "Poemata" type of Buchanan portrait, originally attributed to Porbus, possibly by Adrian Keij.

[*By kind permission.*]

The Dunrobin Portrait of Buchanan of the "Poemata" type, formerly in the possession of Dr Mead, and now in the possession of the Duke of Sutherland.

[*By kind permission.*]

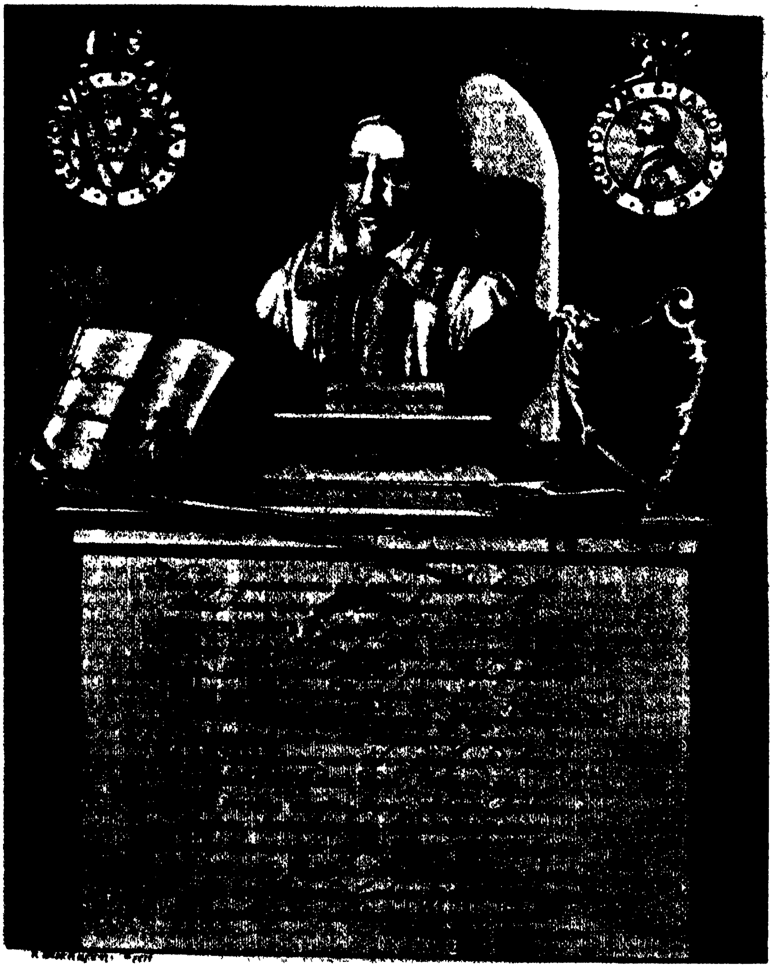
The Houbraken engraving (reduced) of Dr Mead's portrait which identifies the latter with the picture now at Dunrobin Castle.



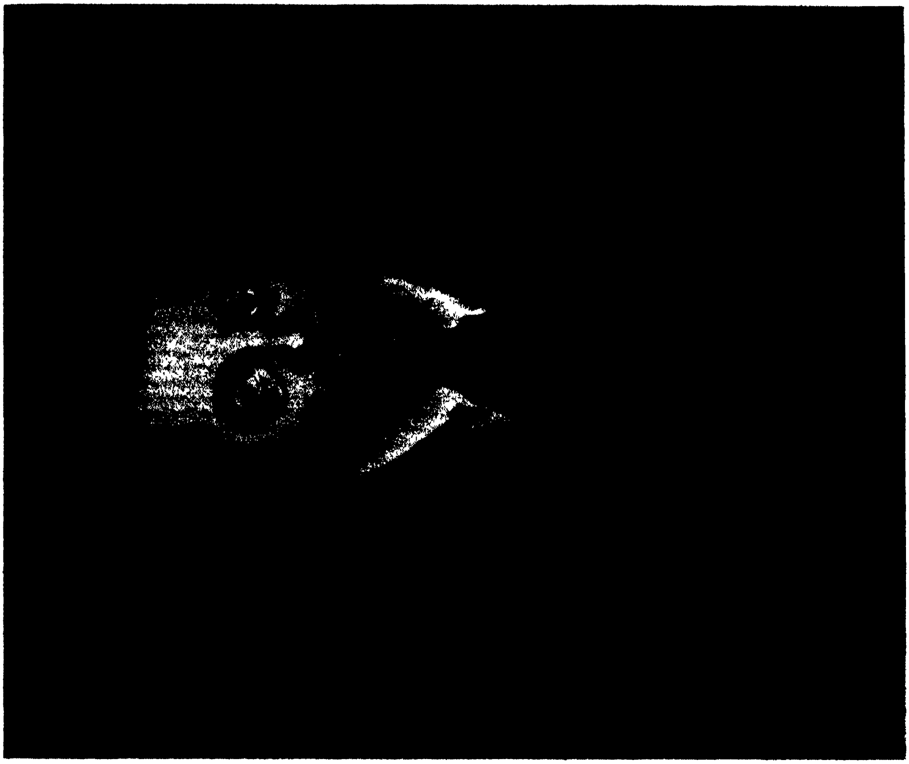
Mr Sowersby's Portrait of Buchanan of the "Poemata" type, said to have been formerly in the possession of the Dowager Countess of Seafield. [*By kind permission.*]

The " Poemata " type of Buchanan portrait, in agreement with the Povey, Mead and Sowersby Portraits.
First issued in 1628 and maintained in the Amsterdam editions of the 17th century.

Title-page of the first Edinburgh Edition of the *Opera Omnia*, by Ruddiman in 1715, showing a highly idealised medallion of Buchanan of the "Poemata" type of portraiture. [Reproduced, much reduced, from a copy in the possession of Dr James Bonar.]



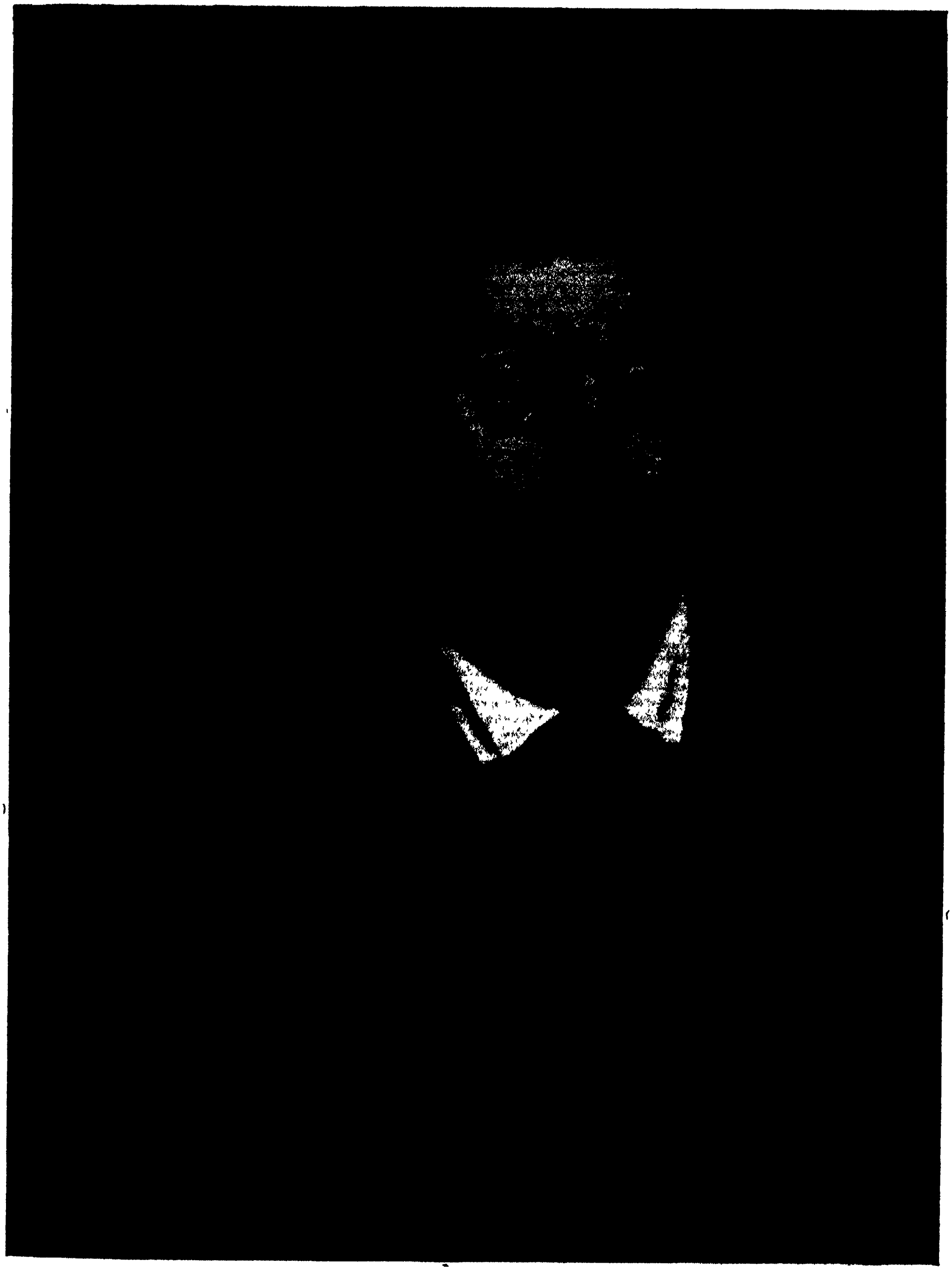
Re-introduction of the Bronckhorst type in the re-issue by Burmann of the *Opera Omnia*, 1725. [Reduced from quarto format.]



The Harding engraving of the youthful Buchanan stated at the time when it was engraved to be at Hamilton Palace. [See p. 254 of the text.]

Miss Christie's Portrait, said to be the middle-aged Buchanan.
[*By kind permission of the owner and of the publishers of the* "Glasgow Quatercentenary Studies."]

Plate XXIV

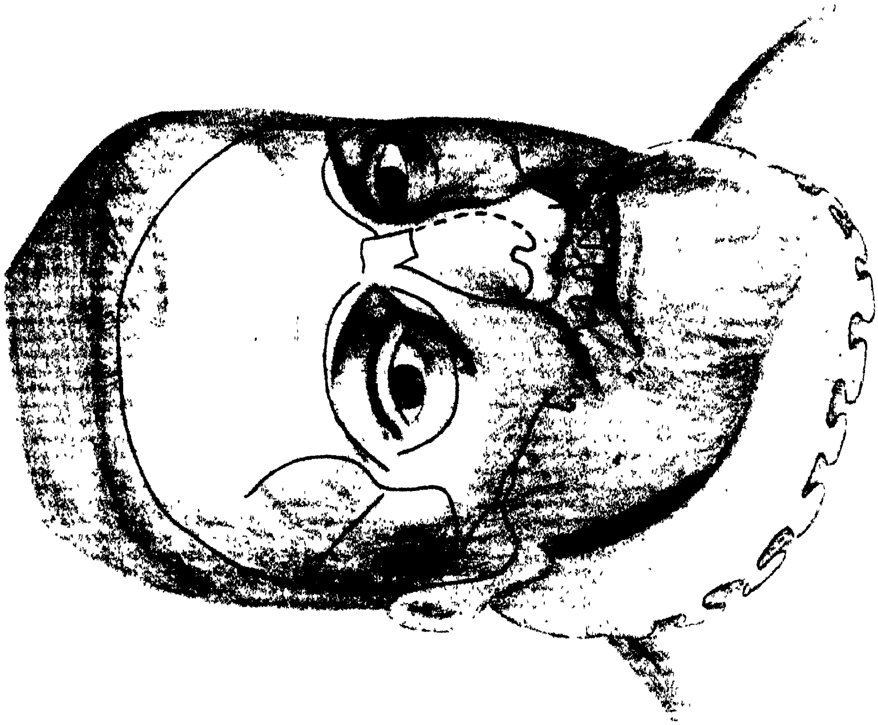


Mr H. B. Buchanan's picture of the youthful Buchanan. This seems to be a more original and less idealised portrait than that of the Glasgow Technical College.

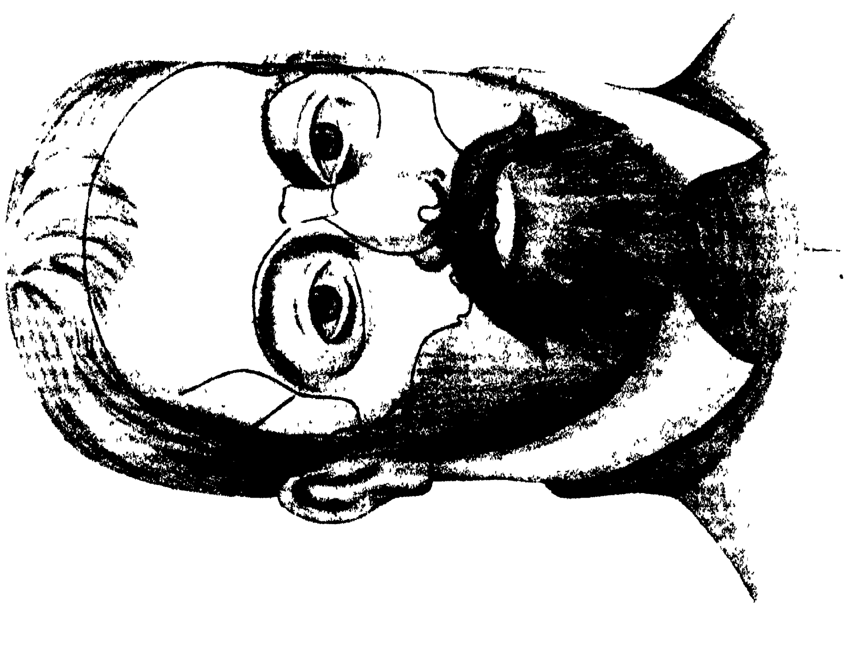
[*By kind permission.*]

Portrait of the youthful Buchanan in the Technical College, Glasgow. Probably a copy of that originally at Hamilton Palace. [*By kind permission.*]

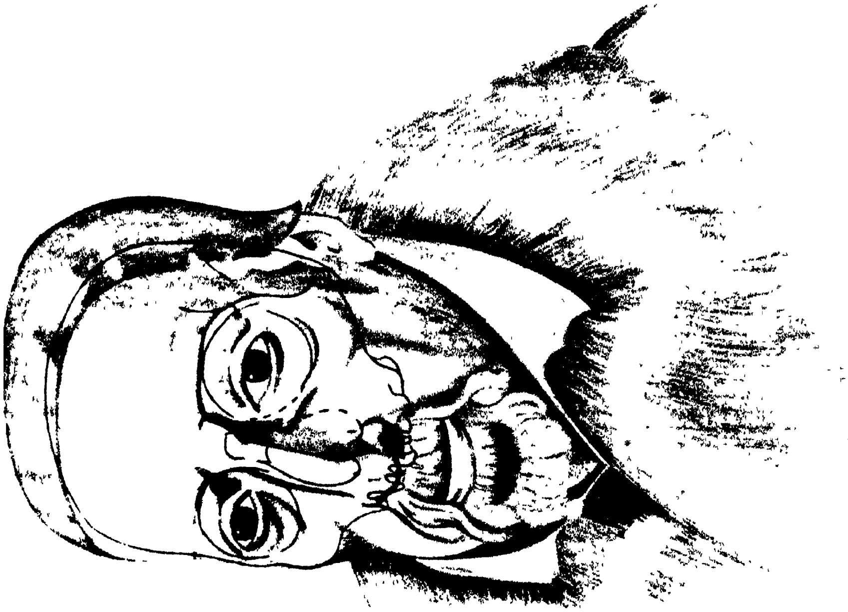
Plate XXVI



George Buchanan
In Possession of Miss Christie



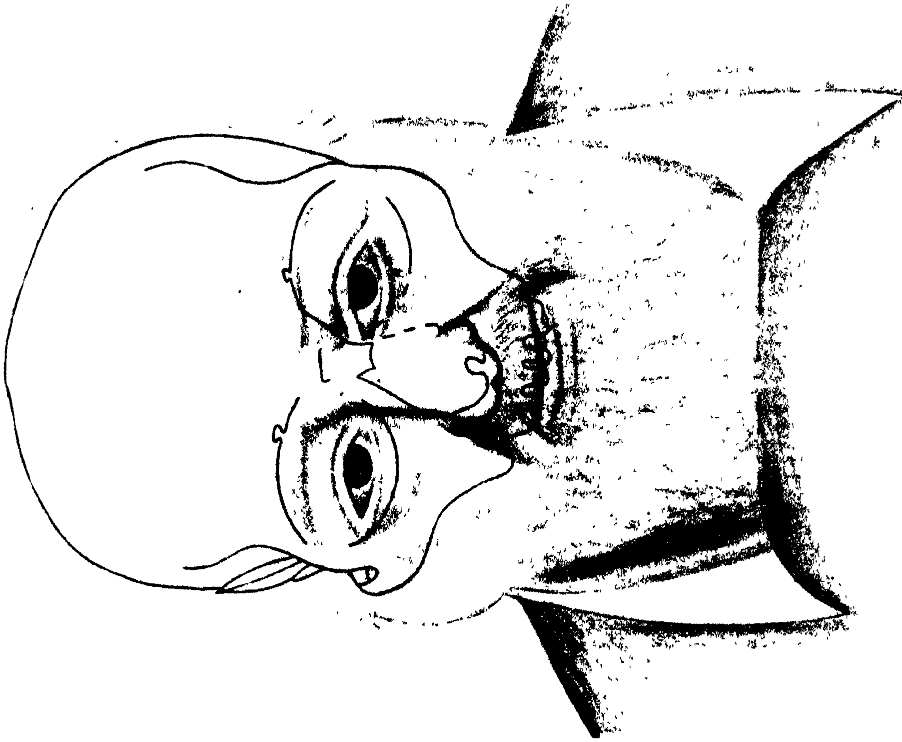
George Buchanan
Technical College Glasgow



George Buchanan
Boissard's Icones

George Buchanan
Supposed Vanson
Chouet Woodcut.

George Buchanan
NATIONAL PORTRAIT GALLERY



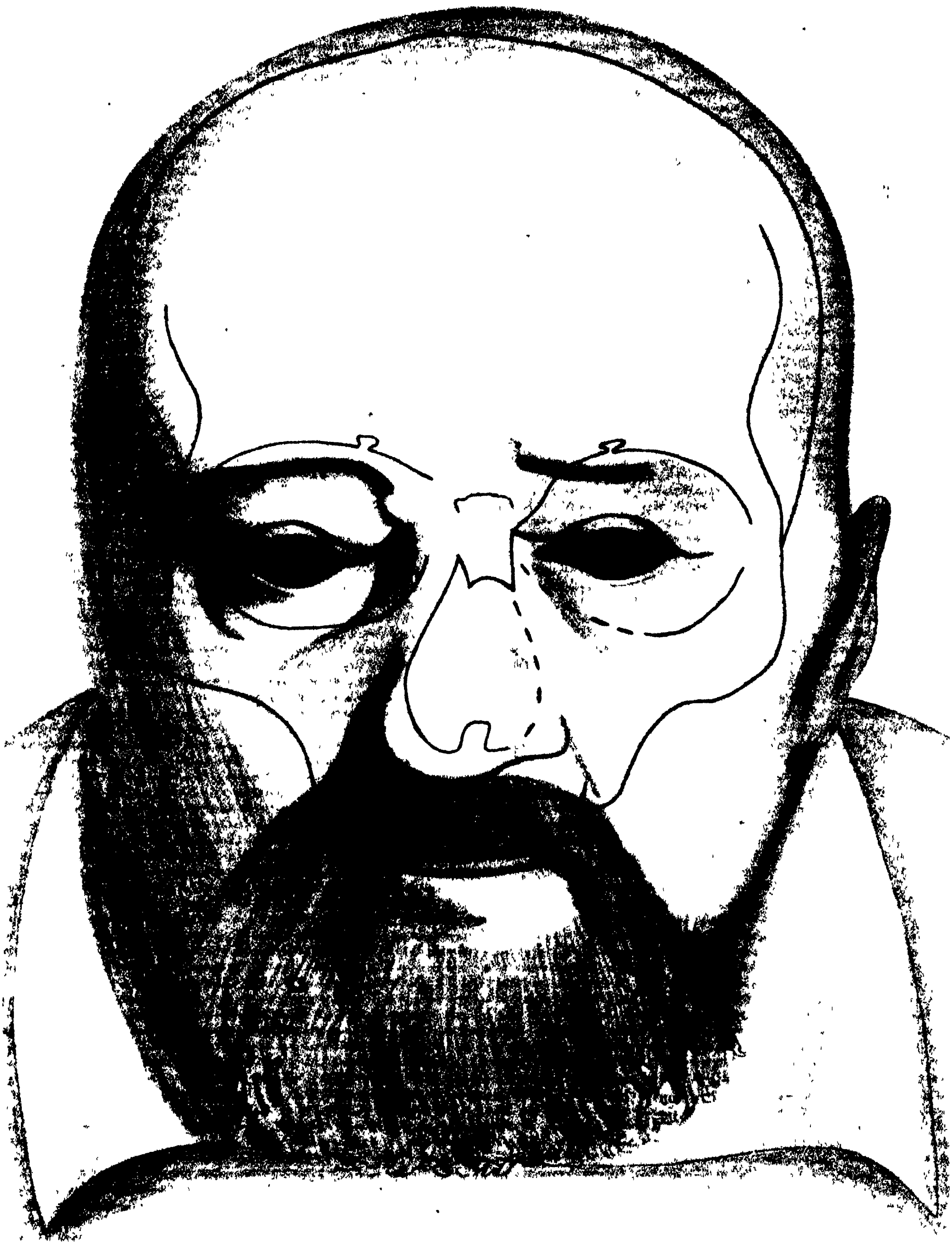
George Buchanan

Belonging to George K Sowersby, Esq

Formerly in the possession of the Dowager Countess of Seafield

George Buchanan

S^T Andrews "Titian"



George Buchanan
ROYAL SOCIETY OF LONDON

he reproduced Beza's *Icones* with 40 new portraits, got a wood-cutter to copy in crude outline without reversing Granthomme's engraving of 1598. It is easy therefore to explain how "one man" has been said to be represented by the Vansom-Beza and the Bronckhorst-Boissard engravings. We do not get proof of the truthfulness of the Bronckhorst portrait if, as I think must be held, the Chouet is merely a crude copy of it. What also becomes of Sir William Hamilton's statement that the Chouet must represent the real man because it fits the skull better? It fits the skull better because the crude wood-cutter has altered the proportions of height to breadth of face. He has given the pear-shaped London face a more apple-shaped form, while preserving the disproportionately long nose, and (for the skull) impossible nasal bridge.

Those who hold that the Chouet woodcut was really prepared from the Vansom painting of 1579, when Buchanan was 73—and when considering this they must remember the Edinburgh balustrade picture, which is given as of age 73—must give up the idea that the London National Gallery picture is the Bronckhorst painting, notwithstanding its date, and suppose it also to belong to the Vansom group. I think they will find it easier to suppose that Chouet copied from Boissard.

We must now ask what is the evidence that associates the London and St Andrews portraits with Bronckhorst, and makes the Edinburgh and Glasgow pictures of this ancestry.

The evidence consists in Bronckhorst's bill to James VI, and his majesty's order for its payment, dated September 1580. The account runs: "Certane portraitures maid by me at his maiesties commaund and delyuerit laitlie to his hienes quherof I have resauit as yit na payement." The first is a portrait of "his maiestie" from the belt, and the price "xvj lib." The second item runs: "Ane other pourtraiet of Mr george buchanane. viii lib." (only half as much for a subject's portrait as for a king's!) and the third item a full length portrait of "his maiestie" xl lib., which shows the value of the royal legs.

The Buchanan item "Ane other pourtraiet" may mean that the first portrait was that of the King, but it might also mean that Bronckhorst had painted a previous portrait of Buchanan. Arnold van Brounckhurst* must have painted his picture or pictures before September 1580. The St Andrews portrait has on it the date 1580, and the age 75; the London portrait is dated 1581, with the age 76. Both portraits are clearly closely related to each other, the St Andrews being in a much worse state of preservation.

In 1580 James was only a boy 14 years of age, and he may or may not have wished for a portrait of his tutor. The order for payment is countersigned by Angus and Argyle, who then dominated the situation and may themselves have desired the portrait of Buchanan. What James thought of Buchanan and his works in his later years is well known, and in the lists of paintings belonging to Charles I prepared after his execution, no portrait of Buchanan appears in royal ownership.

* Thus the name seems to be spelt in the Scottish records. This artist was probably related to the better known and better spelt Bronckhorst of the seventeenth century.

Now without denying that the whole group of portraits to which we are referring may have originated in the Bronckhorst painting, there is still something obscure about them. In the first place there is the inscription on three of them, which runs: "Sic Bvchananvs ora, sic vvltvm tvlit. Pete scripta et astra, nosse si mentem cvpis."

Mr Carruthers translates this as follows: "This is the face and these are the features of Buchanan. If you want to know his mind, search his writings and the stars."

Now I have forgotten the little Latin I ever knew, but I should translate the sentences: "Thus Buchanan bore his features, thus his face. If you seek his spirit, search his writings and heaven."

In other words the verb is in the perfect and strongly suggests that Buchanan was dead at the time the writing was affixed to the picture. His "mens," the "divina mens," is to be found only in his writings and heaven. Further, Buchanan died on September 28, 1582, and is said to have been born in February 1506, or if that be Old Style, February 1507; he could therefore have only been 75 or at most 76 in the year of his death, 1582, while the London portrait makes him 76 in 1581 and the St Andrews portrait 75 in 1580. Thus the dates on these pictures appear to be erroneous, and I believe the dates and inscriptions must have been put on, and probably the pictures painted, after Buchanan's death. They may all have been copied from an original Bronckhorst which has not survived to our day. If they are copies and not only copies from the Bronckhorst, but one from another, we can easily explain the growth of idealised features and the emphasis of an intellectual spirituality, which may have been far less intensely present in the actual man himself.

In connection with this type of portrait two engravings occur, copies of which are in the Print Room of the British Museum (see Plate XII). The first of these is closely related to the Edinburgh "balustrade" picture. The portrait has an ornamental border by Karel van Sichem, and it is said to be painted by an artist whose monogram, there given, has not been identified. The engraving itself is by a second van Sichem, possibly Christoph, the son of the former. The original ornamental border enclosed an oval portrait which has been cleaned off and replaced by a square frame containing Buchanan. It has been suggested that the print was prepared in Cologne early in the 17th century. One would suspect that it was issued as a frontispiece to some edition of one or other of Buchanan's works, published in Cologne or the Netherlands, but I have not hitherto been able to identify it. It has the usual lines "*Sic Buchananus ora*, etc." A second print (see also Plate XII), obviously a copy of the van Sichem, was drawn by Elias Boschius (van der Bosche?) and engraved by Johan Bussemecher. This is still the Bronckhorst type, but a very inferior copy of the van Sichem. If the statement on the van Sichem plate be correct, the Edinburgh balustrade portrait, however much it resembles the Bronckhorst type, was not itself painted by Bronckhorst, but by the hitherto undetermined artist of the monogram.

We will shortly determine to what extent the skull fits the Bronckhorst group of portraits. Meanwhile we ask what other portraits or pseudo-portraits are available. A very useful graphic index to the various types of portraiture of Buchanan was provided by Mr Maitland Anderson in his pamphlet issued for the St Andrews Quatercentenary*, which he has kindly permitted me to reproduce. It shows four pictures of the Bronckhorst type, two of the Buchan type, and two of the Povey type. These will be found useful for showing at a glance the different types of portraiture with which we have to deal (see Plate XIV).

In the first place there is the Earl of Buchan's so-called "Titian" and its reproductions by Raeburn and others. This picture is at St Andrews, and is not to be confused with the St Andrews picture of the Bronckhorst type. The person represented is not the Buchanan of any other type of portraiture; the face does not fit the skull, and there is little doubt that it is a copy of the portrait of President Jeannin of Dijon which hangs in the Musée of that city (see Plate XV). We can therefore at once dismiss the "Titian" type as no portrait of Buchanan at all.

The next type requires far more consideration. We are certain of at least three portraits of this character, which I will call the "Royal Society or Povey type." Thomas Povey was a member of the Long Parliament in 1647, so it is reasonable to suppose that he was born between 1610 and 1620, i.e. from 30 to 40 years after the death of Buchanan. He was a friend of Pepys and an original member of the Royal Society from 1663. To that Society he presented a portrait of George Buchanan, which has painted on it: Georgivs Bvchanan Scotvs (see Plate XVI).

The costume is that of Buchanan's date. The face is wholly unlike that of the Bronckhorst type. Now the portrait has been some 260 years in the possession of the Royal Society, and therefore the ascription to Buchanan is not a recent one. It carries us back to a date, when probably some of those living had seen Buchanan. Povey was a favourite at Court, and Charles II, the grandson of Buchanan's pupil, who frequented the meetings of the Royal Society, could hardly fail to have seen portraits of his grandfather's tutor. Mr Carruthers, who dismisses this portrait, says that George Buchanan was a common name in Scotland; that two George Buchanans might have had their portraits painted, and that the inscription only says "George Buchanan, *a* Scot." I think it would be more reasonable to translate it: "George Buchanan, *the* Scot," and I think there was only one man of that name who reached fame at that date. Now it is quite clear that the word *Scotus* is significant. If this George Buchanan is not the real man, but some rich merchant who visited the Netherlands and had his portrait painted there, as Mr Carruthers suggests, why was *Scotus* put on the picture? It was hardly needful to identify him in the family portrait gallery! Is it not probable that this portrait, whoever it may be of, was being sent abroad—to England, to Denmark, or even to Geneva,—in which case the *Scotus* was appropriate?

Further, in all the editions of Buchanan's works issued in his life, the author on the title page appears as "Georgius Buchanan Scotus," which is certainly "George

* *The Writings and Portraits of George Buchanan*, University Press, St Andrews, 1906.

Buchanan, *the* Scot." But Povey's George Buchanan does not stand as a sole representative of this type. Dr Richard Mead, the celebrated physician, born in 1673 (d. 1754) was a man of wide knowledge, who had studied under Graevius in Utrecht and Boerhaave in Leyden. He had travelled in Italy and in other parts of Europe and amassed a great library, a gallery of well-chosen pictures and a collection of coins and other antiquities. His library alone contained 10,000 volumes and his pictures, books, etc. sold for more than £16,000 on his death—a very large sum for those days. He built for his pictures and antiquities a gallery attached to his house in Great Ormond Street, London. Among his pictures was a portrait of George Buchanan. This painting is almost certainly the one now in possession of the Duke of Sutherland. It was engraved by Houbraken in 1741 and the picture was then attributed to Porbus (see Plates XVII and XVIII). I do not think this portrait was a copy made for Dr Mead of the Royal Society picture, but it is undoubtedly of the same man and must, judging from the hair, have been taken at a somewhat later time. Why should the supposititious rich merchant of Edinburgh have had no less than three paintings of himself? How did two of them come to London to fall into the hands of two fellows of the Royal Society?—I think these portraits tend mutually to strengthen each other, and to give the Royal Society type a strong claim to be considered when we compare skull and portraits.

Still further confirmation of the originality of this type of portrait has turned up since this lecture was given in Edinburgh, namely a picture in the possession of Mr George K. Sowersby, which is also a portrait of early date with the same expression of features but without any title (see Plate XIX). Mr Carruthers' worthy merchant of Edinburgh must indeed have been fond of having paintings made of himself. It is difficult to place exactly Mr Sowersby's portrait. It is certainly not a copy of the Royal Society portrait, being much closer to the Dunrobin picture. There is no doubt that it is the same man, but it would be unwise to assert at present which is the original.

Like Dr Mead's portrait the Royal Society's has been attributed to Porbus, a Flemish artist, but more recently to Adrian Keij. It is clearly Flemish work. There is no evidence that Porbus, either father or son, came to Scotland, and little certain evidence that Buchanan visited the Netherlands. But the difficulty about a Flemish painter is not really great, as Flemish painters were by no means unknown in Edinburgh. Now we know that Tycho Brahé in his Observatory at the Uranienberg fitted up for him by Fredrik II of Denmark had a portrait of George Buchanan. Brahé and Buchanan had been in correspondence in 1576. Tycho Brahé travelled in Germany and Switzerland in 1575, and possibly then made the acquaintance of Beza and through him of Buchanan. It must also be remembered that from 1560 onwards Buchanan had been busy with his poem *De Sphæra*, which takes the anti-Copernican view—what we might in modern language term the "Fundamentalist" view—of the universe. This was also Brahé's standpoint, and Buchanan may well have sought his advice and interpretation against the Copernican views of the more advanced school of heretics. Buchanan's pupil James VI recognised this portrait of his old tutor, when on going

to fetch his wife Anne of Denmark he visited Tycho Brahé at the Uranienberg. Brahé left Denmark in 1597 and died in Prag in 1601. Buchanan's portrait has been sought in vain in Denmark and Bohemia. If it was left at the Uranienberg it might well pass back to the Danish Royal Family with the observatory, and may have been sent to Queen Anne as of more interest to her husband than to the Danes. But what has this to do with Povey? Just this, Povey was son of Justinian Povey who was treasurer to Anne of Denmark, that frivolous but kindly-hearted queen, who never could keep her expenditure within her income, and was continually pledging her jewels and valuables even to her treasurer. If Buchanan's portrait came from the Uranienberg to Queen Anne it may very possibly have passed to her treasurer Justinian Povey, and thence to his son Thomas who ultimately gave it to the Royal Society*. All these stages are only conjectures, but not very improbable conjectures. It has been suggested that Peter Young, who was in Denmark in 1585 and 1589 about the marriage of James to Anne, presented a portrait of Buchanan to Tycho Brahé, but it is not very clear why he should have done so either 3 or 7 years after Buchanan's death. It seems more reasonable to suppose that Buchanan himself sent his portrait to Brahé. The picture, while it may represent a man well on in the sixties, is more appropriate to that age than to the seventies.

Whence Dr Mead obtained his portrait it is difficult to suggest; if as seems probable, it is the one now in the possession of the Duke of Sutherland at Dunrobin Castle, then the picture is of the same man as the Royal Society portrait, but as I have said he is several years older, for the hair of the head, moustache and beard are all grey. Is it conceivable that Dr Mead's portrait was the Vansom sent to Beza? If that were so, then the Tycho Brahé and the Beza portraits need no longer be treated as missing, and we should have a second type of Buchanan very different from the Bronckhorst type.

Mr Carruthers speaks of the Povey-Mead type of Buchanan portrait as if the tradition of this portrait type had started with some error of assignment by Povey in the last quarter of the 17th century, and that from Povey's error had arisen the whole tradition of this portrait type representing Buchanan. Mr Carruthers seems to have entirely overlooked the facts that: (i) the Povey and Mead portraits represent the same man at *different* ages and (ii)—what is far more important—the so-called Povey type is the type which appears in the Elzevir edition of the *Poemata* in 1628, which is obviously engraved from a picture of the Povey type, if not from the Povey picture itself. In 1628—James I only died in 1625—there must have been persons still living who had known Buchanan in the flesh, yet this passed as Buchanan's likeness not only in this edition of his poems, but in those issued at Amsterdam in 1676 and 1687, until it appears in a highly idealised form in Ruddiman's great folio edition of the *Opera Omnia* published in Edinburgh

* An absurd statement occurs on p. 584 of the *Glasgow Quatercentenary Studies* (George Buchanan), 1906, where we are told Povey presented the picture to the Royal Society in 1633, i.e. about 30 years before the foundation of that Society and when Povey was a very young man just beginning to study for the Bar! He lived on into James II's reign, and probably died towards the end of the century.

in 1715 (see Plates XX and XXI). In other words the Royal Society type was as well known and as often—perhaps more often—published in the century which followed Buchanan's death as the Bronckhorst type. Despite Mr Carruthers, traditional evidence for the Povey type is as strong as for the Boissard-Bronckhorst type. It is not until Burmann's re-issue of Ruddiman in 1725 that the Boissard-Bronckhorst type replaces the Povey type in the *Opera* (see Plate XXII).

It has been said that there was no opportunity for a painting being made of Buchanan when he was about 60 years of age. This overlooks the fact that in 1565—6 Buchanan visited Paris to see his Latin paraphrase of the Psalms passed through the press by Henri Estienne and his brother *. On the title page of that work Buchanan appears as Georgius Buchananus Scotus† "poetarum nostri saeculi facilé princeps." A man with such a reputation might well have his portrait painted. How Buchanan travelled to Paris we do not know, very possibly through the Netherlands, and there is a remarkable letter of Languet to Buchanan stating that he lives in sight of Rotterdam which reminds him not only of Erasmus but also of Buchanan. This suggests that he had met Buchanan in Rotterdam.

While the Bronckhorst and Royal Society types show Buchanan as "aged" if not "agéd," there exist two other portrait types quite incompatible with the Bronckhorst type (but more or less compatible with the Royal Society type) which represent, or profess to represent, Buchanan aged about 30 and 50 respectively. One of the former type was originally at Hamilton Palace, and from it Harding made an engraving for Pinkerton's "Scottish Gallery" (see Plate XXIII). By courtesy of the Duke of Hamilton, I have been informed: (1) by the Hamilton Estates Trustees that no such portrait now exists in Hamilton possession, and (2) by Messrs Christie, Manson & Woods, that no such portrait was sold with the Hamilton Palace collections. A portrait of this type on canvas has, however, turned up in the possession of Mr H. B. Buchanan, which is undoubtedly an old painting. It has been in the possession of his family for many years and may possibly have reached them by purchase or gift from Hamilton Palace. I am very grateful for the permission to reproduce it in this paper. A second, somewhat idealised, version of this portrait is now in the Technical College at Glasgow, which again may be the original Hamilton Palace picture (see Plates XXIII, Right, XXIV and XXV).

One of the second type of portrait—i.e. of Buchanan about 50 years of age—is in the possession of Miss Christie of Bedlay. By her permission and that of the publishers of the *Glasgow Quatercentenary Studies* (George Buchanan), I have been allowed to reproduce it (see Plate XXIII, Left).

While the skull of an adult changes to some slight extent with age the amount is so small as to be quite insignificant compared with the personal equation of an

* Ruddiman in his *Anticrisis*, 1754, writes: "I have heard it related a hundred times that Buchanan when Principal of St Leonard's College, at St Andrews, did make such a voyage to France."

† It must be emphasised that the word *Scotus* almost invariably follows George Buchanan's name in any of his works published in his lifetime, and is thus in keeping with the title on the Royal Society portrait.

artist. His deviations from truth, even in the case of the most conscientious of the craft, are of quite a different order of magnitude. We can have no hesitation then in applying the Edinburgh skull to the portraits reputed to be those of Buchanan in the years from 30 to 50.

I will now run rapidly through the portraits. I have to thank Miss Ida McLearn for measured drawings of the portraits to each of which we have attached outlines of photographs of the skull in the corresponding attitude.

(1) *Portrait of a Young Man. Glasgow Technical College Portrait.*

With a slight allowance for artistic inaccuracy the skull might almost fit this portrait. We cannot, however, raise the alveolar margin adequately without getting the orbits and nasal bridge out of position. The frontals of the skull are scarcely high enough for the forehead of the portrait (see Plate XXVI, Left).

Allowing for artistic inaccuracies I do not think we must venture to say on the authority of the skull that this type of portrait does *not* represent Buchanan when young.

(2) *Miss Christie's Portrait of a Man between* 40 *and* 50.

Again this is not such a *very* bad fit. Forehead too high and broad. Cheek bones about right and mastoid fairly good. Alveolar border too low and cannot be raised without sending orbits out of position (see Plate XXVI, Right).

It cannot be affirmed on the evidence of the skull that this is certainly not Buchanan.

(3) *Chouet* (*? Vansom*). Right (? left *reversed*) mastoid out of position, left cheek bone projects beyond portrait without flesh. Alveolar margin is under nose instead of under upper lip. Excess of right frontal. Would be a very bad portrait of the owner of the skull, if portrait at all (see Plate XXVII, Left).

(4) *George Buchanan from Boissard's Icones*, 1598. Left mastoid out of position, right cheek bone protruding beyond portrait without flesh. Alveolar margin of upper jaw under nose instead of under upper lip. Excess of right frontal. All the defects of the Chouet portrait, of course reversed (see Plate XXVII, Right).

(5) *George Buchanan, National Portrait Gallery.* This may stand as representative of the two Edinburgh, the St Andrews, the two Aberdeen and the Glasgow portraits. They are all of the same, the Bronckhorst, type (see Plate XXVIII).

Left mastoid out of place, nose much too long, so that it reaches to alveolar margin. Right cheek bone without flesh outside limits of portrait. Right frontal in excess. It is impossible to defend this portrait as a truthful portrait of the owner of the skull. Those who accept this portrait as a representation of George Buchanan must either discard the skull, or be willing to admit and approve extreme artistic licence.

(6) *Earl of Buchanan's "Titian," St Andrews* (*Copies at the Buchan Club, etc.*). I do not know whether it is worth the energy it has taken to dismiss this portrait

of President Jeannin, but I have done so. The orbits are out of position, the alveolar margin absurd, and the forehead ridiculous for the skull (see Plate XXIX, Left).

(7) *The Royal Society's Portrait.* The orbits are reasonable, so is the nose. The alveolar margin is in the true position; the cheek bones practically where they should be, and the frontals excellent. If the skull is to be our guide, this could hardly be bettered (see Plate XXX). On Plate XXIX, Right, will be found another illustration of this type, namely Mr Sowersby's portrait fitted with the skull. As in the case of the Royal Society picture the skull fits well.

We are, I think, driven to the conclusion that the skull accords with the Povey-Mead type of portrait, the type of which our earliest trace of authenticated date is the Elzevir *Poemata* of 1628.

Now what exactly does this mean for us as phrenologists? Let us look on the two portraits once more.

Let us take the highly idealised Bronckhorst portrait in Edinburgh as one type (see Plate XIII). Note the high forehead of great breadth, the marked bridge and long nose, the projecting cheek bones and stern look. We have here all the signs of the earnest religious reformer, the ascetic scholar and the stringent moralist; Buchanan as a second Cato.

Now look at the Royal Society picture (see Plate XVI). We see the exaggerated right orbit, the Socratic nose, and all the rotundity which identify man and skull. The portrait suggests the intimate of Muretus, the man who preferred a pawky story to true history, the man who could write poems worthy of Catullus; the man who could appeal alike to the surpassing talents of Henry VIII and of the Regent Moray, or sing the virtues of Mary Stewart and of Elizabeth Tudor, according as the moment was propitious for receiving their aid. The two portraits express the two attitudes that historians have taken to Buchanan, and whether we like it or not, each one of us has got to take his choice between Cato and Catullus. For as Buchanan's friend Muretus laid it down:

> "Quisquis versibus exprimit Catullum,
> Raro moribus exprimit Catonem."

The man who in his verses propounds Catullus, will scarcely exemplify Cato in his morality.

Here is a choice between alternatives*. Either you must reject the Cato portrait as no truthful likeness, or if you accept the Cato portrait as a *truthful* representation, you must return the skull to the Greyfriars as not that of George Buchanan, although it fits excellently that other portrait of "Georgivs Bvchanan Scotvs." The alternative lies in how you translate "Scotvs."

* The Chairman, Sir E. A. Schäffer, suggested at the end of the lecture that the two portraiture alternatives were those of Don Quixote and Sancho Panza. This somewhat apt description of the Edinburgh "Scroll" and the Royal Society pictures has really no application to the alternative interpretations of the indisputable mental ability of George Buchanan.

FURTHER NOTES ON NON-LINEAR REGRESSION.

By J. NEYMAN, Ph.D., Warsaw.

1. The theory of non-linear regression has been discussed by Prof. K. Pearson in his memoir *On the General Theory of Skew Correlation and Non-Linear Regression* (London, 1905), and in a paper in *Biometrika*, Vol. XIII. pp. 296—300. In these two papers the general method of calculation of polynomials, which are approximations to the unknown skew regression line, is given and formulae are deduced expressing the coefficients of such polynomials as far as the fourth order in terms of the frequency constants of the population considered.

The results of Heine* combined with those of H. Hamburger†, who worked with the theory of continued fractions, allow us to put the results of K. Pearson into a more general form.

We shall do it, however, without any appeal to the theory of continued fractions.

In a further paper we shall apply these results to the correlation between the standard deviation squared and the mean in a sample drawn from an indefinitely large population.

Higher Parabolae of Regression and Higher Correlation.

2. Let x and y be two variates, and $z = f(x, y)$ the function representing their frequency distribution. The two variates may have their possible values forming a continuous set of points (as it is, e.g., in the case when their distribution is normal), or not (as it is in the case when the distribution is, e.g., binomial). We shall use nevertheless the notation of $\int$, which in the last case is to be considered as the integral of Stieltjes, i.e. as a sum.

The integrals over the entire range of each of the two variates we shall denote by $\int_{-\infty}^{+\infty}$; as in the case when the range is limited, it is always possible to suppose that outside this range

$$f(x, y) = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(1).$$

Further, we shall suppose that the integral of $f(x, y)$, taken over the whole of the ranges of x and y, is equal to unity.

* *Handbuch der Kugelfunctionen*, 1878—1881, I. p. 292.

† "Ueber die Erweiterung des Stieltjesschen Momentenproblems," *Math. Annalen*, Bd. LXXXI. Heft 2—4, 1920.

Let $\bar{y}_x$ be the mean value of y corresponding to a given value of x. We have

$$\bar{y}_x = \frac{\int_{-\infty}^{+\infty} y f(x, y)\, dy}{\int_{-\infty}^{+\infty} f(x, y)\, dy} = \frac{\phi(x)}{p(x)} \qquad (2),$$

if

$$\int_{-\infty}^{+\infty} y f(x, y)\, dy = \phi(x) \qquad (3)$$

and

$$\int_{-\infty}^{+\infty} f(x, y)\, dy = p(x) \qquad (4).$$

In trying to approximate to the generally unknown function $\frac{\phi(x)}{p(x)}$, we shall calculate a polynomial of the n-th degree

$$\tilde{y} = P_n = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots + a_n x^n \qquad (5),$$

satisfying the condition, that the integral

$$I = \int_{-\infty}^{+\infty} \left[\frac{\phi(x)}{p(x)} - P_n\right]^2 p(x)\, dx \qquad (6)$$

is a minimum. If we observe, that $p(x)\, dx$ is nothing but the probability that the variate x should lie in the subrange dx, the meaning of this integral is clear.

As I is always a differentiable function of the a's, we use the ordinary method for finding minima, and solve the system of $n+1$ linear equations in the a's, given by equating to zero the consecutive derivates $\frac{\partial I}{\partial a_k}$ $(k = 0, 1, 2 \ldots n)$, namely,

$$a_0 \mu_k + a_1 \mu_{k+1} + a_2 \mu_{k+2} + \ldots + a_k \mu_{2k} + \ldots + a_n \mu_{k+n} = \lambda_k \qquad (7),$$

where $\mu_k = \int_{-\infty}^{+\infty} x^k p(x)\, dx, \quad \lambda_k = \int_{-\infty}^{+\infty} x^k \phi(x)\, dx = \int_{\infty}^{+\infty}\int_{-\infty}^{+\infty} y x^k f(x, y)\, dx dy \ldots (8).$

If the origin of the coordinates be at the mean of the two variates, we have $\mu_1 = \lambda_0 = 0$. We notice also that $\mu_0 = 1$.

As the determinant of the system of equations (7)

$$\Delta_n = \begin{vmatrix} \mu_0, & \mu_1, & \ldots \mu_k, & \ldots \mu_n \\ \mu_1, & \mu_2, & \ldots \mu_{k+1}, & \ldots \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_k, & \mu_{k+1}, & \ldots \mu_{2k}, & \ldots \mu_{k+n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_n, & \mu_{n+1}, & \ldots \mu_{n+k}, & \ldots \mu_{2n} \end{vmatrix} \qquad (9)$$

is always > 0*, system (9) gives solutions for the a's which are not all equal to zero when, and only when, the λ_k's are not all equal to zero. Such zero solutions have, however, no interest for us. Considering the system of $n+2$ equations

$$\left.\begin{array}{l} -\tilde{y} + a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots + a_n x^n = 0 \\ -\lambda_k + a_0 \mu_k + a_1 \mu_{k+1} + a_2 \mu_{k+2} + \ldots + a_k \mu_{2k} + \ldots + a_n \mu_{k+n} = 0 \\ \hfill (k = 0, 1, 2 \ldots n) \end{array}\right\} \ldots (10),$$

and

* T. J. Stieltjes, "Recherches sur les fractions continues," *Ann. de la fac. des sc. de Toulouse*, 1894.

we can free ourselves of the a's and write the necessary and sufficient conditions that they should be satisfied as

$$\begin{vmatrix} -\tilde{y}, & 1, & x, & x^2, & \dots x^k, & \dots x^n \\ -\lambda_0, & \mu_0, & \mu_1, & \mu_2, & \dots \mu_k, & \dots \mu_n \\ -\lambda_1, & \mu_1, & \mu_2, & \mu_3, & \dots \mu_{k+1}, & \dots \mu_{n+1} \\ -\lambda_k, & \mu_k, & \mu_{k+1}, & \mu_{k+2}, & \dots \mu_{2k}, & \dots \mu_{k+n} \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \mu_{n+2}, & \dots \mu_{n+k}, & \dots \mu_{2n} \end{vmatrix} = 0 \quad \text{.........(11)},$$

which is really the expression of the n-th approximating parabola

$$-\tilde{y} + P_n = 0 \quad \text{..................................(12)}$$

required. It is clear, that the sign of the terms in the first column can be altered.

We can now proceed to the expression of the n-th correlation ratio.

With these words we denote an expression, which is connected with the n-th parabola of regression (12), in the same manner as the coefficient of correlation is connected with the same parabola for $n = 1$ (i.e. with the straight line fitted to the unknown correlation line) and as the well-known correlation ratio η is connected with the actual regression line.

In this way we shall get a measure of the closeness of points (x, y), representing individuals of the population considered, to our n-th parabola of regression.

Such a "natural" measure of closeness is represented by the very familiar integral

$$\Sigma_n^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [y - P_n]^2 f(x, y)\, dx dy \quad \text{.................(13)},$$

in which P_n means the polynomial, giving the n-th parabola of regression. If we suppose for a moment that the coefficients of P_n in (13) are variable, the integral (13), which we may now denote by $\Sigma_n'^2$, will be a function of them, and it is easy to see that if we try to dispose of these coefficients to make $\Sigma_n'^2$ a minimum we shall find the same equations (7) to determine them. Therefore Σ_n^2 is the minimum value of $\Sigma_n'^2$, and we shall call it the mean square deviation of individual observations from the n-th regression-parabola, or shortly, the n-th mean square deviation. To get its expression in terms of μ's and λ's, we consider the following function of the a's and a new variate b,

$$\Sigma_{(b)}^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [by - P_n]^2 f(x, y)\, dx dy \quad \text{.................(14)}.$$

It is clear that $\Sigma_{(b)}^2$ is a homogeneous polynomial of the second degree in the a's and b. Therefore, using the Euler formula,

$$2\Sigma_{(b)}^2 = \frac{\partial \Sigma_{(b)}^2}{\partial b} b + \frac{\partial \Sigma_{(b)}^2}{\partial a_0} a_0 + \frac{\partial \Sigma_{(b)}^2}{\partial a_1} a_1 + \dots + \frac{\partial \Sigma_{(b)}^2}{\partial a_n} a_n \quad \text{...........(15)},$$

for all values of the a's and b. But putting $b = 1$, we have $\Sigma_{(b)}^2 = \Sigma_n^2$, and as the a's satisfy the conditions (7), i.e. as all

$$\frac{\partial \Sigma_n^2}{\partial a_n} = 0 \quad \text{..................................(16)},$$

the equation (15) becomes, for $b=1$,

$$2\Sigma_n^2 = \frac{\partial \Sigma_{(1)}^2}{\partial b} = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\,(y-P_n)\,f(x,y)\,dx\,dy \quad\text{...........(17).}$$

Using in the ordinary notation σ_y for the standard deviation of y, or putting

$$\sigma_y^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2 f(x,y)\,dx\,dy \quad\text{........................(18),}$$

and dividing the two parts of (17) by 2, we have

$$\Sigma_n^2 = \sigma_y^2 - a_0\lambda_0 - a_1\lambda_1 - \dots - a_k\lambda_k - \dots \mp a_n\lambda_k \quad\text{............(19).}$$

Comparing this expression with (12), we see that to get Σ_n^2 we have only to put into (11) σ_y^2 instead of y and λ_k instead of x^k ($k=0, 1, 2 \dots n$), and to divide by the coefficient of $\tilde{y}$, i.e. by $-\Delta_n$. In this way we have

$$\Sigma_n^2 = \frac{1}{\Delta_n}\begin{vmatrix} \sigma_y^2, & \lambda_0, & \lambda_1, & \dots \lambda_k, & \dots \lambda_n \\ \lambda_0, & \mu_0, & \mu_1, & \dots \mu_k, & \dots \mu_n \\ \lambda_1, & \mu_1, & \mu_2, & \dots \mu_{k+1}, & \dots \mu_{n+1} \\ & & & & \\ \lambda_n, & \mu_n, & \mu_{n+1}, & \dots \mu_{n+k}, & \dots \mu_{2n} \end{vmatrix} \quad\text{.(20),}$$

or

$$\Sigma_n^2 = \sigma_y^2\,(1 - H_n^2) \quad\text{..............................(21),}$$

where

$$H_n^2 = \frac{1}{\sigma_y^2\Delta_n}\begin{vmatrix} 0, & \lambda_0, & \lambda_1, & \dots \lambda_k, & \dots \lambda_n \\ -\lambda_0, & \mu_0, & \mu_1, & \dots \mu_k, & \dots \mu_n \\ & & & & \\ -\lambda_k, & \mu_k, & \mu_{k+1}, & \dots \mu_{2k}, & \dots \mu_{k+n} \\ & & & & \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \dots \mu_{n+k}, & \dots \mu_{2n} \end{vmatrix} \quad\text{.(22),}$$

which is the expression of the n-th correlation ratio we wanted to find. It is easy to see that for $n=1$, we have

$$H_1^2 = \frac{\lambda_1^2}{\sigma_y^2\mu_2} \quad\text{...................................(23),}$$

which is the expression of the ordinary coefficient of correlation squared, r_{xy}^2*. We see further, as the equation of the n-th regression parabola (11) can be also written in the following manner,

$$\tilde{y} : \begin{vmatrix} 0, & 1, & x, & x^2, & \dots x^k, & \dots x^n \\ -\lambda_0, & \mu_0, & \mu_1, & \mu_2, & \dots \mu_k, & \dots \mu_n \\ & & & & & \\ -\lambda_k, & \mu_k, & \mu_{k+1}, & \mu_{k+2}, & \dots \mu_{2k}, & \dots \mu_{k+n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \mu_{n+2}, & \dots \mu_{n+k}, & \dots \mu_{2n} \end{vmatrix} \frac{1}{\Delta_n} = \frac{\Delta_n'}{\Delta_n}\ \text{(say)...(24),}$$

* As H_n^2 is nothing but the most natural generalization of the coefficient of correlation squared, it would be perhaps better to call it "the n-th coefficient of correlation." Unfortunately this name is adapted for what I should call "the coefficient of correlation between n-th powers of variates."

that, having the equation of the n-th regression-parabola, *we get at once the expression of H_n^2 by putting H_n^2 instead of $\tilde{y}$ and $\frac{\lambda_k}{\sigma_y^2}$ instead of the k-th power of x ($k = 0, 1 \dots n$). It is necessary to remember that x and y are measured from their means.*

Expansions of P_n and H_n^2.

3. We shall find it of interest to calculate the general expression for the difference

$$Q_n = P_n - P_{n-1} \quad \dots\dots (25).$$

As $P_0 = 0$, we shall be able to express the right-hand side of the equation of the n-th regression-parabola in the form of a sum

$$P_n = Q_1 + Q_2 + \dots + Q_n \quad \dots\dots (26).$$

In this way we shall reach the expressions for successive regression-parabolas, without any appeal to the theory of continued fractions and of orthogonal functions. For $n \leqslant 4$ these expressions have been given by Prof. K. Pearson in *Biometrika*, *loc. cit.*

We shall need here the following property of determinants. Let D be a determinant and $D_{rs\,t|ij\,k}$ the determinant which we obtain from D by cancelling the r-th, s-th...t-th rows and the same number of columns, namely the i-th, j-th...k-th columns.

We have
$$D \times D_{1m|1m} = D_{1|1} D_{m|m} - D_{1|m} D_{m|1} \quad \dots\dots (27),$$
where $m > 1$.

If $D_{1|1}$ and $D_{1m|1m}$ are not equal to zero, we can write

$$\frac{D}{D_{1|1}} - \frac{D_{m|m}}{D_{1m|1m}} = -\frac{D_{m|1} D_{1|m}}{D_{1|1} D_{1m|1m}} \quad \dots\dots (28),$$

the identity which we shall apply to our expressions of H_n^2 and P_n.

Turning back to (25) and putting Δ' instead of Δ_n', we can now write

$$Q_n = P_n - P_{n-1} = \frac{\Delta'}{\Delta'_{1|1}} - \frac{\Delta'_{n|n}}{\Delta'_{1n|1n}} \quad \dots\dots (29),$$

and, using (28),

$$Q_n = -\frac{1}{\Delta_{n-1}\Delta_n} \begin{vmatrix} -\lambda_0, & \mu_0, & \mu_1, & \dots & \mu_{n-1} \\ -\lambda_1, & \mu_1, & \mu_2, & \dots & \mu_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \dots & \mu_{2n-1} \end{vmatrix} \times \begin{vmatrix} 1, & x, & x^2, & \dots & x^n \\ \mu_0, & \mu_1, & \mu_2, & \dots & \mu_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{n-1}, & \mu_n, & \mu_{n+1}, & \dots & \mu_{2n-1} \end{vmatrix} \quad \dots\dots (30),$$

or changing the columns of the first determinant into rows and changing the sign of the λ's:

$$Q_n \quad \frac{1}{\Delta_{n-1}\Delta_n} \begin{vmatrix} \lambda_0, & \lambda_1, & \lambda_2, & \dots & \lambda_n \\ \mu_0, & \mu_1, & \mu_2, & \dots & \mu_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{n-1}, & \mu_n, & \mu_{n+1}, & \dots & \mu_{2n-1} \end{vmatrix} \times \begin{vmatrix} 1, & x, & x^2, & \dots & x^n \\ \mu_0, & \mu_1, & \mu_2, & \dots & \mu_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{n-1}, & \mu_n, & \mu_{n+1}, & \dots & \mu_{2n-1} \end{vmatrix} \quad \dots\dots (31$$

We see that the second determinant can be obtained from the first one by putting x^k for λ^k ($k = 0, 1 \ldots n$). The first determinant we shall denote by V_n, and the second by $V_n(x)$. Now the equation of the n-th regression-parabola can be written as follows:

$$\tilde{y} = \frac{V_1 V_1(x)}{\Delta_0 \Delta_1} + \frac{V_2 V_2(x)}{\Delta_1 \Delta_2} + \ldots + \frac{V_n V_n(x)}{\Delta_{n-1} \Delta_n} \qquad \text{.................(32).}$$

Applying the same method to (22), we easily find the expansion of H_n^2, namely,

$$H_n^2 = \frac{1}{\sigma_y^2}\left[\frac{V_1^2}{\Delta_0\Delta_1} + \frac{V_2^2}{\Delta_1\Delta_2} + \ldots + \frac{V_n^2}{\Delta_{n-1}\Delta_n}\right] \qquad \text{.................(33).}$$

As the calculation of higher parabolas and correlation ratios needs the knowledge of the higher moments of the sampled population, the above results will be of interest in theoretical research, where the higher moments may be supposed to be known. Such a case arises, for instance, when we consider the dependency of the standard deviation upon the mean of a sample.

The expansion of P_n can be used to find the necessary and sufficient conditions which the constants λ and μ must satisfy in order that the actual regression line may be represented by an n-th order parabola.

Assume that $\qquad \bar{y} = P_n$ ....(34)

represents the actual regression line. Then, using (24), we find

$$\lambda_k = \int_{-\infty}^{+\infty} y_x x^k p(x)\,dx = -\frac{1}{\Delta_n} \begin{vmatrix} 0, & \mu_k, & \mu_{k+1}, & \ldots & \mu_{n+k} \\ \lambda_0, & \mu_0, & \mu_1, & \ldots & \mu_n \\ \lambda_1, & \mu_1, & \mu_2, & \ldots & \mu_{n+1} \\ \\ \lambda_n, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{2n} \end{vmatrix} \qquad \text{.(35),}$$

where the coefficient of μ_{n+k} is not zero, and this is the necessary condition required. Assuming that the expansion (32) is permissible, we shall now prove that this condition is sufficient. In fact, assume that all the λ_k's are linear functions of $\mu_k, \mu_{k+1}, \ldots \mu_{n+k}$ represented by (35). Considering V_{n+s}, where $s \geqslant 1$, we see that the terms of the first line of this determinant can be represented by the *same* linear and homogeneous function (35) of corresponding terms of the $n+1$ following lines of the same determinant. We conclude that $V_{n+s} = 0$. Now if we consider V_n and if we multiply the terms of its t-th row by the coefficient of μ_{k+t-2} ($t = 2, 3 \ldots n+1$) in (35) and subtract the products from the terms of the first line, we shall see that owing to (35) the determinant V_n will be transformed into the product of Δ_n into the coefficient of μ_{n+k} in (35), both of them being different from zero. Therefore $V_n \neq 0$. As the coefficient of x^n in $V_n(x)$ is $\Delta_{n-1} \neq 0$, we see that if all the λ's satisfy the condition (35), where the coefficient of μ_{n+k} is not zero, the actual regression line is represented by an n-th order parabola.

The general condition established can be considered as the generalisation of respective formulae given by K. Pearson for $n \leqslant 4$ (*loc. cit.*), and by A. A. Tchouproff* for $n \leqslant 2$.

* *Grundbegriffe u. Grundprobleme der Korrelationstheorie*, Teubner, 1925, pp. 48—49.

ON THE PROBABLE ERROR OF THE MODE OF SKEW FREQUENCY DISTRIBUTIONS.

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(Thesis approved for the Degree of Doctor of Science in the University of London.)

CONTENTS.

Introduction.

This paper consists of four Chapters. The first treats of the deduction of a formula giving the probable error of the mode of Pearsonian curves as a function of β_1 and β_2, the second deals with the requisite tables, the third represents the probable error of the mode as a surface, while the fourth Chapter is concerned with statistical applications. In spite of the importance of the mode no general investigation of the accuracy with which it can be determined has hitherto been made, although Professors Pearson and Filon* have indicated by a very different method its determination for special curves. I have endeavoured in this paper to find a formula for the probable error of the mode, to make a table of it for an adequate range, to draw a contour diagram and to make a model of the surface in order to visualize the whole field. The contour diagram provides facilities for finding an approximate value; a more accurate value can be obtained by interpolation from the table itself.

* *Phil. Trans.* Vol. 191 (1898), pp. 229—311.

Chapter I. Deduction of a Formula to calculate the Probable Error of the Mode.

Let the differential equation of the frequency curve be assumed to be given by the form:

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x+a}{c_0 + c_1 x + c_2 x^2} \quad \text{...............................(1).}$$

Multiplying (1) by $\qquad yx^s(c_0 + c_1 x + c_2 x^2)\,dx$

and integrating throughout the range b_1 to b_2 we get

$$\int_{b_1}^{b_2} x^s(c_0 + c_1 x + c_2 x^2)\,dy = -\int_{b_1}^{b_2} y\,(x^{s+1} + ax^s)\,dx \quad \text{..............(2).}$$

Integrating the left-hand side of (2) by parts and remembering that usually the expression $yx^s(c_0 + c_1 x + c_2 x^2)$ vanishes at both ends of the range we get:

$$-sc_0\int_{b_1}^{b_2} x^{s-1}\,y\,dx - (s+1)\,c_1\int_{b_1}^{b_2} x^s y\,dx - (s+2)\,c_2\int_{b_1}^{b_2} x^{s+1}\,y\,dx$$

$$= -\int_{b_1}^{b_2} x^{s+1}\,y\,dx - a\int_{b_1}^{b_2} x^s y\,dx \quad \text{...........(3),}$$

or with the usual notation for the moment coefficients

$$sc_0\mu'_{s-1} + (s+1)\,c_1\mu_s' + (s+2)\,c_2\,\mu'_{s+1} = \mu'_{s+1} + a\mu_s' \quad \text{...........(4).}$$

Hence, the origin being any fixed point, we have for:

$$s = 0, \qquad c_1\mu_0' + 2c_2\mu_1' = \mu_1' + a\mu_0',$$

or,

$$c_1 + 2c_2\mu_1' = \mu_1' + a \quad \text{..............................(5),}$$

$$s = 1, \qquad c_0 + 2c_1\mu_1' + 3c_2\mu_2' = \mu_2' + a\mu_1' \quad \text{..............(6),}$$

$$s = 2, \qquad 2c_0\mu_1' + 3c_1\mu_2' + 4c_2\mu_3' = \mu_3' + a\mu_2' \quad \text{..............(7),}$$

$$s = 3, \qquad 3c_0\mu_2' + 4c_1\mu_3' + 5c_2\mu_4' = \mu_4' + a\mu_3' \quad \text{..............(8).}$$

Eliminating c_0, c_1 and c_2 from (5), (6), (7) and (8):

$$a = \frac{\mu_4'(\mu_3' - 13\mu_2'\mu_1' + 12\mu_1'^3) + \mu_3'(8\mu_3'\mu_1' - 20\mu_2'\mu_1'^2 + 3\mu_2'^2) + 9\mu_2'^3\mu_1'}{10\mu_4'\mu_2' - 10\mu_4'\mu_1'^2 - 12\mu_3'^2 + 32\mu_3'\mu_2'\mu_1' - 8\mu_3'\mu_1'^3 - 18\mu_2'^3 + 6\mu_2'^2\mu_1'^2} \equiv \frac{N}{D}.$$

Denote the distance of the mode from any fixed origin by M:

$$M \equiv \frac{N}{D} = a \quad \text{..................................(9),}$$

$$\log M = \log N - \log D \quad \text{.......................(10).}$$

Now assuming the sample so large that statistical differentials may be replaced by mathematical differentials we reach by differentiating (10):

$$\frac{\delta M}{\overline{M}} = \delta\mu_4'\left\{\frac{\overline{\mu}_3' - 13\overline{\mu}_2'\overline{\mu}_1' + 12\overline{\mu}_1'^3}{\overline{N}} - \frac{10\,(\overline{\mu}_2' - \overline{\mu}_1'^2)}{\overline{D}}\right\}$$

$$+\,\delta\mu_3'\left\{\frac{\overline{\mu}_4' + 16\overline{\mu}_3'\overline{\mu}_1' - 20\overline{\mu}_2'\overline{\mu}_1'^2 + 3\overline{\mu}_2'^2}{\overline{N}} + \frac{24\overline{\mu}_3' - 32\overline{\mu}_2'\overline{\mu}_1' + 8\overline{\mu}_1'^3}{\overline{D}}\right\}$$

$$- \delta\mu_2' \left\{ \frac{13\bar{\mu}_4'\bar{\mu}_1' + 20\bar{\mu}_3'\bar{\mu}_1'^2 - 6\bar{\mu}_2'\bar{\mu}_3' - 27\bar{\mu}_2'^2\bar{\mu}_1'}{\bar{N}} + \frac{10\bar{\mu}_4' + 32\bar{\mu}_3'\bar{\mu}_1' - 54\bar{\mu}_2'^2 + 12\bar{\mu}_2'\bar{\mu}_1'^2}{\bar{D}} \right\}$$

$$- \delta\mu_1' \left\{ \frac{13\bar{\mu}_4'\bar{\mu}_2' - 36\bar{\mu}_4'\bar{\mu}_1'^2 - 8\bar{\mu}_3'^2 + 40\bar{\mu}_3'\bar{\mu}_2'\bar{\mu}_1' - 9\bar{\mu}_2'^3}{\bar{N}} + \frac{-20\bar{\mu}_4'\bar{\mu}_1' + 32\bar{\mu}_3'\bar{\mu}_2' - 24\bar{\mu}_3'\bar{\mu}_1'^2 + 12\bar{\mu}_2'^2\bar{\mu}_1'}{\bar{D}} \right\} \quad \ldots\ldots(11),$$

where all the $\bar{\mu}'$'s are for the sampled population; we shall indicate by rules over our symbols that they are the values of the corresponding quantities for this population. If now we refer to the mean of the sampled population as our fixed origin we shall have $\bar{\mu}_1' = 0$ and

$$\bar{N} = \bar{\mu}_4\bar{\mu}_3 + 3\bar{\mu}_2^2\bar{\mu}_3 = \bar{\sigma}^7\sqrt{\bar{\beta}_1}(\bar{\beta}_2 + 3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

$$\bar{D} = 10\bar{\mu}_4\bar{\mu}_2 - 12\bar{\mu}_3^2 - 18\bar{\mu}_2^3 = 2\bar{\sigma}^6(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9) \quad \ldots\ldots\ldots\ldots(13),$$

$$\bar{M} \equiv \frac{\bar{N}}{\bar{D}} = \frac{\bar{\sigma}}{2}\,\frac{\sqrt{\bar{\beta}_1}(\bar{\beta}_2 + 3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Introducing (12), (13) and (14) into (11):

$$\frac{\delta M}{\bar{M}} = \frac{\delta\mu_4'}{\bar{\sigma}^4}\left\{\frac{1}{\bar{\beta}_2 + 3} - \frac{5}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\} + \frac{\delta\mu_3'}{\bar{\sigma}^3}\left\{\frac{1}{\sqrt{\bar{\beta}_1}} + \frac{12\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\}$$

$$\frac{\delta\mu_2'}{\bar{\sigma}^2}\left\{\frac{6}{\bar{\beta}_2 + 3} - \frac{5\bar{\beta}_2 - 27}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\} + \frac{\delta\mu_1'}{\bar{\sigma}}\left\{\frac{8\bar{\beta}_1 - 13\bar{\beta}_2 + 9}{\sqrt{\bar{\beta}_1}(\bar{\beta}_2 + 3)} - \frac{16\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\} \ldots(15),$$

$$\delta M = \tfrac{1}{2}\delta\mu_1'\left\{\frac{8\bar{\beta}_1 - 13\bar{\beta}_2 + 9}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} - \frac{16\bar{\beta}_1(\bar{\beta}_2 + 3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\}$$

$$+ \tfrac{1}{2}\frac{\delta\mu_2'}{\bar{\sigma}}\left\{\frac{6\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} - \frac{\sqrt{\bar{\beta}_1}(\bar{\beta}_2 + 3)(5\bar{\beta}_2 - 27)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\}$$

$$+ \tfrac{1}{2}\frac{\delta\mu_3'}{\bar{\sigma}^2}\left\{\frac{\bar{\beta}_2 + 3}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} + \frac{12\bar{\beta}_1(\bar{\beta}_2 + 3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\}$$

$$+ \tfrac{1}{2}\frac{\delta\mu_4'}{\bar{\sigma}^3}\left\{\frac{\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} - \frac{5\sqrt{\bar{\beta}_1}(\bar{\beta}_2 + 3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\} \quad \ldots\ldots(16).$$

Now if n be the number of individuals in the sample and we represent by curled brackets the mean value for all possible samples of the expression enclosed, then:

$$\{\delta\mu_1'^2\} = \frac{\bar{\sigma}^2}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17), \qquad \{\delta\mu_2'^2\} = (\bar{\beta}_2 - 1)\frac{\bar{\sigma}^4}{n} \quad \ldots\ldots\ldots\ldots\ldots(18),$$

$$\{\delta\mu_3'^2\} = \frac{\bar{\mu}_6 - \bar{\mu}_3^2}{n} = \frac{(\bar{\beta}_4 - \bar{\beta}_1)\bar{\sigma}^6}{n} \ldots\ldots(19), \qquad \{\delta\mu_4'^2\} = \frac{\bar{\mu}_8 - \bar{\mu}_4^2}{n} = \frac{(\bar{\beta}_6 - \bar{\beta}_2^2)\bar{\sigma}^8}{n} \quad \ldots\ldots(20),$$

$$\{\delta\mu_1'\delta\mu_2'\} = \frac{\bar{\mu}_3 - \bar{\mu}_2\bar{\mu}_1}{n} = \frac{\sqrt{\bar{\beta}_1}\,\bar{\sigma}^3}{n} \ldots\ldots\ldots(21), \qquad \{\delta\mu_1'\delta\mu_3'\} = \frac{\bar{\mu}_4 - \bar{\mu}_3\bar{\mu}_1}{n} = \frac{\bar{\beta}_2\bar{\sigma}^4}{n} \quad \ldots\ldots(22),$$

$$\{\delta\mu_1'\,\delta\mu_4'\} = \frac{\bar\mu_5 - \bar\mu_4\bar\mu_1}{n} = \frac{\bar\beta_3}{\sqrt{\bar\beta_1}}\,\frac{\bar\sigma^5}{n} \quad\text{.........(23)},\qquad \{\delta\mu_2'\,\delta\mu_3'\} = \frac{\bar\mu_5 - \bar\mu_2\bar\mu_3}{n} = \frac{\bar\beta_3 - \bar\beta_1}{\sqrt{\bar\beta_1}}\,\frac{\bar\sigma^5}{n} \quad\text{...(24)},$$

$$\{\delta\mu_3'\,\delta\mu_4'\} = \frac{\bar\mu_7 - \bar\mu_3\bar\mu_4}{n} = \frac{\bar\beta_5 - \bar\beta_1\bar\beta_2}{\sqrt{\bar\beta_1}}\,\frac{\bar\sigma^7}{n} \quad\text{...(25)},\qquad \{\delta\mu_2'\,\delta\mu_4'\} = \frac{\bar\mu_6 - \bar\mu_2\bar\mu_4}{n} = \frac{\bar\beta_4 - \bar\beta_2}{n}\,\bar\sigma^6 \quad\text{...(26)},$$

since $\mu_1 = 0$ for the sampled population.

Further let

$$\left.\begin{aligned}
B_1 &= \tfrac{1}{2}\left\{\frac{8\bar\beta_1 - 13\bar\beta_2 + 9}{5\bar\beta_2 - 6\bar\beta_1 - 9} - \frac{16\bar\beta_1(\bar\beta_2+3)}{(5\bar\beta_2 - 6\bar\beta_1 - 9)^2}\right\}\\
B_2 &= \tfrac{1}{2}\left\{\frac{6}{5\bar\beta_2 - 6\bar\beta_1 - 9} - \frac{(\bar\beta_2+3)(5\bar\beta_2 - 27)}{(5\bar\beta_2 - 6\bar\beta_1 - 9)^2}\right\}\\
B_3 &= \tfrac{1}{2}\left\{\frac{\bar\beta_2+3}{5\bar\beta_2 - 6\bar\beta_1 - 9} + \frac{12\bar\beta_1(\bar\beta_2+3)}{(5\bar\beta_2 - 6\bar\beta_1 - 9)^2}\right\}\\
B_4 &= \tfrac{1}{2}\left\{\frac{1}{5\bar\beta_2 - 6\bar\beta_1 - 9} - \frac{5(\bar\beta_2+3)}{(5\bar\beta_2 - 6\bar\beta_1 - 9)^2}\right\}
\end{aligned}\right\}\quad\text{.(27).}$$

Then we may write:

$$\delta M = \delta\mu_1' B_1 + \frac{\delta\mu_2'}{\bar\sigma}\sqrt{\bar\beta_1}\,B_2 + \frac{\delta\mu_3'}{\bar\sigma^2}\,B_3 + \frac{\delta\mu_4'}{\bar\sigma^3}\sqrt{\bar\beta_1}\,B_4.$$

Square, sum for all possible samples, divide by their number, and we have:

$$\begin{aligned}\sigma_M^2 = \frac{\bar\sigma^2}{n}\{&B_1^2 + \bar\beta_1(\bar\beta_2 - 1)B_2^2 + (\bar\beta_4 - \bar\beta_1)B_3^2 + \bar\beta_1(\bar\beta_6 - \bar\beta_2^2)B_4^2 + 2B_1B_2\bar\beta_1\\
&+ 2B_1B_3\bar\beta_2 + 2B_1B_4\bar\beta_3 + 2B_2B_3(\bar\beta_3 - \bar\beta_1) + 2B_2B_4\bar\beta_1(\bar\beta_4 - \bar\beta_2)\\
&+ 2B_3B_4(\bar\beta_5 - \bar\beta_1\bar\beta_2)\}\text{.........(28).}\end{aligned}$$

Hence we deduce that:

Ratio of the probable error of the mode to the probable error of the mean

$$\begin{aligned}= \{&B_1^2 + \bar\beta_1(\bar\beta_2 - 1)B_2^2 + (\bar\beta_4 - \bar\beta_1)B_3^2 + \bar\beta_1(\bar\beta_6 - \bar\beta_2^2)B_4^2 + 2B_1B_2\bar\beta_1\\
&+ 2B_3B_4(\bar\beta_5 - \bar\beta_1\bar\beta_2) + 2B_1B_3\bar\beta_2 + 2B_1B_4\bar\beta_3 + 2B_2B_3(\bar\beta_3 - \bar\beta_1)\\
&+ 2B_2B_4(\bar\beta_4 - \bar\beta_2)\bar\beta_1\}^{\frac{1}{2}}\text{ ......(29).}\end{aligned}$$

Chapter II. Tabling of the Probable Error of the Mode.

(1) *Explanation of Tables.* It will be clear from this result that the determination of the probable error (or of the standard deviation) of the mode for each individual case would involve a large amount of computation. In order to avoid this it is necessary that the purely algebraical formula (29) should be replaced by tables. An examination of the values of B_1, B_2, B_3 and B_4 shows that they involve only β_1 and β_2, but the presence of the factor $1/(5\bar\beta_2 - 6\bar\beta_1 - 9)$ indicates that the probable error of the mode will become excessive in the neighbourhood of the line $5\bar\beta_2 - 6\bar\beta_1 - 9 = 0$. But this is the "Rectangular Line" passing through the "Rectangular Point" $\beta_1 = 0$, $\beta_2 = 1{\cdot}8$, and is the so-called "axis" of the biquadratic which encloses J-curves and separates limited range curves from U-curves. When we get curves falling into the J-area our formula must break down, for the curve

starts either with a finite or an infinite ordinate, which in a different sense is then the "mode." The only variation in the position of this maximum ordinate is that due to change in the terminal of the range from sample to sample; but this is not given by our formula, which now gives a spurious mode lying outside the curve range. On the border of the biquadratic curve which encloses true J-curves the spurious mode becomes real. Along the "rectangular line" itself the curve suddenly changes from a right-handed J to a left-handed L, and under such circumstances the spurious mode, as we approach the rectangular line, swings across and its probable error becomes infinite. This is the meaning of the factor $1/(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)$ in the B-terms, but the interest of this spurious mode is purely mathematical and not statistical. The biquadrate bounding the area of the spurious mode is not shown on Diagram I for it is too crowded; it passes near to contour 15 above and enclosing the asymptote reaches to about contour 7 below it.

Passing now to a further point we notice that $\bar{\beta}_3$, $\bar{\beta}_4$, $\bar{\beta}_5$ and $\bar{\beta}_6$ occur in the values of the mean powers and mean products of the moment coefficients. It was, therefore, needful to compute these, or to use the values given in *Biometrika*, Vol. VII. by A. J. Rhind for these quantities, of course assuming always that a Pearson frequency curve might be used. But on examination it was found (a) that there were certain errors in Rhind's Tables, and (b) what is more important, that smaller intervals of the argument β_2 were needful. Accordingly new Tables of the values of β_3, β_4, β_5 and β_6 for given values of β_1 and β_2 were computed from the difference formulae:

$$\beta_{2s} = (2s+1)\{\tfrac{1}{2}\beta_{2s-1} + (1+\tfrac{1}{2}\alpha)\beta_{2s-2}\}/\{1-\tfrac{1}{2}(2s-1)\alpha\},$$

$$\beta_{2s+1} = (2s+2)\{\tfrac{1}{2}\beta_1\beta_{2s} + (1+\tfrac{1}{2}\alpha)\beta_{2s-1}\}/\{1-\tfrac{1}{2}(2s)\alpha\},$$

where

$$\alpha = (2\beta_2 - 3\beta_1 - 6)/(\beta_2 + 3) \quad \text{.......................(30).}$$

(2) The calculation of this new Table I, which the author believes will be of service for other purposes, occupied several months of strenuous labour.

The next stage in the work was the computing of B_1, B_2, B_3 and B_4; their values are not reproduced here, because it is not clear that they will be of service for other investigations. The values of the standard deviations and mean products are so easily obtained from Equations (17)—(26) that it does not seem worth while providing tables of them, although they occur frequently in practical work.

(3) By aid of the Tables of the β's and the B's Table II giving the ratio of the probable error of the mode to that of the mean was constructed. It will be noted that in the final formula (29) there is no occurrence of $\sqrt{\bar{\beta}_1}$, which is found in (21), (23), (24) and (25), but drops out on the squaring of δM and the substitution of the products of the moment coefficients.

I suggest that the Table now provided will enable future workers to give the standard deviation of the mode with the same ease as in the past the standard deviation of the mean has been given, and thus to complete the specification of the constants of skew frequency distributions for *large* samples. I recognise that my work does not enable one to determine the frequency distribution of the mode, but

TABLE I*.

Values of β_2.

β_1.

β_2.	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*
1·8	0	·41440	·78154	1·10667	1·39429	1·64888	1·87200	2·06839
1·9	0	·44892	·84727	1·20075	1·51422	1·79186	2·03730	2·25370
2·0	0	·48493	·91579	1·29873	1·63902	1·94118	2·20909	2·44615
2·1	0	·52250	·98720	1·40077	1·76889	2·09643	2·38759	2·64600
2·2	0	·56169	1·06162	1·50701	1·90400	2·25783	2·57302	2·85348
2·3	0	·60257	1·13918	1·61763	2·04456	2·42561	2·76565	3·06886
2·4	0	·64522	1·22000	1·73280	2·19077	2·60000	2·96571	3·29241
2·5	0	·68971	1·30423	1·85270	2·34286	2·78125	3·17349	3·52442
2·6	0	·73612	1·39200	1·97753	2·50105	2·96962	3·38927	3·76518
2·7	0	·78455	1·48348	2·10750	2·66560	3·16538	3·61333	4·01500
2·8	0	·83508	1·57882	2·24282	2·83676	3·36883	3·84600	4·27422
2·9	0	·88781	1·67821	2·38371	3·01479	3·58026	4·08759	4·54317
3·0	0	·94286	1·78182	2·53043	3·20000	3·80000	4·33846	4·82222
3·1	0	1·00032	1·88985	2·68324	3·39268	4·02838	4·59896	5·11175
3·2	0	1·06033	2·00250	2·84239	3·59314	4·26575	4·86947	5·41215
3·3	0	1·12300	2·12000	3·00818	3·80174	4·51250	5·15040	5·72385
3·4	0	1·18847	2·24258	3·18092	4·01882	4·76901	5·44216	6·04727
3·5	0	1·25690	2·37049	3·36094	4·24478	5·03571	5·74521	6·38289
3·6	0	1·32842	2·50400	3·54857	4·48000	5·31304	6·06000	6·73120
3·7	0	1·40321	2·64339	3·74419	4·72492	5·60147	6·38704	7·09270
3·8	0	1·48145	2·78897	3·94820	4·98000	5·90149	6·72686	7·46795
3·9	0	1·56333	2·94105	4·16100	5·24571	6·21364	7·08000	7·85750
4·0	0	1·64906	3·10000	4·38305	5·52258	6·53846	7·44706	8·26197
4·1	0	1·73885	3·26618	4·61483	5·81115	6·87656	7·82866	8·68200
4·2	0	1·83294	3·44000	4·85684	6·11200	7·22857	8·22545	9·11826
4·3	0	1·93160	3·62189	5·10964	6·42576	7·59516	8·63815	9·57147
4·4	0	2·03510	3·81231	5·37382	6·75310	7·97705	9·06750	10·04239
4·5	0	2·14375	4·01176	5·65000	7·09474	8·37500	9·51429	10·53182
4·6	—	2·25787	4·22080	5·93887	7·45143	8·78983	9·97935	11·04062
4·7	—	—	4·44000	6·24115	7·82400	9·22241	10·46360	11·56969
4·8	—	—	4·67000	6·55765	8·21333	9·67368	10·96800	12·12000
4·9	—	—	—	6·88920	8·62038	10·14464	11·49356	12·69258
5·0	—	—	—	7·23673	9·04615	10·63636	12·04138	13·28852
5·1	—	—	—	—	9·49176	11·15000	12·61263	13·90900
5·2	—	—	—	—	9·95840	11·68679	13·20857	14·55525
5·3	—	—	—	—	—	12·24808	13·83055	15·22862
5·4	—	—	—	—	—	12·83529	14·48000	15·93053
5·5	—	—	—	—	—	—	15·15849	16·66250
5·6	—	—	—	—	—	—	15·86769	17·42618
5·7	—	—	—	—	—	—	—	18·22333
5·8	—	—	—	—	—	—	—	19·05585
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

* In the values given for β_5 and β_6 complete reliability beyond the eighth figure is not to be depended on.

TABLE I—(*continued*).

Values of β_2.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	β_2.
2·24000	2·38909	2·51765	2·62743	2·72000	2·79676	2·85895	2·90769	1·8
2·44379	2·61000	2·75446	2·87904	2·98542	3·07509	3·14938	3·20948	1·9
2·65532	2·83918	3·00000	3·13981	3·26038	3·36330	3·45000	3·52174	2·0
2·87484	3·07688	3·25455	3·41000	3·54514	3·66167	3·76108	3·84474	2·1
3·10261	3·32337	3·51837	3·68990	3·84000	3·97047	4·08291	4·17876	2·2
3·33890	3·57894	3·79175	3·97980	4·14524	4·29000	4·41578	4·52411	2·3
3·58400	3·84387	4·07500	4·28000	4·46118	4·62057	4·76000	4·88108	2·4
3·83820	4·11848	4·36842	4·59082	4·78812	4·96250	5·11589	5·25000	2·5
4·10182	4·40308	4·67234	4·91258	5·12640	5·31612	5·48377	5·63119	2·6
4·37517	4·69800	4·98710	5·24562	5·47636	5·68176	5·86400	6·02500	2·7
4·65860	5·00360	5·31304	5·59032	5·83837	6·05980	6·25692	6·43178	2·8
4·95247	5·32023	5·65055	5·94702	6·21278	6·45060	6·66291	6·85189	2·9
5·25714	5·64828	6·00000	6·31613	6·60000	6·85455	7·08235	7·28571	3·0
5·57301	5·98814	6·36180	6·69804	7·00042	7·27204	7·51564	7·73365	3·1
5·90049	6·34024	6·73636	7·09319	7·41447	7·70351	7·96320	8·19612	3·2
6·24000	6·70500	7·12414	7·50200	7·84258	8·14938	8·42545	8·67353	3·3
6·59200	7·08289	7·52558	7·92494	8·28522	8·61011	8·90286	9·16634	3·4
6·95696	7·47439	7·94118	8·36250	8·74286	9·08617	9·39588	9·67500	3·5
7·33538	7·88000	8·37143	8·81517	9·21600	9·57806	9·90500	10·20000	3·6
7·72779	8·30025	8·81687	9·28349	9·70517	10·08630	10·43074	10·74184	3·7
8·13474	8·73570	9·27805	9·76800	10·21091	10·61143	10·97362	11·30103	3·8
8·55680	9·18692	9·75556	10·26929	10·73379	11·15400	11·53419	11·87812	3·9
8·99459	9·65455	10·25000	10·78795	11·27442	11·71461	12·11304	12·47368	4·0
9·44877	10·13921	10·76203	11·32463	11·83341	12·29386	12·71077	13·08830	4·1
9·92000	10·64160	11·29231	11·88000	12·41143	12·89241	13·32800	13·72258	4·2
10·40901	11·16243	11·84156	12·45475	13·00916	13·51093	13·96539	14·37717	4·3
10·91657	11·70247	12·41053	13·04962	13·62732	14·15012	14·62364	15·05275	4·4
11·44348	12·26250	13·00000	13·66538	14·26667	14·81071	15·30345	15·75000	4·5
11·99059	12·84338	13·61081	14·30286	14·92800	15·49349	16·00558	16·46966	4·6
12·55881	13·44600	14·24384	14·96289	15·61215	16·19927	16·73082	17·21250	4·7
13·14909	14·07130	14·90000	15·64640	16·32000	16·92889	17·48000	17·97931	4·8
13·76246	14·72029	15·58028	16·35432	17·05247	17·68325	18·25398	18·77093	4·9
14·40000	15·39403	16·28571	17·08767	17·81053	18·46329	19·05366	19·58824	5·0
15·06286	16·09364	17·01739	17·84750	18·59520	19·27000	19·88000	20·43214	5·1
15·75226	16·82031	17·77647	18·63493	19·40757	20·10442	20·73400	21·30361	5·2
16·46951	17·57531	18·56418	19·45143	20·24877	20·96763	21·61671	22·20366	5·3
17·21600	18·36000	19·38182	20·29739	21·12000	21·86080	22·52923	23·13333	5·4
17·99322	19·17581	20·23077	21·17500	22·02254	22·78513	23·47273	24·09375	5·5
18·80276	20·02426	21·11250	22·08537	22·95771	23·74192	24·44842	25·08608	5·6
19·64632	20·90700	22·02857	23·03000	23·92696	24·73250	25·45760	26·11154	5·7
20·52571	21·82576	22·98065	24·01046	24·93176	25·75831	26·50162	27·17143	5·8
21·44291	22·78241	23·97049	25·02844	25·97373	26·82086	27·58192	28·26711	5·9
—	23·77895	25·00000	26·08571	27·05455	27·92174	28·70000	29·40000	6·0
—	24·81750	26·07119	27·18419	28·17600	29·06265	29·85746	30·57162	6·1
—	—	27·18621	28·32590	29·34000	30·24537	31·05600	31·78356	6·2
—	—	28·34737	29·51300	30·54857	31·47182	32·29739	33·03750	6·3
—	—	—	30·74780	31·80387	32·74400	33·58353	34·33521	6·4
—	—	—	32·03276	33·10820	34·06406	34·91642	35·67857	6·5
—	—	—	—	34·46400	35·43429	36·29818	37·06957	6·6
—	—	—	—	35·87390	36·85710	37·73108	38·51029	6·7
—	—	—	—	—	38·33508	39·21750	40·00299	6·8
—	—	—	—	—	39·87100	40·76000	41·55000	6·9
—	—	—	—	—	—	42·36129	43·15385	7·0

TABLE I—(*continued*).

Values of β_1.

β_1.

β_2	0	·1	·2	·3	·4	·5	·6	·7
1·8	3·85714	4·07037	4·20199	4·26974	4·28739	4·26566	4·21297	4·13599
1·9	4·40244	4·64442	4·79441	4·87269	4·89492	4·87323	4·81715	4·73416
2·0	5·00000	5·27357	5·44352	5·53307	5·55998	5·53802	5·47791	5·38807
2·1	5·65385	5·96205	6·15366	6·25517	6·28681	6·26413	6·19921	6·10151
2·2	6·36842	6·71453	6·92954	7·04376	7·08000	7·05602	6·98537	6·87863
2·3	7·14865	7·53619	7·77640	7·90384	7·94461	7·91858	7·84109	7·72393
2·4	8·00000	8·43274	8·70000	8·84126	8·88615	8·85714	8·77143	8·64228
2·5	8·92857	9·41053	9·70672	9·86955	9·91071	9·87753	9·81065	9·63899
2·6	9·94118	10·47664	10·80364	10·97368	11·02497	10·98615	10·87892	10·71985
2·7	11·04545	11·63897	11·99861	12·18325	12·23629	12·19003	12·06882	11·89115
2·8	12·25000	12·90637	13·30040	13·49946	13·55280	13·49688	13·35901	13·15981
2·9	13·56452	14·28876	14·71881	14·93176	14·98353	14·91525	14·66914	14·53334
3·0	15·00000	15·79734	16·26482	16·49068	16·53846	16·45455	16·27321	16·02004
3·1	16·56897	17·44479	17·95077	18·18802	18·22872	18·12523	17·91578	17·62896
3·2	18·28571	19·24547	19·79058	20·03698	20·06672	19·93875	19·69601	19·37010
3·3	20·16667	21·21577	21·80000	22·05242	22·06630	21·90809	21·62578	21·25445
3·4	22·23077	23·37444	23·99693	24·25111	24·34303	24·04755	23·71830	23·29418
3·5	24·50000	25·74304	26·40178	26·65201	26·61435	26·37315	25·98826	25·50270
3·6	27·00002	28·34641	29·03790	29·27665	29·20000	28·90287	28·45200	27·89494
3·7	29·76087	31·21363	31·93214	32·14962	32·02226	31·65688	31·12780	30·47745
3·8	32·81818	34·37829	35·11555	35·29902	35·10645	34·65795	34·03614	33·29869
3·9	36·21429	37·88011	38·62415	38·75716	38·48143	37·93182	37·20000	36·34923
4·0	40·00002	41·76595	42·50000	42·56138	42·18020	41·50769	40·64530	39·66217
4·1	44·23684	46·09165	46·79249	46·75499	46·24069	45·41884	44·40133	43·26338
4·2	49·00000	50·92418	51·56000	51·38852	50·70667	49·70330	48·50130	47·18203
4·3	54·38235	56·34460	56·87196	56·52129	55·62888	54·40472	52·98300	51·45104
4·4	60·50000	62·45189	62·81163	62·22322	61·06648	59·57347	57·78962	56·10775
4·5	—	69·36821	69·47964	68·57759	67·08882	65·26786	63·27068	61·19457
4·6	—	77·24602	76·99870	75·68397	73·77764	71·55580	69·18323	66·75995
4·7	—	—	85·52000	83·66259	81·23000	78·51680	75·69329	72·85939
4·8	—	—	95·23182	92·65977	89·56191	86·24448	82·87765	79·55676
4·9	—	—	—	102·85541	98·91311	94·84982	90·82608	86·92588
5·0	—	—	—	114·47304	109·45344	104·46524	99·64417	95·05252
5·1	—	—	—	—	121·39118	115·25000	109·45684	104·03682
5·2	—	—	—	—	134·98447	127·39721	120·41296	113·99636
5·3	—	—	—	—	—	141·14317	132·69118	125·07009
5·4	—	—	—	—	—	156·78005	146·50770	137·42323
5·5	—	—	—	—	—	—	162·12647	151·25376
5·6	—	—	—	—	—	—	179·87296	166·80075
5·7			—	—	—	—	—	184·35534
5·8	—	—	—	—	—	—	—	204·27568
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

TABLE I—(*continued*).

Values of β_4.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	β_2
4·04000	3·92924	3·80711	3·67634	3·53913	3·39727	3·25220	3·10510	1·8
4·63021	4·51000	4·37733	4·23523	4·08613	3·93202	3·77448	3·61479	1·9
5·27513	5·14437	5·00000	4·84537	4·68319	4·51562	4·34441	4·17097	2·0
5·97844	5·83588	5·67849	5·51000	5·33338	5·15101	4·96481	4·77632	2·1
6·74410	6·58831	6·41643	6·23257	6·04000	5·84134	5·63869	5·43372	2·2
7·57640	7·40576	7·21771	7·01679	6·80660	6·59000	6·36928	6·14625	2·3
8·48000	8·29265	8·08654	7·86667	7·63697	7·40059	7·16000	6·91718	2·4
9·45993	9·25376	9·02746	8·78650	8·53521	8·27699	8·01455	7·75000	2·5
10·52168	10·29429	10·04538	9·78094	9·50570	9·22337	8·93686	8·64846	2·6
11·67120	11·41986	11·14560	10·85499	10·55318	10·24419	9·93116	9·61657	2·7
12·91498	12·63657	12·33387	12·01406	11·68273	11·34426	11·68273	10·65861	2·8
14·26010	13·95159	13·61641	13·26400	12·89988	12·52874	12·15424	11·77918	2·9
15·71429	15·37056	15·00000	14·61114	14·21053	13·80322	13·39311	12·98319	3·0
17·28600	16·90296	16·49198	16·06234	15·62111	15·17370	14·72426	14·27595	3·1
18·98452	18·55685	18·10036	17·62506	17·13857	16·64667	16·15375	15·66314	3·2
20·82000	20·34167	19·83386	19·30739	18·77043	18·22912	17·68811	17·15037	3·3
22·80364	22·26770	21·70199	21·11814	20·52486	19·92866	19·33441	18·74572	3·4
24·94775	24·34628	23·71517	23·06690	22·41071	21·75348	21·10026	20·45478	3·5
27·26593	26·58983	25·88479	25·16414	24·43765	23·71249	22·99392	22·28571	3·6
29·77323	29·01036	28·22335	27·42130	26·61616	25·81537	25·02430	24·24676	3·7
32·48632	31·62803	30·74459	29·85091	28·95770	28·07261	27·20107	26·34686	3·8
35·42369	34·45448	33·46360	32·46669	31·47479	30·49567	29·53475	28·59568	3·9
38·60597	37·50988	36·39706	35·28370	34·18110	33·09702	32·03672	31·00367	4·0
42·05617	40·81478	39·56337	37·94426	37·09164	35·89027	34·71942	33·58221	4·1
45·80000	44·39200	42·98291	41·58947	40·22286	38·89031	37·59636	36·34362	4·2
49·86636	48·26703	46·67828	45·11667	43·59285	42·11346	40·68231	39·30132	4·3
54·28773	52·46836	50·67464	48·92245	47·22155	45·57759	43·99340	42·46993	4·4
59·10079	57·02793	55·00000	53·03157	51·13095	49·30236	47·54727	45·86538	4·5
64·34706	61·98164	59·68567	57·47152	55·34637	53·30941	51·36330	49·50509	4·6
70·07371	67·37000	64·76674	62·27294	59·89174	57·62259	55·46274	53·40809	4·7
76·33454	73·23883	70·28261	67·47004	64·80000	62·26828	59·86897	57·59525	4·8
83·19114	79·64013	76·27770	73·10111	70·10345	67·27568	64·60777	62·08948	4·9
90·71429	86·63316	82·80220	79·20913	75·83926	72·67717	69·70761	66·91597	5·0
98·98572	94·28567	89·91304	85·84250	82·04900	78·50882	75·20000	72·10244	5·1
108·10025	102·67545	97·67502	93·05590	88·77931	84·81081	81·11987	77·67953	5·2
118·16846	111·89225	106·16215	100·91122	96·08264	91·62805	87·50607	83·68109	5·3
129·32001	122·04000	115·45933	109·47890	104·01818	99·01089	94·40185	90·14466	5·4
141·70781	133·23978	125·66434	118·83929	112·65295	107·01593	101·85554	97·11190	5·5
155·51348	145·63333	136·89033	129·08444	122·06304	115·70695	109·92127	104·62922	5·6
170·95414	159·38758	149·26891	140·32027	132·33522	125·15610	118·65982	112·74843	5·7
188·29143	174·70041	162·95392	152·66919	143·56876	135·44525	128·13965	121·52747	5·8
207·84335	191·80793	178·12626	166·27335	155·87777	146·66761	138·43810	131·03138	5·9
—	210·99415	195·00000	181·29870	169·39394	158·92977	149·64286	141·33333	6·0
—	232·60341	213·83011	197·94009	184·27000	172·35402	161·85366	152·51594	6·1
—	—	234·92263	216·42764	200·68400	187·08139	175·18440	164·67274	6·2
—	—	258·64778	237·03500	218·84465	203·27519	189·76568	177·91006	6·3
—	—	—	260·08994	238·99803	221·12555	205·74781	192·34915	6·4
—	—	—	285·98810	261·43629	240·85495	223·30470	208·12889	6·5
—	—	—	—	286·50857	262·72535	242·63850	225·40894	6·6
—	—	—	—	314·63551	287·04710	263·98543	244·37376	6·7
—	—	—	—	—	314·19047	287·62313	265·23750	6·8
—	—	—	—	—	344·60055	313·88000	288·25000	6·9
—	—	—	—	—	—	343·14704	313·70445	7·0

TABLE I—(*continued*).

Values of β_3.

β_1.

β_2	0	·1	·2	·3	·4	·5	·6	·7
1·8	0	1·41563	2·55335	3·45274	4·14938	4·67447	5·05504	5·31428
1·9	0	1·68269	3·03698	4·11024	4·94488	5·57799	6·04158	6·36300
2·0	0	1·99086	3·59423	4·86684	5·85930	6·61567	7·17383	7·56603
2·1	0	2·34609	4·23542	5·73605	6·90840	7·80484	8·47016	8·94237
2·2	0	2·75531	4·97250	6·73343	8·11022	9·16522	9·95138	10·51347
2·3	0	3·29646	5·81933	7·87680	9·48543	10·71931	11·64114	12·30359
2·4	0	3·76936	6·79200	9·18694	11·05781	12·49286	13·56632	14·34023
2·5	0	4·39479	7·90935	10·69172	12·85476	14·51538	15·78319	16·65462
2·6	0	5·11605	9·19345	12·40745	14·90800	16·82084	18·25025	19·28231
2·7	0	5·94878	10·67033	14·37849	17·25435	19·44838	21·08439	22·26384
2·8	0	6·91168	12·37077	16·63931	19·93669	22·44332	24·30640	25·64561
2·9	0	8·02724	14·33140	19·23517	23·00517	25·85827	27·87491	29·48081
3·0	0	9·32265	16·59605	22·21964	26·51868	29·75455	32·13621	33·83063
3·1	0	10·83106	19·21744	25·65651	30·54669	34·20394	36·87766	38·76566
3·2	0	12·59314	22·25945	29·62217	35·17151	39·29040	42·27778	44·36758
3·3	0	14·65928	25·80000	34·20857	40·49118	45·11360	48·43448	50·73124
3·4	0	17·09242	29·93481	39·52721	46·62315	51·79124	55·46278	57·96718
3·5	0	19·97203	34·78258	45·71414	53·70889	59·46367	63·49841	66·20474
3·6	0	23·39961	40·49179	52·93674	61·92000	68·29905	72·70228	75·59586
3·7	0	27·50649	47·25012	61·40265	71·46601	78·50005	83·26613	86·30244
3·8	0	32·46494	55·29753	71·37178	82·60477	90·31249	95·41972	98·58916
3·9	0	38·50483	64·94474	83·17295	95·65633	104·03669	109·44000	112·65722
4·0	0	45·93863	76·60000	97·22686	111·02153	120·04231	125·66296	128·82752
4·1	0	55·20045	90·80847	114·07864	129·20765	138·78825	144·49908	147·46592
4·2	0	66·90894	108·31200	134·44499	150·86399	160·84945	166·45371	169·01630
4·3	0	81·97288	130·14217	159·28370	176·83235	186·95369	192·15427	194·02097
4·4	0	101·77708	157·77045	189·89859	208·21978	218·03271	222·14023	223·14757
4·5	0	128·52933	193·36018	228·10415	246·50617	255·29464	258·14675	257·22494
4·6	—	165·95745	240·21033	276·48965	293·70715	300·32869	300·70801	297·29164
4·7	—	—	303·58400	338·86359	352·62801	355·25957	351·72287	344·66229
4·8	—	—	392·37271	421·03600	427·27315	422·98141	413·36471	401·01993
4·9	—	—	—	532·27910	523·53374	507·52256	488·53923	468·54685
5·0	—	—	—	688·26156	650·39838	614·63369	581·20426	550·11337
5·1	—	—	—	—	822·21327	752·77500	696·86837	649·55652
5·2	—	—	—	—	1063·22283	934·85457	843·39459	772·10198
5·3	—	—	—	—	—	1181·47331	1032·34962	925·02170
5·4	—	—	—	—	—	1527·44262	1281·37852	1118·69384
5·5	—	—	—	—	—	—	1618·63791	1368·38182
5·6	—	—	—	—	—	—	2091·69833	1697·36503
5·7	—	—	—	—	—	—	—	2142·78154
5·8	—	—	—	—	—	—	—	2767·36099
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

TABLE I—(*continued*).

Values of β_3.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	
5·47200	5·54505	5·54774	5·49223	5·38882	5·21032	5·07194	4·87218	1·8
6·56556	6·66900	6·69008	6·64297	6·53966	6·39032	6·20353	5·98658	1·9
7·81963	7·95777	8·00000	7·96281	7·86015	7·70376	7·50359	7·26805	2·0
9·25357	9·43082	9·49691	9·47100	9·36924	9·20521	8·99040	8·73446	2·1
10·88912	11·10996	11·20249	11·18895	11·08800	10·91533	10·68412	10·40546	2·2
12·75081	13·01968	13·14100	13·14055	13·03987	12·85700	12·60708	12·30280	2·3
14·86629	15·18749	15·33963	15·35250	15·25093	15·05566	14·78400	14·45047	2·4
17·26684	17·64442	17·82888	17·85464	17·75028	17·53956	17·24229	16·87500	2·5
19·98792	20·42546	20·64305	20·68041	20·57039	20·34018	20·01238	19·60577	2·6
23·06977	23·57017	23·82076	23·86733	23·74757	23·49257	23·12805	22·67532	2·7
26·55821	27·12336	27·40557	27·45754	27·32248	27·03588	26·62694	26·11977	2·8
30·50551	31·13665	31·44672	31·49852	31·34075	31·01385	30·55093	29·97924	2·9
34·97143	35·66579	36·00000	36·04377	35·85359	35·47542	34·94677	34·29832	3·0
40·02452	40·77897	41·12872	41·15378	40·91866	40·47551	39·86670	39·12670	3·1
45·74361	46·55101	46·90489	46·89705	46·60097	46·07576	45·36914	44·51979	3·2
52·21964	53·06850	53·41068	53·35129	52·97396	52·34552	51·51961	50·53845	3·3
59·55782	60·43103	60·73997	60·60490	60·12077	59·36297	58·39163	57·25491	3·4
67·88034	68·75344	69·00044	68·75867	68·13571	67·21640	66·06791	64·74368	3·5
77·32958	78·16863	78·31586	77·92779	77·12604	76·00572	74·64162	73·09286	3·6
88·07212	88·82758	88·82894	88·24432	87·21400	85·84425	84·21798	82·40021	3·7
100·30364	100·91991	100·70481	99·86000	98·53927	96·86080	94·91600	92·77596	3·8
114·25501	114·64607	114·13512	112·94981	111·26193	109·20215	106·87064	104·34450	3·9
130·19998	130·25632	129·34314	127·71610	125·56589	123·03600	120·23522	117·24648	4·0
148·46471	148·04207	146·59001	143·45076	141·66316	138·55442	135·18444	131·64135	4·1
169·44000	168·34874	166·18242	163·23621	159·79886	155·97808	151·91782	147·71022	4·2
193·59693	191·58787	188·48207	184·62341	180·25740	175·56124	170·66381	165·65935	4·3
221·50693	218·25251	213·91770	208·87083	203·36995	197·59782	191·68480	185·72423	4·4
253·86810	248·93685	243·00000	236·44133	229·52365	222·42874	215·28302	208·17445	4·5
291·53977	284·36158	276·34066	267·85963	259·17616	250·45088	241·80774	233·31950	4·6
335·58868	325·40700	314·67673	303·75068	292·85347	282·12801	271·66398	261·51581	4·7
387·35126	373·15656	358·90198	344·86280	331·20000	318·00442	305·32303	293·17509	4·8
448·51903	428·95489	410·10784	392·09743	374·96309	358·72156	343·33557	328·77466	4·9
521·25714	494·48636	469·63736	446·54746	425·06230	405·03908	386·34752	368·86975	5·0
608·37258	571·88229	539·15698	509·54711	482·57201	457·86117	435·12000	414·10888	5·1
713·55655	663·87024	620·75355	582·73768	548·81752	518·26993	490·55405	465·25256	5·2
841·74254	773·98556	717·06701	668·15520	625·41017	587·56812	553·72181	523·19655	5·3
999·64805	906·87600	831·47510	768·34849	714·33162	667·33415	625·90596	589·00103	5·4
1196·61837	1068·75185	968·35433	886·54107	818·03932	759·49378	708·65022	663·92692	5·5
1445·98797	1268·06663	1133·45664	1026·85589	939·60867	866·41460	803·82428	749·48239	5·6
1767·36192	1516·57773	1334·46556	1194·63332	1082·92777	991·03145	913·70857	847·48183	5·7
2190·63315	1831·05543	1581·83626	1396·88998	1252·96795	1137·01614	1041·10567	960·12198	5·8
2763·47753	2236·14894	1890·10532	1642·99492	1456·16614	1309·00889	1189·48840	1090·01813	5·9
—	2769·43027	2279·99999	1926·12857	1700·97663	1512·94076	1363·20000	1240·65000	6·0
—	3490·80240	2781·97487	2322·69706	1998·68407	1756·48819	1567·72730	1415·90414	6·1
—	—	3442·43442	2799·25452	2364·63286	2049·72980	1810·08041	1620·93898	6·2
—	—	4335·33729	3412·44751	2820·14136	2406·11294	2099·32960	1862·18916	6·3
—	—	—	4218·75744	3395·58295	2843·91584	2447·37773	2147·87250	6·4
—	—	—	5308·17016	4135·55270	3388·52216	2870·10032	2488·61733	6·5
—	—	—	—	5107·94743	4076·08374	3389·06759	2898·36583	6·6
—	—	—	—	6420·90275	4959·65704	4034·22238	3395·70595	6·7
—	—	—	—	—	6119·99250	4848·19168	4005·88251	6·8
—	—	—	—	—	7685·64544	5893·51196	4763·92498	6·9
—	—	—	—	—	—	7265·32153	5719·68219	7·0

TABLE I—(*continued*).

Values of β_6.

β_1.

	0	·1	·2	·3	·4	·5	·6
1·8	9·00000	10·56988	11·37904	11·65931	11·57571	11·24612	10·75489
1·9	11·26008	13·23077	14·24766	14·60515	14·51125	14·11351	13·51686
2·0	14·00000	16·46165	17·72924	18·17685	18·06655	17·58286	16·85605
2·1	17·31490	20·37698	21·94568	22·49677	22·36051	21·76742	20·87902
2·2	21·32037	25·12011	27·00353	27·71253	27·53558	26·80218	25·71216
2·3	26·15756	30·98446	33·20564	34·00193	33·76303	32·84853	31·50583
2·4	32·00000	37·78431	40·64836	41·58212	41·24961	40·10000	38·43902
2·5	39·06250	46·18140	49·64275	50·74739	50·24583	48·78960	46·83136
2·6	47·61300	56·36215	60·52268	61·73207	61·05669	59·19886	56·60720
2·7	55·66909	68·73087	73·70383	75·02652	74·05525	71·66945	68·44455
2·8	70·61765	83·79934	89·70741	91·09765	89·70029	86·61786	82·56030
2·9	86·04990	102·22137	109·19212	110·56655	108·55927	104·55444	98·96957
3·0	105·00000	124·84101	132·99779	134·21520	131·33846	126·10769	119·60133
3·1	128·40949	152·76061	162·20552	163·03622	158·92300	152·05685	143·76373
3·2	157·53846	187·43873	198·22205	198·30084	192·43054	183·37199	172·75506
3·3	194·10417	230·83408	242·90000	241·65332	233·28402	221·27382	207·61816
3·4	240·49650	285·62144	298·71190	295·24380	283·31160	267·30520	249·65531
3·5	300·12500	355·52475	369·00857	361·91859	344·88577	323·43535	300·50441
3·6	378·00027	445·84514	458·40994	445·49756	421·12000	392·20036	362·24145
3·7	481·75409	564·33393	573·41334	551·18736	516·15119	476·89985	437·52021
3·8	623·54548	722·67424	723·36878	686·21072	635·55341	581·87572	529·76495
3·9	823·87500	939·16548	922·09827	860·79199	786·95638	712·91403	643·44000
4·0	1120·00044	1243·83108	1190·70000	1089·74754	980·99234	877·83511	784·43293
4·1	1586·99669	1688·93436	1562·65014	1395·14604	1232·78583	1087·37952	960·60823
4·2	2401·00000	2372·80866	2093·67200	1810·95208	1564·37061	1356·56923	1182·62312
4·3	4092·27183	3501·41300	2882·34489	2391·55352	2008·75892	1706·85856	1465·15573
4·4	9317·00007	5580·42447	4117·65621	3228·42286	2617·09069	2169·64147	1826·36689
4·5	—	10228·34035	6204·67651	4485·37892	3471·86981	2792·18750	2303·06656
4·6	—	26594·29656	10169·64897	6481·22602	4713·08621	3648·13982	2931·31183
4·7	—	—	19470·30398	9912·98937	6594·13318	4857·09743	3779·10320
4·8	—	—	56371·22036	16599·18193	9614·97614	6623·62278	4949·12279
4·9	—	—	—	32999·08240	14889·79442	9321·73705	6609·16370
5·0	—	—	—	107695·38628	25413·19110	13699·35294	9048·42104
5·1	—	—	—	—	52445·38896	21455·52501	12800·88518
5·2	—	—	—	—	198872·14895	37301·06052	18948·75669
5·3	—	—	—	—	—	80097·49338	29999·60559
5·4	—	—	—	—	—	372908·59530	53139·45765
5·5	—	—	—	—	—	—	119233·63920
5·6	—	—	—	—	—	—	759289·60401
5·7	—	—	—	—	—	—	—
5·8	—	—	—	—	—	—	—
5·9	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—

TABLE I—(*continued*).

Values of β_6.

β_1.

·9	1·0	1·1	1·2	1·3	1·4	1·5	
8·83433	8·15118	7·47777	6·82451	6·16894	5·60360	5·04289	1·8
11·17900	10·34692	9·52638	8·72995	7·96593	7·23956	6·55383	1·9
14·01129	13·00000	12·00305	11·03545	10·10708	9·22407	8·38990	2·0
17·41688	16·18967	14·98100	13·80869	12·68431	11·61499	10·60464	2·1
21·49603	20·00831	18·54532	17·12800	15·76982	14·47888	13·25951	2·2
26·36630	24·56380	22·79495	21·08421	19·44700	17·89242	16·42508	2·3
32·16574	29·98227	27·84518	25·78270	23·81232	21·94400	20·18247	2·4
39·05687	36·41153	33·83049	31·34598	28·97749	26·73564	24·62500	2·5
47·23137	44·02511	40·90805	37·91659	35·07200	32·38521	29·86015	2·6
56·91597	53·02714	49·26193	45·66075	42·24625	39·02907	36·01182	2·7
68·37949	63·65838	59·10810	54·77260	50·67518	46·82519	43·22304	2·8
81·94402	76·20351	70·70058	65·47942	60·56257	55·95689	51·65914	2·9
97·98379	91·00000	84·33882	78·04766	72·14637	66·63722	61·51151	3·0
116·96246	108·44918	100·37681	92·79065	85·70487	79·11430	73·00209	3·1
139·42608	129·02981	119·23414	110·07767	101·56439	93·67759	86·38868	3·2
166·03608	153·31496	141·40973	130·34509	120·10836	110·66555	101·96895	3·3
197·59354	181·99321	167·49870	154·11009	141·78858	130·47480	120·09977	3·4
235·07313	215·89514	198·21347	181·98750	167·13881	153·57131	141·18296	3·5
279·66656	256·02699	234·41002	214·71069	196·79148	180·50386	165·70000	3·6
332·82366	303·61339	277·12110	253·15751	231·49838	211·92066	194·21319	3·7
396·40001	360·15227	327·59804	298·38281	272·15609	248·58964	227·38490	3·8
472·60352	427·48564	387·36420	351·65936	319·83784	291·42344	265·99719	3·9
564·27028	507·89191	458·28341	414·52944	375·83325	341·51036	310·97634	4·0
674·95976	604·20700	538·52748	488·87120	441·69836	400·15311	363·42307	4·1
809·19917	719·98521	643·29726	576·98306	519·31919	468·91698	424·64982	4·2
972·79761	859·71417	763·76288	681·69350	610·99251	549·69077	496·22740	4·3
1173·28180	1029·10666	908·47666	806·50403	719·52993	644·76391	580·04326	4·4
1420·50928	1235·50000	1083·03746	955·77730	848·39240	756·92502	678·37500	4·5
1727·54471	1488·41118	1294·57784	1135·00321	1001·86621	889·58832	793·98353	4·6
2111·93585	1800·31916	1552·26683	1351·05386	1185·29254	1046·95774	930·23215	4·7
2597·60657	2187·78540	1868·00882	1612·80000	1405·37863	1234·24080	1091·23937	4·8
3217·73134	2673·08744	2257·43040	1931·57581	1670·60729	1457·93113	1282·07756	4·9
4019·21171	3286·65035	2741·29638	2322·13115	1991·80039	1726·18369	1509·03180	5·0
5069·85289	4070·74603	3347·57848	2803·84815	2382·89098	2049·32000	1779·94211	5·1
6470·26979	5085·27254	4114·54025	3402·51244	2862·00092	2440·51504	2104·65868	5·2
8374·45168	6417·05252	5095·44479	4152·90618	3452·96676	2916·74377	2495·65464	5·3
11027·01600	8195·32568	6365·92882	5102·68429	4187·53643	3500·10258	2968·86115	5·4
14834·80538	10618·63416	8035·91250	6318·30392	5108·59113	4219·68376	3544·81789	5·5
20514·98864	14003·83814	10269·52127	7894·34108	6274·97282	5114·27804	4250·28332	5·6
29432·15808	18881·04131	13319·83857	9968·58982	7768·88564	6236·34746	5120·52074	5·7
44471·78151	26191·91258	17592·66245	12747·42898	9707·56565	7657·98899	6202·60000	5·8
72771·12146	37747·20230	23770·94842	16550·29979	12262·25509	9480·10426	7559·91373	5·9
137288·89025	57434·99984	32803·61819	21891·83037	15690·22476	11846·90000	9281·23336	6·0
368137·95820	95110·70470	47889·60904	29643·43991	20391·20888	14969·55326	11488·51177	6·1
—	183958·54967	73394·60360	41378·15038	27012·30625	19166·32023	14358·25475	6·2
—	537373·87106	123089·42788	60189·03145	36656·13405	24933·62294	18149·09652	6·3
—	—	244764·82983	92941·09117	51329·91458	33079·04770	23252·05212	6·4
—	—	796873·64937	158003·73264	75027·70957	44987·59385	30279·45386	6·5
—	—	—	324150·69600	116774·84051	63202·70610	40232·52177	6·6
—	—	—	1217690·26020	201460·19097	92848·88108	54839·83220	6·7
—	—	—	—	428250·86517	145728·18199	77304·82418	6·8
—	—	—	—	1965948·20210	255463·29045	114166·67456	6·9
—	—	—	—	—	565740·80342	180792·85424	7·0

TABLE II.

Ratio of the Probable Error of the Mode to the Probable Error of the Mean.

β_1.

β_2	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*
1·8	∞	21·3313	4·2009	2·1183	1·2174	·7269	·4193	·2072
1·9	7·7368	457·8417	12·9077	4·3927	2·1953	1·2549	·7466	·4311
2·0	4·3012	33·2972	169·7287	12·4767	4·4553	2·2432	1·2838	·7642
2·1	3·1616	8·4333	84·9902	96·6366	11·7814	4·4488	2·2717	1·3064
2·2	2·5970	4·2726	14·0362	242·9030	65·5132	12·0059	4·4041	2·2867
2·3	2·2634	3·1503	6·5493	22·8026	1167·9192	48·8362	10·3622	4·3384
2·4	2·0460	2·6578	4·2238	9·1310	36·8018	∞	38·6385	9·7353
2·5	1·8957	2·2921	3·1915	5·4143	12·6243	61·2427	1517·6874	31·8420
2·6	1·7881	2·0583	2·6332	3·8779	6·9579	17·4427	108·2317	434·5838
2·7	1·7095	1·8984	2·2922	3·0755	4·7362	8·9118	24·2562	216·1278
2·8	1·6516	1·7842	2·0664	2·6007	3·6263	5·7946	11·4150	34·2005
2·9	1·6103	1·7004	1·9086	2·2942	2·9870	4·2963	7·1814	14·6528
3·0	1·5811	1·6381	1·7942	2·0841	2·5828	3·4550	5·1003	8·6834
3·1	1·5622	1·5918	1·7093	1·9340	2·3105	2·9332	4·0108	6·0586
3·2	1·5520	1·5578	1·6455	1·8238	2·1187	2·5871	3·3476	4·6629
3·3	1·5493	1·5337	1·5977	1·7417	1·9797	2·3467	2·9140	3·8294
3·4	1·5534	1·5180	1·5625	1·6806	1·8776	2·1746	2·6166	3·2922
3·5	1·5637	1·5094	1·5378	1·6359	1·8026	2·0498	2·4061	2·9282
3·6	1·5799	1·5073	1·5223	1·6051	1·7490	1·9595	2·2550	2·6725
3·7	1·6018	1·5114	1·5161	1·5866	1·7133	1·8963	2·1471	2·4899
3·8	1·6293	1·5220	1·5178	1·5803	1·6938	1·8556	2·0726	2·3622
3·9	1·6624	1·5397	1·5301	1·5867	1·6902	1·8351	2·0257	2·2740
4·0	1·7014	1·5665	1·5549	1·6080	1·7033	1·8343	2·0032	2·2188
4·1	1·7464	1·6062	1·5967	1·6477	1·7358	1·8541	2·0041	2·1924
4·2	1·7978	1·6667	1·6633	1·7120	1·7917	1·8969	2·0289	2·1931
4·3	1·8561	1·7653	1·7687	1·8106	1·8777	1·9669	2·0798	2·2207
4·4	1·9221	1·9436	1·9381	1·9588	2·0031	2·0704	2·1558	2·2770
4·5	1·9965	2·3211	2·2203	2·1819	2·1838	2·2167	2·2820	2·3652
4·6	—	3·4126	2·7193	2·5226	2·4412	2·4189	2·4376	2·4905
4·7	—	—	3·7109	3·0601	2·8113	2·6967	2·6538	2·6602
4·8	—	—	6·4341	3·9632	3·3544	3·0796	2·9428	2·8847
4·9	—	—	—	5·7005	4·1910	3·6148	3·3289	3·1786
5·0	—	—	—	10·6664	5·6242	4·3834	3·8501	3·5624
5·1	—	—	—	—	8·1895	5·5415	4·5669	4·0669
5·2	—	—	—	—	16·5324	7·4444	5·5857	4·7390
5·3	—	—	—	—	—	11·1952	7·1126	5·6557
5·4	—	—	—	—	—	24·9391	9·6439	6·9532
5·5	—	—	—	—	—	—	14·7870	8·9040
5·6	—	—	—	—	—	—	38·3578	12·1682
5·7	—	—	—	—	—	—	—	19·0756
5·8	—	—	—	—	—	—	—	20·9345
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

TABLE II—(*continued*).

Ratio of the Probable Error of the Mode to the Probable Error of the Mean.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	β_2
0	—	—	—	—	—	—	—	1·8
·2137	0	—	—	—	—	—	—	1·9
·4418	·2199	0	—	—	—	—	—	2·0
·7794	·4516	·2257	0	—	—	—	—	2·1
1·3238	·7927	·4608	·2313	0	—	—	—	2·2
2·2924	1·3375	·8044	·4693	·2366	0	—	—	2·3
4·2616	2·2914	1·3481	·8148	·4772	·2417	0	—	2·4
9·2167	4·1795	2·2856	1·3563	·8241	·4846	·2466	0	2·5
27·3963	8·6715	4·0960	2·2766	1·3627	·8324	·4916	·2514	2·6
214·2686	23·4630	8·2243	4·0130	2·2652	1·3675	·8399	·4982	2·7
536·8306	132·4970	20·7275	7·8248	3·9320	2·2521	1·3711	·8468	2·8
49·3159	2365·9521	92·5373	18·5702	7·4670	3·8536	2·2381	1·3737	2·9
18·8973	73·5471	∞	69·7007	16·6666	7·1439	3·7785	2·2233	3·0
10·6407	24·5570	115·3008	2797·9031	55·2575	15·3999	6·8551	3·7067	3·1
7·1985	13·0637	32·2652	195·0629	754·5445	45·4504	14·2059	6·5923	3·2
5·4232	8·5553	16·0888	43·0412	373·6907	360·1882	38·4326	13·1965	3·3
4·3841	6·3070	10·1749	19·9059	58·6048	902·3059	216·7873	33·2041	3·4
3·7238	5·0188	7·3339	12·1171	24·7849	82·0317	3859·0983	147·8862	3·5
3·2808	4·2123	5·7431	8·5285	14·4603	31·1179	119·2455	∞	3·6
2·9731	3·6767	4·7652	6·5690	9·9220	17·3081	39·4906	182·8220	3·7
2·7556	3·3080	4·1186	5·3807	7·5111	11·5534	20·7994	50·8071	3·8
2·6024	3·0487	3·6782	4·6092	6·0748	8·5876	13·4722	25·1232	3·9
2·4975	2·8664	3·3701	4·0854	5·1532	6·8542	9·8206	15·7414	4·0
2·4316	2·7414	3·1538	3·6557	4·5324	5·7559	7·7303	11·2373	4·1
2·3991	2·6619	3·0049	3·4647	4·1019	5·0222	6·4239	8·7171	4·2
2·3972	2·6207	2·9087	3·2876	3·7999	4·5157	5·5588	7·1650	4·3
2·4252	2·6139	2·8560	3·1713	3·5899	4·1606	4·9647	6·1470	4·4
2·4841	2·6398	2·8417	3·1043	3·4497	3·9126	4·5484	5·4516	4·5
2·5767	2·6988	2·8630	3·0798	3·3655	3·7440	4·2566	4·9654	4·6
2·7072	2·7928	2·9196	3·0943	3·3282	3·6396	4·0565	4·6235	4·7
2·8822	2·9256	3·0127	3·1462	3·3334	3·5871	3·9275	4·3865	4·8
3·1106	3·1026	3·1451	3·2360	3·3784	3·5806	3·8567	4·2298	4·9
3·4051	3·3318	3·3217	3·3659	3·4629	3·6166	3·8362	4·1379	5·0
3·7831	3·6239	3·5490	3·5398	3·5883	3·6938	3·8611	4·1008	5·1
4·2700	3·9939	3·8366	3·7633	3·7576	3·8129	3·9289	4·1125	5·2
4·9030	4·4629	4·1971	4·0446	3·9757	3·9760	4·0402	4·1696	5·3
5·7402	5·0710	4·6485	4·3947	4·2495	4·1872	4·1957	4·2711	5·4
6·8793	5·8339	5·2155	4·8287	4·5884	4·4523	4·3988	4·4175	5·5
8·4860	6·8516	5·9343	5·3673	5·0054	4·7796	4·6543	4·6113	5·6
10·9159	8·2317	6·6993	6·0399	5·5178	5·1798	4·9694	4·8565	5·7
15·0461	10·1883	8·0738	6·8654	6·1501	5·6681	5·3533	5·1589	5·8
24·2232	13·1659	9·7223	7·9782	6·9362	6·2646	5·8189	5·5274	5·9
—	18·3208	12·0456	9·4138	7·9259	6·9971	6·3809	5·9709	6·0
—	30·4688	15·6688	11·3534	9·1941	7·9053	7·0692	6·5057	6·1
—	—	22·0403	14·1326	10·8607	9·0461	7·9092	7·1508	6·2
—	—	38·1802	18·4461	13·1310	10·5089	8·9484	7·9325	6·3
—	—	—	26·2704	16·3951	12·4328	10·2534	8·8874	6·4
—	—	—	47·9585	21·5227	15·0620	11·9257	10·0672	6·5
—	—	—	—	31·0936	18·8677	14·1298	11·5483	6·6
—	—	—	—	60·8817	24·9280	17·1544	13·4480	6·7
—	—	—	—	—	36·6168	21·5651	15·9472	6·8
—	—	—	—	—	79·1531	28·6967	19·4168	6·9
—	—	—	—	—	—	42·9814	24·5042	7·0

its standard deviation is at least an element of importance. It is possible, of course, that the distribution of the mode may be skew. The arithmetical work has been extremely laborious, but will have been repaid, if it enables statistical workers to obtain rapidly an important measure of accuracy by a minute or two of additional computation.

CHAPTER III. GRAPHIC REPRESENTATION OF THE RATIO OF THE PROBABLE ERROR OF THE MODE TO THE PROBABLE ERROR OF THE MEAN BY A SURFACE.

(1) *Contour Lines of the Surface of the Probable Error of the Mode.* It will be seen from Table II that the standard deviation of the mode is usually greater than that of the mean and therefore the mode cannot in general be determined with the same accuracy as the mean; within however a large range of values of β_1, β_2, such as most frequently occur in statistical practice, it can be found with *adequate* accuracy, if the size of the sample be reasonably large. For small samples, not only would our method of approaching the problem be illegitimate, but the mode would have no practical value at all.

It will be noted, however, that in a part of the U-curve area the mode, in this case an "antimode," can be determined with greater accuracy than the mean, and it is easy to see that this must physically be so.

In order to get some idea of the nature of the distribution of accuracy of determination of the mode, we may denote the ratio of the standard deviation of the mode to that of the mean by z, and consider the geometry of the surface:

$$z = f(\beta_1, \beta_2),$$

which our Table II represents.

It is clear that there are two asymptotic planes: the first, $5\beta_2 - 6\beta_1 - 9 = 0$, is the vertical plane through the rectangular line, and the other will arise when $\beta_6 = \infty$ or $8\beta_2 - 15\beta_1 - 36 = 0$. In actual distributions, of course, such a value of β_6 cannot occur, but when we fit a Pearson curve by the first four moments $\beta_6 = \infty$ is possible. The surface asymptotes on both sides to the first plane, and it is sufficient to limit it by the second plane for practical purposes, as values of β_1 and β_2 beyond this are very rare. In order to obtain the contour lines of the surface a double series of sections were drawn: i.e. those for β_1 constant and those for β_2 constant. Additional points on these sections were obtained, by interpolation, where needful. From these sections the points corresponding to a constant value of z were read off and plotted to obtain a given contour. In this manner Diagram I was obtained.

(2) *Minimal Lines.* Considerable attention has been devoted to the points on both sections where the series of tangents to the curves are horizontal, i.e. to the points on the curves where z is a minimum. The locus of these points on the contour diagram may be termed a "minimal line." The minimal line for the β_1-sections may be termed the β_1-minimal line and that for the β_2-sections the β_2-minimal line.

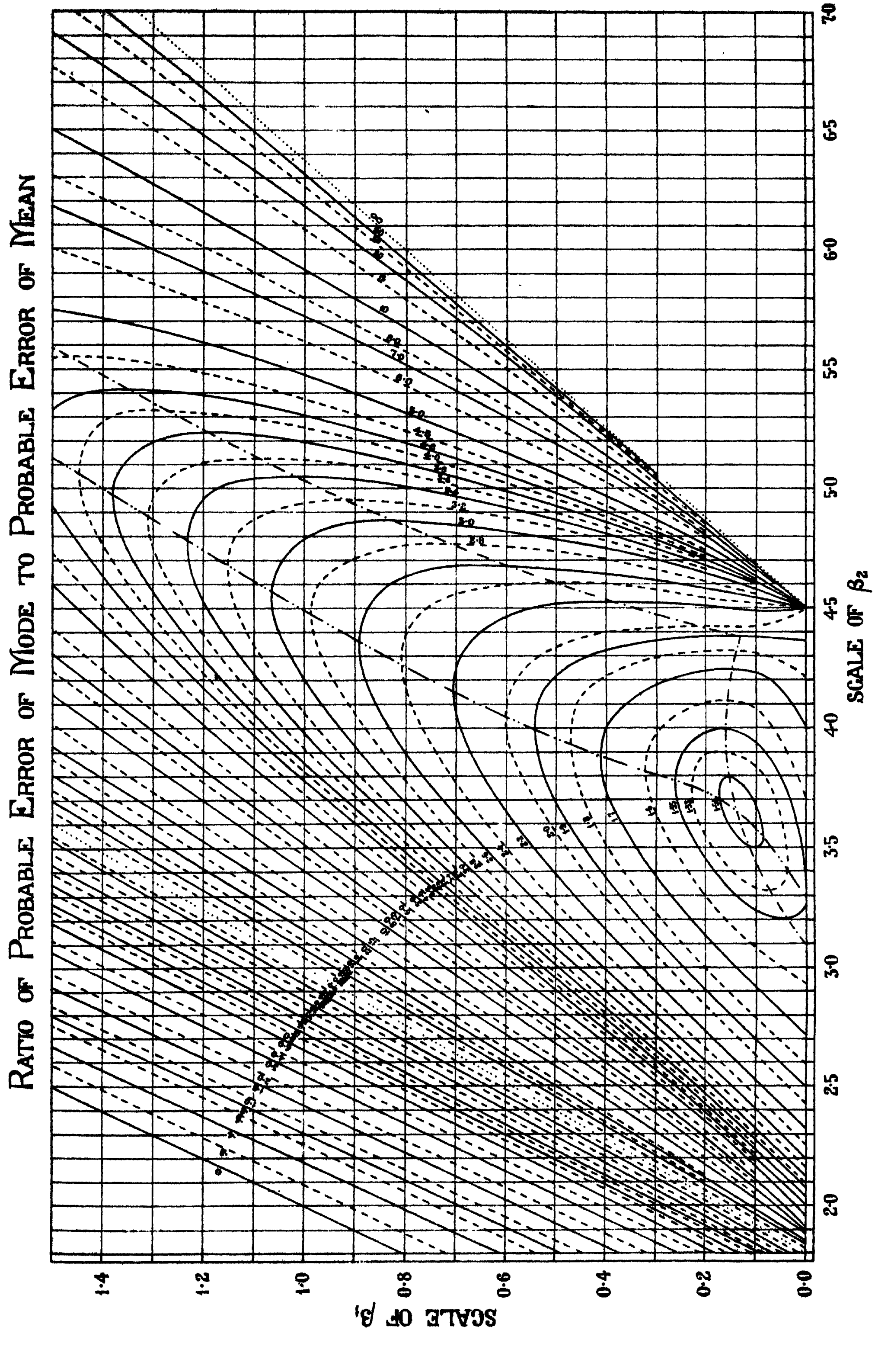

RATIO OF PROBABLE ERROR OF MODE TO PROBABLE ERROR OF MEAN
SCALE OF β_2
2·0 2·5 3·0 3·5 4·0 4·5 5·0 5·5 6·0 6·5 7·0
SCALE OF β_1
0·0 0·2 0·4 0·6 0·8 1·0 1·2 1·4

Diagram I

For determining with adequate accuracy the position of the minimum point of any section, the six-point method in most cases, but in a few the five or four-point method, as given in *Tracts for Computers*, No. II, p. 36, was adopted. A brief description of the method is as follows:

Let the ordinates at six equidistant points x_1, x_2, x_3, x_4, x_5 and x_6 be z_1, z_2, z_3, z_4, z_5 and z_6 respectively, the origin being mid-way between x_3 and x_4.

The abscissa $\bar{x}$ and the ordinate $\bar{z}$ of the minimum point can be found by the following equations:

$$\bar{x}^2 \{z_2 - z_1 - 2(z_4 - z_3) + z_6 - z_5\} + \bar{x}\{z_2 - z_1 - 6(z_3 - z_2) - 6(z_4 - z_5) - (z_6 - z_5)\}$$
$$+ \tfrac{1}{12}\{-(z_2 - z_1) + 98(z_4 - z_3) - (z_6 - z_5)\} = 0 \quad \text{...........(31).}$$

Writing this equation $\qquad a\bar{x}^2 + b\bar{x} + c = 0,$

we have: $$\bar{z} = \tfrac{1}{2}(z_3 + z_4) - \tfrac{1}{64}b + \tfrac{1}{12}c\bar{x} + \tfrac{1}{48}b\bar{x}^2 \quad \text{....................(32).}$$

The β_1-minimal line is fairly straightforward and the minimum values of z and the corresponding values of β_2 being obtained, the minimal curve can be plotted: see Diagram II. It was clear that this curve could be represented fairly well by a

MINIMAL LINES

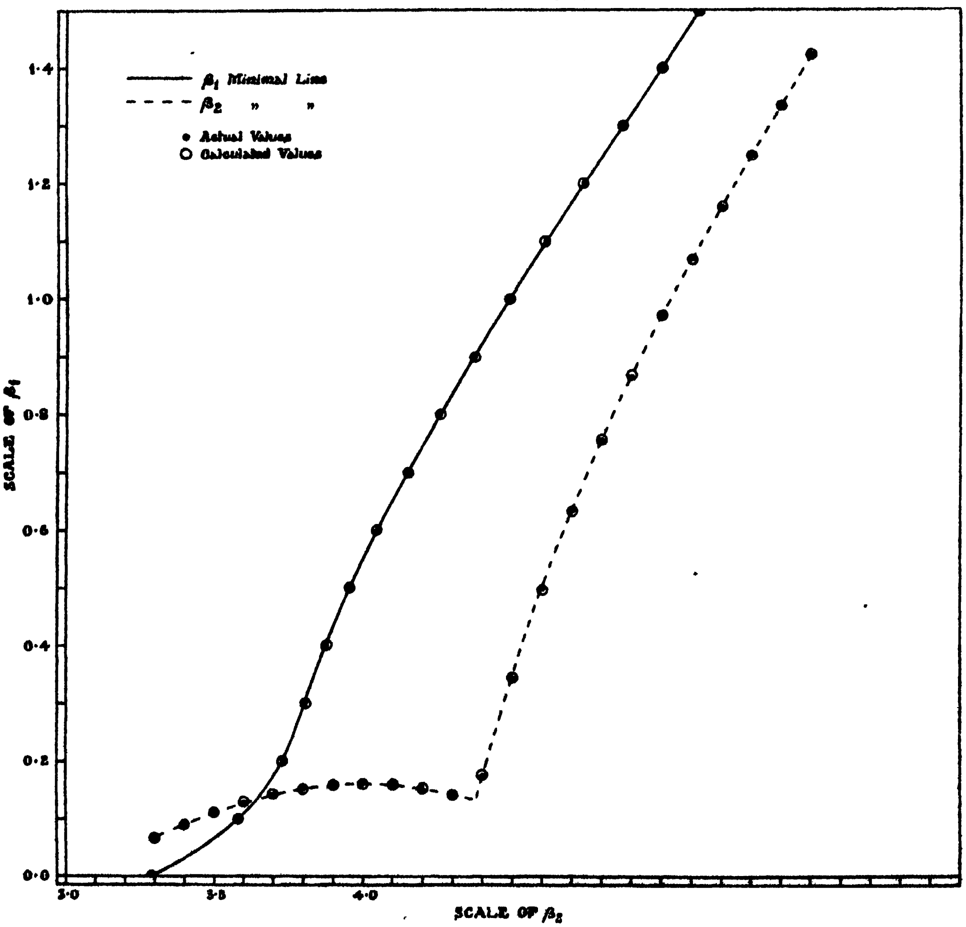


Diagram II.

parabola of the sixth degree. Such a parabola was determined by the method of least squares and its equation was

$$y = 4{\cdot}202293 + {\cdot}05608703x + {\cdot}0005887224x^2 - {\cdot}0001020906x^3$$
$$+ {\cdot}000006487695x^4 + {\cdot}0000005543824x^5 - {\cdot}00000004007236x^6 \ldots\ldots(33),$$

where y is the calculated value of β_2 and $x = 20(\beta_1 - {\cdot}75)$. Table III compares the values of β_2 found from this curve with the plotted values as obtained by computation from Table I.

TABLE III.

β_1	x	Minimum value z	β_2	Calculated by (33)
0	−15	1·5492	3·2884	3·2890
·1	−13	1·5071	3·5833	3·5830
·2	−11	1·5157	3·7318	3·7272
·3	− 9	1·5803	3·8019	3·8082
·4	− 7	1·6896	3·8729	3·8751
·5	− 5	1·8323	3·9541	3·9510
·6	− 3	2·0007	4·0462	4·0424
·7	− 1	2·1894	4·1476	4·1469
·8	1	2·3944	4·2562	4·2589
·9	3	2·6125	4·3707	4·3737
1·0	5	2·8414	4·4892	4·4898
1·1	7	3·0796	4·6119	4·6089
1·2	9	3·3254	4·7371	4·7343
1·3	11	3·5779	4·8647	4·8679
1·4	13	3·8355	5·0043	5·0043
1·5	15	4·0994	5·1249	5·1245

The accordance is within about 1 per cent. and this may be considered reasonable for present purposes. The β_2-minimal line is less simple, because for the portion of the surface dealt with in our diagram it consists of two separate parts, each of which has been fitted by an independent parabola of the third degree. The results are exhibited in Tables IV and V. The intersection of the β_1-minimal and the β_2-minimal curves gives the "bottom of the basin" in the solid surface, and this point has been determined as

$$z = 1{\cdot}505, \quad \text{at} \quad \beta_1 = {\cdot}134, \quad \beta_2 = 3{\cdot}64.$$

This point corresponds on the (β_1, β_2) diagram* to a curve of Pearson's Type IV, a skew curve, and not, as might *à priori* have been supposed, to a normal curve. We see accordingly that for points in the "basin" the mode as determined by the method of moments is always 50 per cent. more inaccurate than the mean, and usually much more than this. Still the mean can be so accurate for large samples that we may learn something from determining the mode even when its probable error is a considerable multiple of that of the mean.

The equation of the parabola fitted to one part of the β_2-minimal line is

$$y = {\cdot}153384 + {\cdot}00831818x - {\cdot}00195078x^2 - {\cdot}00004203768x^3 \quad \ldots(34).$$

* See *Tables for Statisticians*, p. 66.

TABLE IV.

β_2	x	Minimum value z	β_1	y calculated by (34)
3·3	−5	1·5298	·06848	·06828
3·4	−4	1·5177	·09166	·09159
3·5	−3	1·5089	·11141	·11201
3·6	−2	1·5039	·12936	·13078
3·7	−1	1·5041	·14279	·14316
3·8	0	1·5099	·15395	·15338
3·9	1	1·5241	·16023	·15971
4·0	2	1·5495	·16207	·16039
4·1	3	1·5906	·15927	·15965
4·2	4	1·6554	·15181	·15275
4·3	5	1·7598	·14161	·14095

y = computed value of β_1.
$x = 10(\beta_2 - 3\cdot 8)$.

It will be seen that the deviation of β_1 from its computed value is as a rule less than 1 per cent.

The equation of the parabola fitted to the other part of the β_2-minimal line is

$$y = \cdot 9205583 + \cdot 0516564x - \cdot 00100021x^2 + \cdot 0000422167x^3 \quad \ldots\ldots(35).$$

TABLE V.

β_2	x	Minimum value z	β_1	y calculated by (35)
4·4	−11	1·9368	·17386	·17512
4·5	− 9	2·1787	·34301	·34386
4·6	− 7	2·4189	·49925	·49547
4·7	− 5	2·6512	·63419	·63199
4·8	− 3	2·8774	·75425	·75545
4·9	− 1	3·0995	·86453	·86786
5·0	1	3·3189	·96775	·97126
5·1	3	3·5362	1·07096	1·06767
5·2	5	3·7524	1·15870	1·15911
5·3	7	3·9678	1·24915	1·24762
5·4	9	4·1825	1·33688	1·33523
5·5	11	4·3971	1·42205	1·42395

y = computed value of β_1.
$x = 10(\beta_2 - 4\cdot 95)$.

(3) *Model of the Surface.* The surface was modelled by cutting out the contours and building them up at the appropriate values of z. It is seen to be of a very striking and suggestive kind. Photographs of it are provided in Plates I and II.

The reader will find it desirable to compare the contour diagram and the surface. He will then notice that the surface consists essentially of a first portion which we have termed the "basin." This, after a fairly flat portion, rises abruptly to the two asymptotic planes. The asymptotic plane to the right of the figures in Plate I, but to the left in Plate II, is that which bounds the area where the

moments of Type IV curve may become infinite and which is marked on the (β_1, β_2) diagram as the heterotypic area. The boundary of the "basin," on the left of the photographs on Plate I and on the right of the photographs on Plate II, is the vertical plane through the "rectangular line" and beyond this we have another portion of the surface of a scroll-like form which sinks below the bottom of the "basin" and covers the types of curves which may be found above the "rectangular line," including the U- and some J-curves. The portion of the surface shown is limited by three considerations: (1) all real values are limited by $\beta_1 = 0$, hence the surface is not shown below this plane; again (2) the tables have been limited by $\beta_1 = 1{\cdot}5$. which cuts off the back part of the "basin" as shown in Plate I; unlike the front part of the "basin," the value here of the Probable Error of the Mode would be real but some limit had to be given to the tables and the model; (3) a similar limit to the tables is given by $\beta_2 = 7{\cdot}0$. On the "scroll" side a limitation has been made at $\beta_2 = 1{\cdot}8$, although it would have been more proper to have carried it to $\beta_2 = 1$, and limited it by the straight line $\beta_2 - \beta_1 - 1 = 0$. This is the contour line marked 0 in Diagram I of which however only a portion is shown. The contour lines on this portion of the scroll are practically straight lines and for the purposes of the model it was desirable to limit the size. (4) For values lying between $\beta_2 = 1$ and $1{\cdot}8$, and the corresponding value of β_1 falling below the line $\beta_2 - \beta_1 - 1 = 0$, the contour lines of Diagram I are so approximately straight that for practical purposes the value of the probable error may be found by producing them.

CHAPTER IV. STATISTICAL APPLICATIONS.

I propose now to consider by a few examples the application of the above results to determine the probable error of the mode. I shall use several cases of skew distribution, which have already been adopted in earlier researches to illustrate various statistical points. By hatched spaces on the diagrams I shall indicate the relative significance of the probable errors of mode and mean.

Example I. Barometric Height at Laudale (4018 *observations*)*.

Barometric Height	*27·9*	*28·0*	*28·1*	*28·2*	*28·3*	*28·4*	*28·5*	*28·6*	*28·7*	*28·8*
Frequency	1	1	0	0	·5	2·5	2	4·5	8·5	12
Barometric Height	*28·9*	*29·0*	*29·1*	*29·2*	*29·3*	*29·4*	*29·5*	*29·6*	*29·7*	*29·8*
Frequency	32·5	47·5	72·0	87·5	125·5	187·5	255·5	304	346·5	394
Barometric Height	*29·9*	*30·0*	*30·1*	*30·2*	*30·3*	*30·4*	*30·5*	*30·6*	*30·7*	*30·8*
Frequency	380·5	400	361	376	266·5	182	94	50·5	21	2

* *Phil. Trans.* Vol. 190 (1897), pp. 423—469.

The moments were calculated using Sheppard's Corrections. We find:

$$\begin{array}{llll} \text{Mean} = & 29{\cdot}85699, & \beta_1 = & {\cdot}2039448, \\ \mu_2 = & 14{\cdot}78622, & \beta_2 = & 3{\cdot}200531, \\ \mu_3 = & -\,25{\cdot}67687, & \sigma = & {\cdot}3845285. \\ \mu_4 = & 699{\cdot}7395, & & \end{array}$$

The distance between the mean and mode:

$$d = {\cdot}093161,$$

and

$$\begin{array}{llll} \text{Mode} = & 29{\cdot}95015, & & \\ \kappa = & -\,{\cdot}7629264, & \text{so Type I curve was used,} & \\ r = & 56{\cdot}83636, & \epsilon = 458{\cdot}09798, & \\ b = & 77{\cdot}65624, & & \\ m_1 = & 46{\cdot}11299, & m_2 = 8{\cdot}72339, & \\ a_1 = & 65{\cdot}30266, & a_2 = 12{\cdot}35358, & \\ y_0 = & 421{\cdot}22. & & \end{array}$$

The equation to the curve is

$$y = 421{\cdot}22\left(1 + \frac{x}{65{\cdot}3027}\right)^{46{\cdot}1130}\left(1 - \frac{x}{12{\cdot}3536}\right)^{8{\cdot}7234}.$$

In finding the ratio of the probable error of the mode to the probable error of the mean the interpolation was made from Table II, using the Mid-Panel Central Difference Formula in *Tracts for Computers*, No. III, p. 7. Let the interpolate divide the square formed by the four nearest interpolants in the x-ratio θ ($\phi = 1 - \theta$), and the y-ratio χ ($\psi = 1 - \chi$), then

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\ & - \tfrac{1}{6}\chi\psi\{(1+\psi)(\phi\delta^{2'} z_{0,0} + \theta\delta^{2'} z_{1,0}) + (1+\chi)(\phi\delta^{2'} z_{0,1} + \theta\delta^{2'} z_{1,1})\} \quad \ldots(36). \end{aligned}$$

	3·1	*3·2*	*3·3*	*3·4*
·1		1·5578 $z_{-1,0}$	1·5337 $z_{-1,1}$	
·2	1·7093 $z_{0,-1}$	1·6455 $z_{0,0}$	1·5977 $z_{0,1}$	1·5625 $z_{0,2}$
·3	1·9340 $z_{1,-1}$	1·8238 $z_{1,0}$	1·7417 $z_{1,1}$	1·6806 $z_{1,2}$
·4		2·1187 $z_{2,0}$	1·9797 $z_{2,1}$	

$$\begin{array}{ll} \theta = {\cdot}039448, & \phi = {\cdot}960552, \\ \chi = {\cdot}005312, & \psi = {\cdot}994688. \\ \delta^2 z_{0,0} = {\cdot}0906, & \delta^{2'} z_{0,0} = {\cdot}0160, \\ \delta^2 z_{1,0} = {\cdot}1166, & \delta^{2'} z_{1,0} = {\cdot}0281, \\ \delta^2 z_{0,1} = {\cdot}0800, & \delta^{2'} z_{0,1} = {\cdot}0126, \\ \delta^2 z_{1,1} = {\cdot}0940, & \delta^{2'} z_{1,1} = {\cdot}0210. \end{array}$$

Whence from equation above $z_{\theta, \chi} = 1{\cdot}6503$.

$$\text{Probable error of mean} = {\cdot}00410,$$
$$\text{,, \quad ,, \quad mode} = {\cdot}00675.$$
$$\text{Mean} = 29{\cdot}8570 \pm {\cdot}0041,$$
$$\text{Mode} = 29{\cdot}9502 \pm {\cdot}0068.$$

In the accompanying Diagram III the ranges of *likely* variation of mean and mode are represented by belts of $2\frac{1}{2}$ times the probable error plotted on either side of the mean and mode.

BAROMETRIC HEIGHT AT LAUDALE

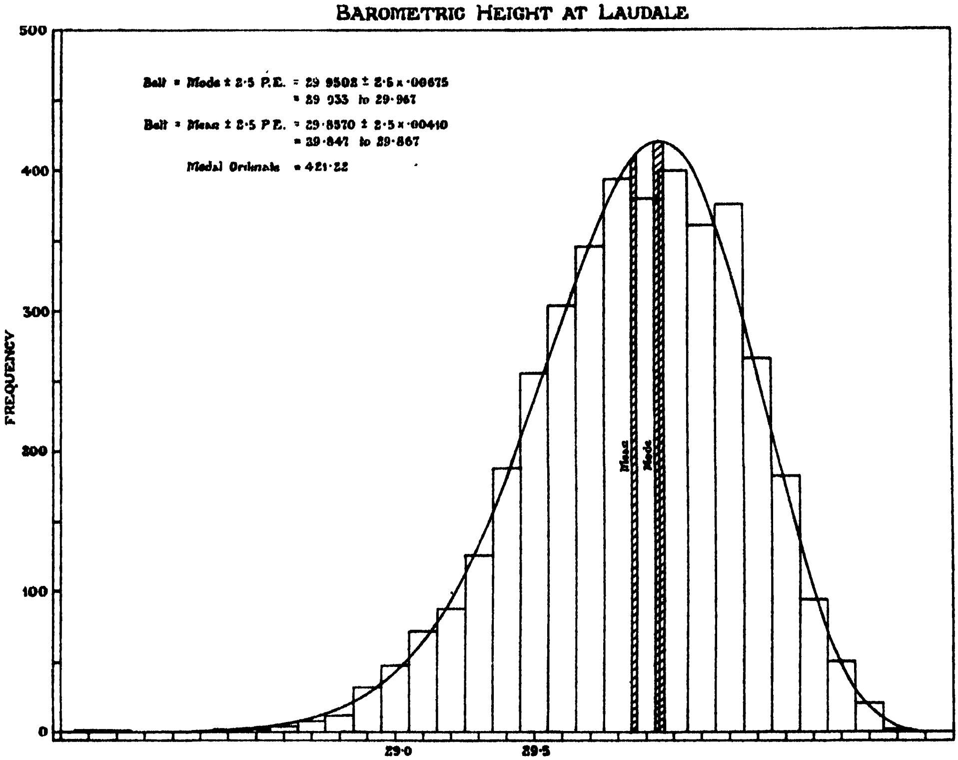


Diagram III.

Example II. Glands of the Fore-Legs of Swine. Right Leg ♀.

In the *Proceedings of the American Academy of Arts and Science*, Vol. XXXII. p. 87, 1896, C. B. Davenport and C. Bullard treated the variation in number of the Müllerian glands in the fore-legs of 4000 swine, and Professor Karl Pearson discussed their material in a memoir in the *Phil. Trans.* Vol. 191 A, pp. 289—296.

The following table gives the distribution for 2000 sows:

Number of Glands	0	1	2	3	4	5	6	7	8	9	10
Frequency per mille	7·5	104·5	182·5	241	207	138·5	67	36	11	4	1

In computing the moments no correction was made for grouping because this is a case of frequencies concentrated at mid-ordinates. We have

$$\text{Mean} = 3{\cdot}50100, \qquad \beta_1 = {\cdot}2591775,$$
$$\mu_2 = 2{\cdot}824999, \qquad \beta_2 = 3{\cdot}110825,$$
$$\mu_3 = 2{\cdot}417275, \qquad \kappa = -{\cdot}37311.$$
$$\mu_4 = 24{\cdot}82631,$$

The frequency distribution is of Type I:

$$r = 19{\cdot}98608, \qquad \epsilon = 72{\cdot}72580,$$
$$m_1 = 3{\cdot}78391, \qquad m_2 = 14{\cdot}20217,$$
$$a_1 = 3{\cdot}79632, \qquad a_2 = 14{\cdot}24873,$$
$$b = 18{\cdot}04505, \qquad y_0 = 237{\cdot}2621,$$
$$d = {\cdot}522985,$$
$$\text{Mode} = 2{\cdot}97802.$$

The equation to the curve is

$$y = 237{\cdot}262\left(1 + \frac{x}{3{\cdot}7963}\right)^{3{\cdot}7839}\left(1 - \frac{x}{14{\cdot}2487}\right)^{14{\cdot}2022}.$$

By an interpolation from my Table II the ratio of the probable errors was found to be 1·8155.

$$\text{Probable error of mean} = {\cdot}67449\sqrt{\frac{2{\cdot}824999}{2000}}$$
$$= {\cdot}02535,$$
$$\text{Probable error of mode} = {\cdot}02535 \times 1{\cdot}8155$$
$$= {\cdot}04602.$$

Hence

$$\text{Mean} = 3{\cdot}5010 \pm {\cdot}0253,$$
$$\text{Mode} = 2{\cdot}9780 \pm {\cdot}0460.$$

The result is indicated in Diagram IV. The modal and mean belts or zones are clearly separated and the modal belt sufficiently narrow to be of value.

Example III. Professor Weldon's Crab Measurements No. 4 *.

We will now turn to a nearly symmetrical curve given by the following series of 999 observations:

Index	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency	1	3	5	11	40	55	98	121	152	147
Index	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Frequency	126	82	72	41	28	8	7	0	0	2

* *Phil. Trans.* Vol. 185 A (1894), p. 96.

Moments were computed applying Sheppard's Adjustments. I found

$$\mu_2 = 7{\cdot}425852, \qquad \beta_1 = {\cdot}029370,$$
$$\mu_3 = 3{\cdot}467966, \qquad \beta_2 = 3{\cdot}137155,$$
$$\mu_4 = 172{\cdot}9930, \qquad \kappa = {\cdot}119199.$$

Type IV curve fits best.

$$r = 67{\cdot}92055, \qquad m = 34{\cdot}96028,$$
$$\nu = -24{\cdot}98612, \qquad a = 22{\cdot}15894,$$
$$d = {\cdot}2201479, \qquad \text{Origin} = 1{\cdot}533014.$$
$$\text{Mean} = 9{\cdot}684685, \qquad \text{Mode} = 9{\cdot}464537.$$
$$y_0 = 1{\cdot}76447, \qquad \sigma = 2{\cdot}72504.$$

MÜLLERIAN GLANDS IN THE FORE LEGS OF SWINE (DAVENPORT & BULLARD)

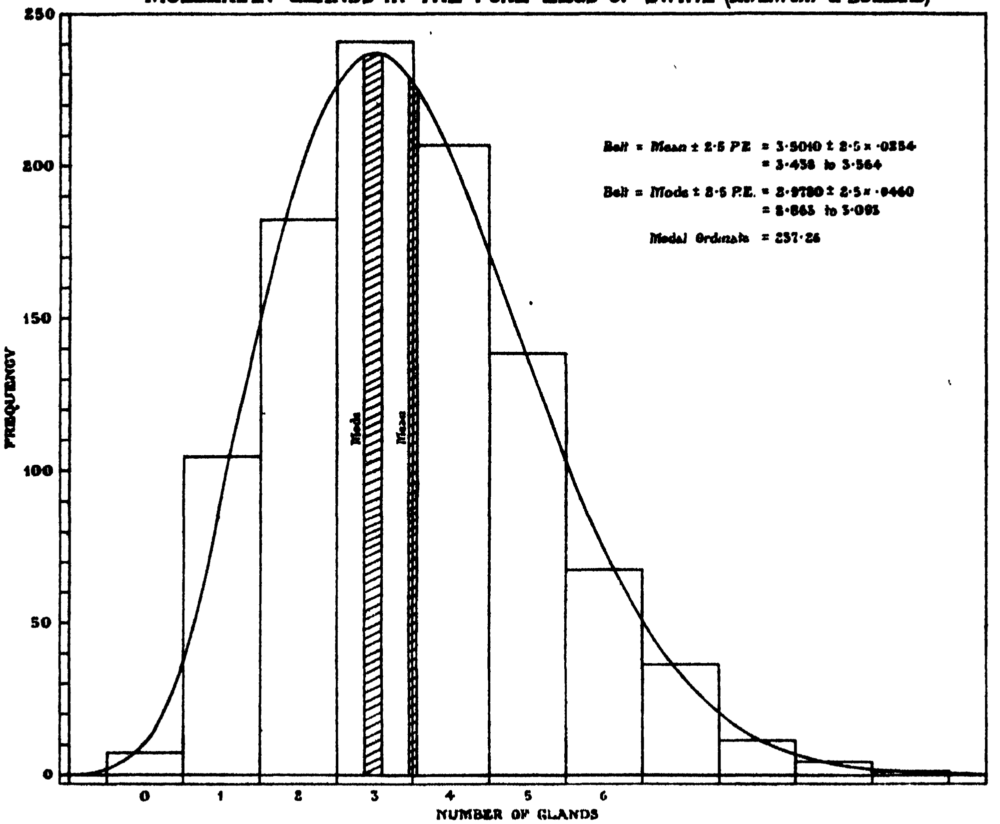


Diagram IV.

The ratio of probable errors was calculated to be 1·5623.

$$\text{Probable error of mean} = {\cdot}0582,$$
$$\text{,,} \quad \text{,,} \quad \text{mode} = {\cdot}0909.$$
$$\text{Mean} = 9{\cdot}6847 \pm {\cdot}0582,$$
$$\text{Mode} = 9{\cdot}4645 \pm {\cdot}0909.$$

The equation to the curve is

$$y = 1{\cdot}76447 \left(1 + \frac{x^2}{(22{\cdot}1589)^2}\right)^{-24{\cdot}9603} e^{24{\cdot}9661 \tan^{-1} \frac{x}{22{\cdot}1589}}.$$

This curve was selected because it is very nearly symmetrical, and illustrates a case in which the "modal belt" (see Diagram V) actually overlaps a portion of the "mean belt," and thus we cannot, without further investigation, assert that mean and mode are not identical.

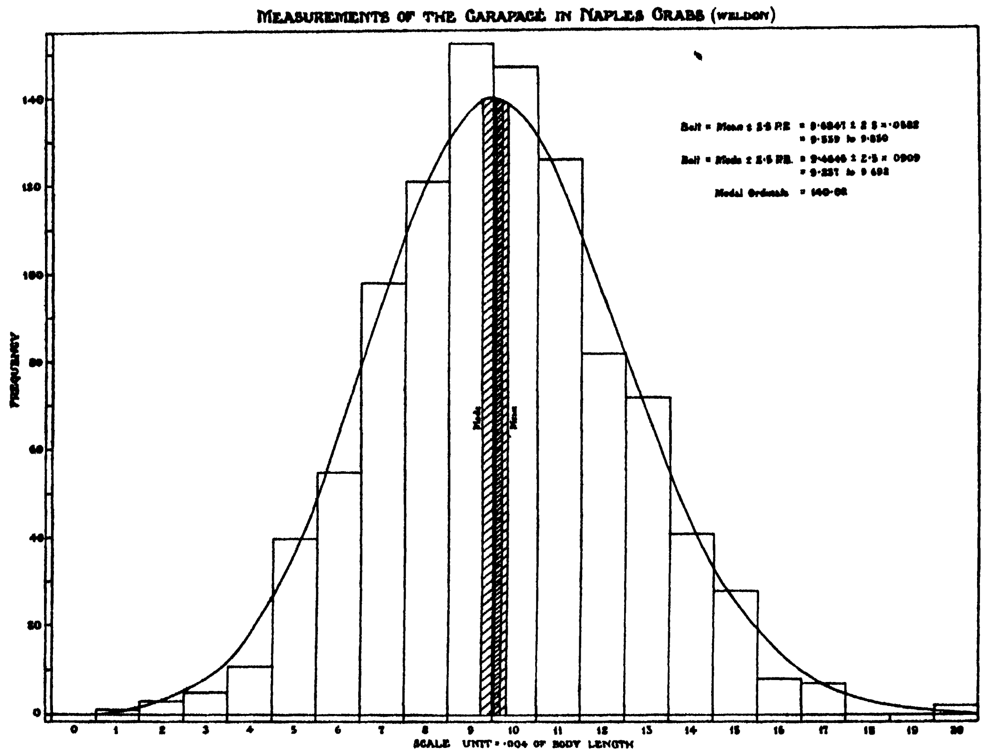


Diagram V.

Example IV. Distribution of 8689 *cases of Enteric Fever received into the Metropolitan Asylums Board Fever Hospitals,* 1871—93.

Age	Number of cases	Age	Number of cases
Under *5*	266	*35—40*	299
5—10	1143	*40—45*	163
10—15	2019	*45—50*	98
15—20	1955	*50—55*	40
20—25	1319	*55—60*	14
25—30	857	Above *60*	13
30—35	503		

I adopted Professor Karl Pearson's redistribution of the tail* dividing the group "Above *60*" as follows:

60—65, 8; *65—70*, 4; *70—75*, 1.

The above frequency distribution has high contact at the old age end but not at the childhood end, thus abruptness corrections are necessary for the calculation of the moments, and for that purpose it becomes desirable to know the sub-divided distribution of the group "Under *5*" into, say, 5 yearly groups.

No data being available for the sub-division for the years 1871—93, the age incidence in the first five years of life provided by the Reports of the Medical Officer of Health for the County of London, for the years 1909—14, which provided 253 cases, were taken, and formed suitable material for adjusting the 266 cases in the above data. They run:

Age ...	*0—1*	*1—2*	*2—3*	*3—4*	*4—5*
Number	14	21	48	78	92

Sub-dividing the group of 266 under 5 proportionally to the above we get:

Age ...	*0—1*	*1—2*	*2—3*	*3—4*	*4—5*
Number	15	22	50	82	97

INCIDENCE OF ENTERIC FEVER AT VARIOUS AGES FROM 1871 – 1893

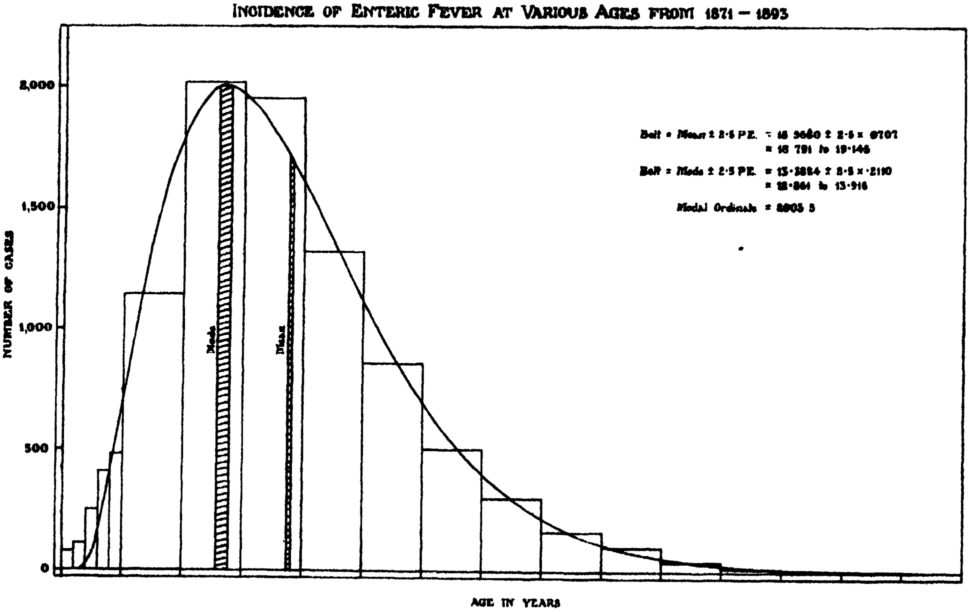


Diagram VI.

The Moments were found, using abruptness corrections† and taking five years as the unit,

$$\mu_2 = 3{\cdot}821776, \qquad \beta_1 = 1{\cdot}033412,$$
$$\mu_3 = 7{\cdot}595118, \qquad \beta_2 = 4{\cdot}348825,$$
$$\mu_4 = 63{\cdot}51882, \qquad \kappa = -2{\cdot}424407.$$

* See *Phil. Trans.* Vol. 186 A (1895), pp. 390—392.

† *Biometrika*, Vol. XII. pp. 231—258.

Type I must be used.

$$r = 34{\cdot}50820, \qquad \epsilon = 108{\cdot}69359,$$
$$b = 43{\cdot}11413, \qquad \sigma = 1{\cdot}954936,$$
$$m_1 = 1{\cdot}736247, \qquad m_2 = 30{\cdot}77195,$$
$$a_1 = 2{\cdot}302704, \qquad a_2 = 40{\cdot}81143,$$
$$d = 1{\cdot}115930, \qquad y_0 = 2003{\cdot}526.$$

Mean = 18·9680 years, Mode = 13·3884 years.

The equation to the curve is

$$y = 2003{\cdot}53\left(1 + \frac{x}{2{\cdot}3027}\right)^{1{\cdot}7362}\left(1 - \frac{x}{40{\cdot}8114}\right)^{30{\cdot}7720}.$$

The ratio of probable errors was found to be 2·9825.

Probable error of mean = ·014146 in working units,
= ·070728 years,

Probable error of mode = ·21095 years.

$$\text{Mean} = 18{\cdot}9680 \pm {\cdot}0707,$$
$$\text{Mode} = 13{\cdot}3884 \pm {\cdot}2109.$$

The accompanying Diagram VI gives the "likely" belts of variation. It will be seen that mode and mean are quite distinct, and if the modal belt be larger than the mean belt, it still represents quite valuable information as to the likelihood of modal variations.

Example V. Cloudiness at Breslau, 1876—85 (3653 *observations*)*.

We will now take an illustration which indicates how an "antimode" can be more precise than a mean. The following data provide a U-curve.

Degrees of Cloudiness at Breslau.

Degree	0	1	2	3	4	5	6	7	8	9	10
Frequency (days)	751	179	107	69	46	9	21	71	194	117	2089

From the above data the following values result:

$$\mu_2 = 18{\cdot}29987, \qquad \beta_1 = {\cdot}6112252,$$
$$\mu_3 = -61{\cdot}20297, \qquad \beta_2 = 1{\cdot}741445,$$
$$\mu_4 = 583{\cdot}1839, \qquad \kappa = -{\cdot}1538507.$$

Type I is the suitable curve.

$$r = {\cdot}1795807, \qquad \epsilon = {\cdot}0069873,$$
$$b = 9{\cdot}981426, \qquad \sigma = 4{\cdot}277834,$$
$$m_1 = -{\cdot}8774224, \qquad m_2 = -{\cdot}9429969.$$

* *Proc. R. S. London*, Vol. XII. pp. 287—290. Data taken from Hugo Meyer's *Anleitung zur Bearbeitung meteorologischer Beobachtungen für die Klimatologie*, Berlin, 1891, S. 108.

Both m_1 and m_2 being negative, the curve is of U-shape.

$$a_1 = 4{\cdot}810939, \quad a_2 = 5{\cdot}170487,$$
$$d = 2{\cdot}002148, \quad y_0 = 50{\cdot}74810,$$
$$\text{Mean} = 6{\cdot}829181, \quad \text{Antimode} = 4{\cdot}827034.$$

Hence the equation to the curve is

$$y = 50{\cdot}7481\left(1 + \frac{x}{4{\cdot}8109}\right)^{-{\cdot}8774}\left(1 - \frac{x}{5{\cdot}1705}\right)^{-{\cdot}9430}.$$

CLOUDINESS AT BRESLAU, 3,653 DAYS

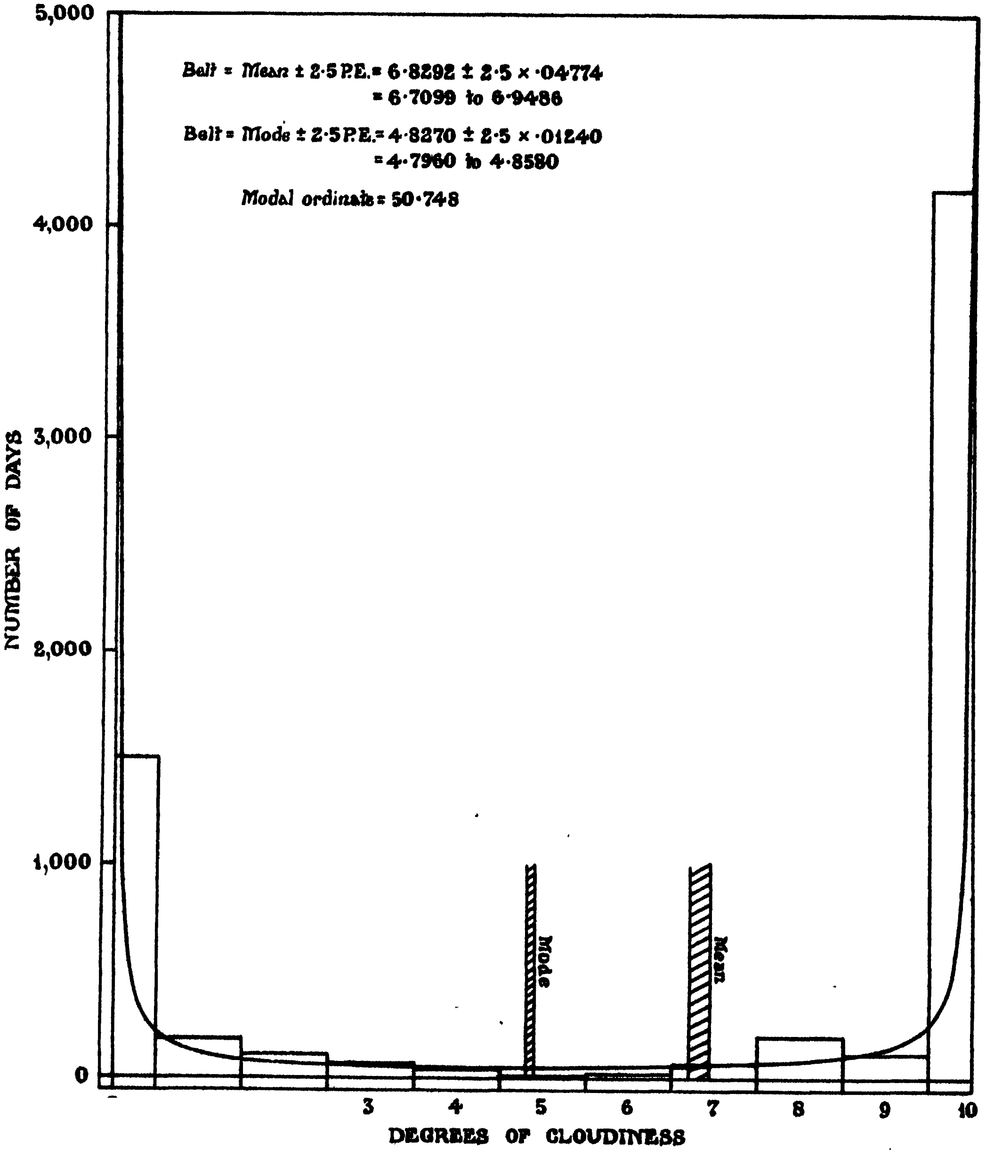


Diagram VII.

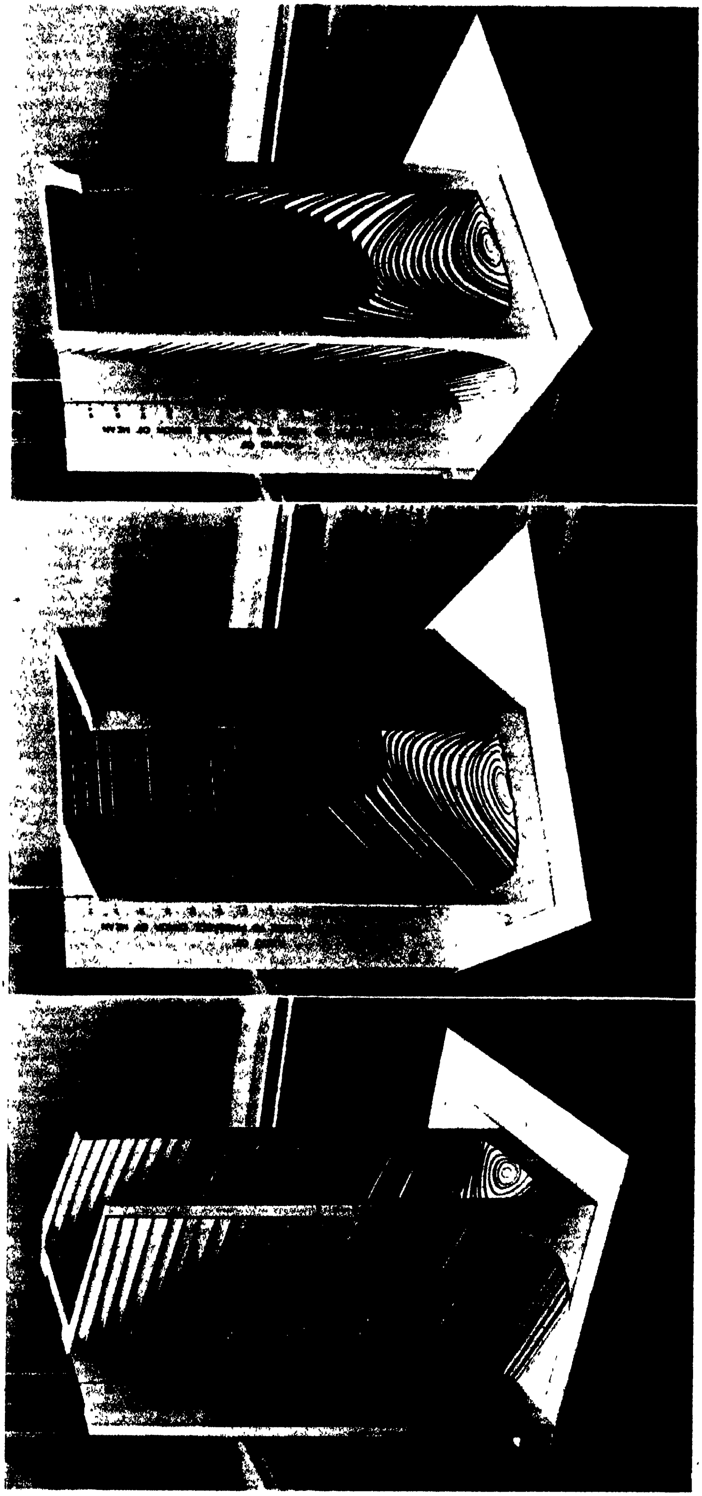
The ratio of the probable errors was calculated to be ·25974.

$$\begin{aligned} \text{Probable error of mean} &= \cdot 04774, \\ \text{,, \quad ,, \quad antimode} &= \cdot 01240. \\ \text{Mean} &= 6{\cdot}8292 \pm \cdot 0477, \\ \text{Antimode} &= 4{\cdot}8270 \pm \cdot 0124. \end{aligned}$$

It is clear that the standard deviation of the antimode is only about a quarter of that of the mean, and the ratio in this case corresponds to a value of z on the "scroll" part of the surface (Plate I, A), lying below the bottom of the "basin."

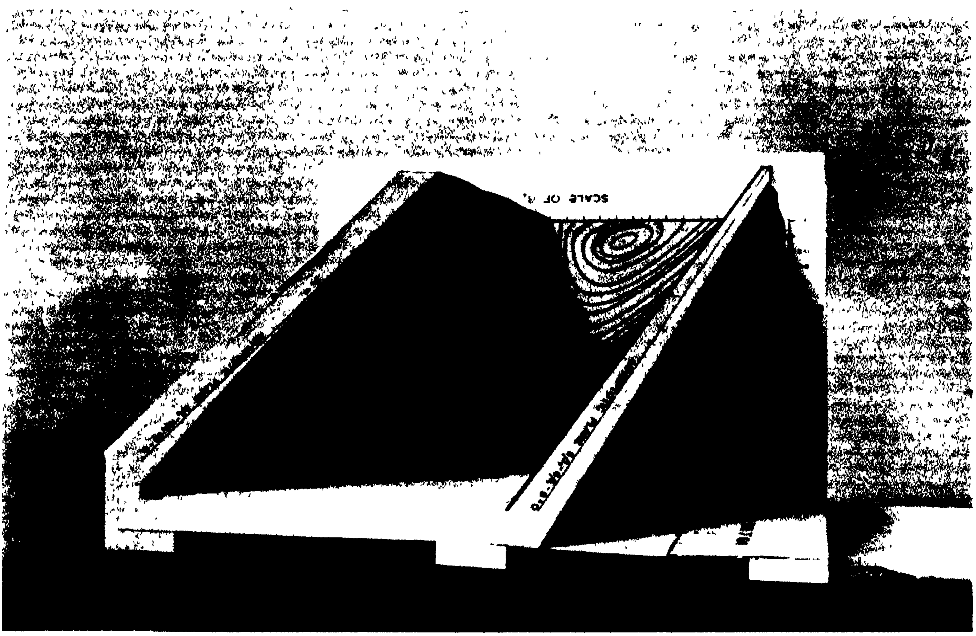
The above give illustrations of the error that is likely to occur in the position of the mode as determined from the four-moment formula. On the corresponding diagrams belts are indicated with ranges of 5 times the probable error round the mean and the mode. It will be seen that while the belt about the mean is very much smaller in most cases than the belt about the mode, yet the latter is sufficiently narrow to show that the mode, as determined by the four-moment formula, has practical significance. In every case such displacements as are suggested as probable by the modal belt are not such as would upset any reasoning which is likely to be based in practice on the observed position of the mode.

C B A

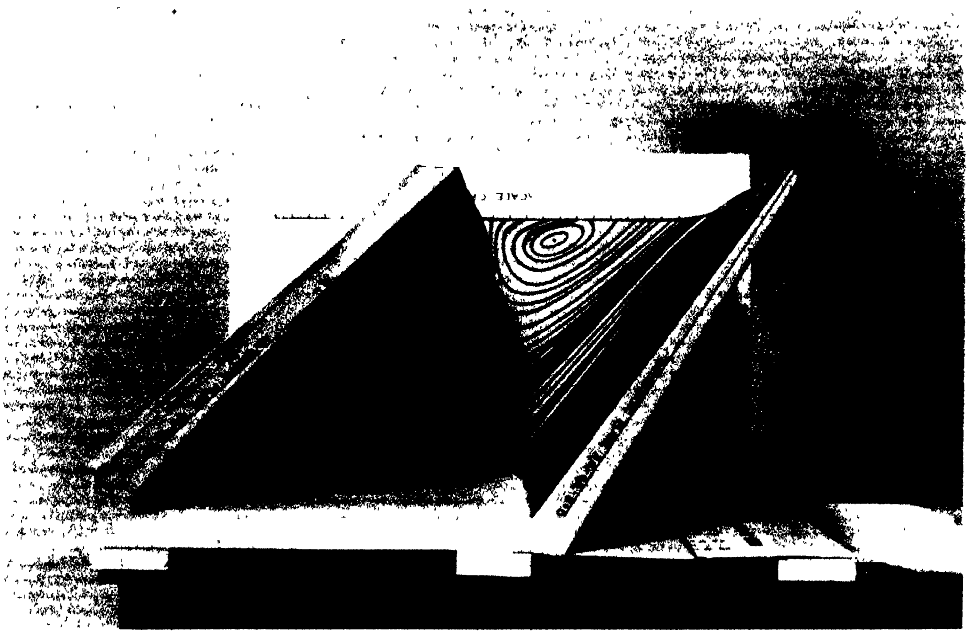


Photographs of the surface which gives the Probable Error of the mode, from three aspects. **B** and **C** show the "basin." **A** shows the portion of the surface lying outside the asymptotic plane through the "rectangular line."

D



E



D and **E**. Photographs of the vertical aspect of the surface, showing the "basin" from above and the two asymptotic planes.

UEBER DIE ANWENDUNG DER DIFFERENZENMETHODE ("VARIATE DIFFERENCE METHOD") BEI REIHENAUSGLEICHUNGEN, STABILITÄTSUNTERSUCHUNGEN UND KORRELATIONSMESSUNGEN.

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(1) BEIM Studium statistischer Reihen und, insbesondere, bei Korrelationsmessungen, kommt es nicht selten vor, daß man sich veranlaßt sieht, die betreffenden Reihen in einzelne Komponenten zu zerlegen. So unterscheidet man bei Reihen, welche die Evolution der Ernteerträge einer gewissen Landschaft darstellen,—eine "säkulare," "evolutorische," oder überhaupt "glatte" Komponente, die den Einfluß des Fortschrittes landwirtschaftlicher Technik wiedergibt, und eine "unregelmäßige," "oszillierende," "zufällige" Komponente, die unter dem Einfluß des Klimas oder der Witterung eines gewissen Jahres steht. Bei "Preiskurven" unterscheidet man oftmals eine "säkulare" und eine wellenartige "Konjunktur-" Komponente, ferner "Saison"-Komponente und eine unregelmäßige "restliche" Komponente, hervorgerufen durch Zufälligkeiten des Börsenspiels, Paniken, Kriege, u.s.w. Es ist eine Anzahl verschiedener "Zerlegungsmethoden" für statistische Reihen vorgeschlagen und mit größerem oder geringerem Erfolge erprobt worden. Als eine derselben kann auch die Differenzenmethode (Variate-Difference-Method) angesehen werden. Das, was letztere, äußerlich, von den übrigen Verfahren unterscheidet, ist der Umstand, daß sie nur bestimmte lineare Funktionen der gesuchten "zufälligen" Reihenkomponente ergeben kann, während erstere die Anmaßung haben, die Reihe in ihre "wahren" Komponenten mehr oder minder genau zu zerlegen. Ob mit Recht—soll im weiteren untersucht werden.

(2) Die bei Zerlegung statistischer Reihen gewöhnlich benutzten Methoden laufen letzten Endes darauf hinaus, das eine oder andere hauptsächlich für Sterblichkeitsmessungen ausgearbeitete "Tafel-Ausgleichungsverfahren" anzuwenden. Ein Charakteristikum dieser Verfahren besteht darin, daß sie immer die "glatte" (wellenartige oder evolutorische) Komponente zu erfassen suchen, das heißt, nur deren "wahrscheinlichsten" oder "vorteilhaftesten" Wert im Bereiche gewisser Fehlergrenzen bestimmen. Die "restliche" Komponente wird dann durch Subtraktion der "ausgeglichenen" von den ursprünglichen Werten der Reihe gewonnen.

Drei Methoden kommen in erster Linie bei solchen "Ausgleichungen" in Betracht*. *Erstens*, kann man eine hypothetisch angenommene oder (in seltenen

* Vergl. E. Czuber: *Wahrscheinlichkeitsrechnung*, 2 Aufl., 1910, II. Band, S. 171.

Fällen) à priori gegebene Funktion der Beobachtungsreihe anzupassen suchen. Gewöhnlich nimmt man an, daß die "glatte" Komponente entweder durch eine ganze rationale Funktion (eine Gerade oder eine Parabel mehr oder minder hoher Ordnung) darstellbar ist, oder aber auch durch eine mehr oder weniger komplizierte trigonometrische Funktion (Fourier Analyse in ihren verschiedenen Formen*). Wird dabei die Zahl der Parameter dieser Funktionen geringer, als diejenige der Reihenglieder, genommen, so müssen die "vorteilhaftesten" Werte der Parameter gefunden werden. Gewöhnlich geschieht das durch Anwendung der Methode der kleinsten Quadrate oder eines ihrer Derivate: Methode der Momente, Methode der Flächen (Cantelli), u.s.w.

Zweitens, kann man "mechanische Ausgleichungsmethoden" gebrauchen. Diese "gehen von der Erwägung aus, daß die *Störung* eines Einzelwertes (das heißt in unserem Falle: der "glatten Komponente") in einer Beobachtungsreihe sich auch noch in den beiderseits benachbarten Gliedern bemerkbar mache, daß jedoch eine sehr geringe Wahrscheinlichkeit dafür bestehe, es werde eine längere Folge von Einzelwerten in gleichem Sinne beeinflußt sein; viel eher sei zu erwarten, daß sich durch den Wechsel des Sinnes auch in kürzeren oder längeren Abschnitten ein Ausgleich im Verlaufe der Erscheinung vollziehe†." Von Wittstein und Woolhouse bis W. F. Sheppard und E. C. Rhodes ist hier eine Anzahl verschiedener Verfahren vorgeschlagen worden, von denen einige die auszugleichende Reihe durch eine Anzahl von ineinandergreifenden Parabeln darzustellen versuchten; in vielen Fällen wurde auch zur Methode der kleinsten Quadrate gegriffen.

Drittens, werden vielfach graphische Ausgleichungsmethoden geübt, wobei "die zusammengehörigen Wertepaare durch Punkte in einem Koordinatensystem dargestellt und nun eine Kurve verzeichnet (wird), die dem Zuge der Punkte möglichst getreu folgt und ihnen 'möglichst nahe' liegt‡." Da ein solches Verfahren, obwohl es in geschickten Händen zu sehr befriedigenden Resultaten führen kann, schließlich nur auf Vertrauen zur Persönlichkeit des Forschers beruht und auch eine wirkliche Bestimmung der möglichen Fehlergrenzen der gefundenen Resultate kaum zuläßt, so werden wir es im weiteren unberücksichtigt lassen.

(3) Wie oben schon angedeutet wurde, beruhen die gebräuchlichen Zerlegungsmethoden statistischer Reihen auf einer *erweiterten Auslegung* der Ausgleichungsverfahren: es wird nämlich stillschweigend angenommen, daß nicht nur die "glatte" Komponente (oder die "glatt*en*" Komponent*en*) einer Reihe durch die gefundenen "ausgeglichenen" Werte ungefähr darstellbar seien, sondern daß auch die Annäherungswerte der "restlichen" Komponente durch eine gewöhnliche Subtraktion der ausgeglichenen von den ursprünglichen Werten gewonnen werden können. Hierbei wird das relative Größenverhältniss der "glatten" und "restlichen" Elemente zu einander nicht besonders berücksichtigt, obgleich die letzteren gewöhnlich als die beträchtlich kleineren gedacht werden. Unseres Wissens, ist bis jetzt die

* Natürlich, kann man trigonometrische Funktionen auch mit ganzen rationalen kombinieren.

† Czuber, *loc. cit.*, S. 185.

‡ Ebenda, S. 197.

logische Berechtigung eines solchen Verfahrens von niemanden besonders beanstandet worden, obgleich seine Korrektheit, wie wir gleich sehen werden, stark bezweifelt werden könnte.

Es sei, um die der Ausgleichungstheorie eigentümliche Ausdrucksweise zu gebrauchen, je eine Bestimmung der N Größen $G_1, G_2, G_3, \dots G_N$ gemacht worden, wobei die Werte $u_1, u_2, u_3, \dots u_N$ gefunden wurden. Die *wahren Fehler* ϵ dieser Bestimmungen sind durch das System (1) gegeben:

$$\left.\begin{array}{l} \epsilon_1 = -u_1 + G_1 \\ \epsilon_2 = -u_2 + G_2 \\ \dots\dots\dots\dots \\ \epsilon_N = -u_N + G_N \end{array}\right\} \quad \dots\dots\dots\dots (1).$$

G soll hier die "glatte" und $(-\epsilon)$ die *wahre* "restliche" Komponente bedeuten. Auf eine, fürs erste nicht genauer zu definierende, Weise sind nun die "angenäherten ausgeglichenen" Werte von $G_1, G_2, \dots G_N$ bestimmt worden. Es seien das die Größen $g_1, g_2, \dots g_N$. Dann sind die *scheinbaren Fehler* λ dieser Größen durch folgendes System ausgedrückt:

$$\left.\begin{array}{l} \lambda_1 = -u_1 + g_1 \\ \lambda_2 = -u_2 + g_2 \\ \dots\dots\dots\dots \\ \lambda_N = -u_N + g_N \end{array}\right\} \quad \dots\dots\dots\dots (2).$$

Eliminiert man u aus beiden Gleichungssystemen, so erhält man unmittelbar:

$$\left.\begin{array}{l} \epsilon_1 = \lambda_1 + G_1 - g_1 \\ \epsilon_2 = \lambda_2 + G_2 - g_2 \\ \dots\dots\dots\dots \\ \epsilon_N = \lambda_N + G_N - g_N \end{array}\right\} \quad \dots\dots\dots\dots (3).$$

Damit also die scheinbaren Fehler den uns einzig interessierenden wahren Fehlern gleich seien, wie es gewöhnlich angenommen wird, müssen alle $G_i - g_i$ von $i = 1$ bis $i = N$ verschwinden.

Nun sei aber folgendes festgestellt. Von seltenen Ausnahmen abgesehen können die "ausgeglichenen" Werte $g_1, g_2, \dots g_N$ nur aus der Reihe $u_1, u_2, \dots u_N$ berechnet werden, und zwar bestimmen die meisten Ausgleichungsmethoden, mit gutem Grund, den "ausgeglichenen" Wert g_i als eine *lineare Funktion* F entweder *aller* (klassische Form der Methode der kleinsten Quadrate) oder nur *einiger* (mechanische Ausgleichung) Werte der beobachteten Reihe u. Nehmen wir an, k sei die Anzahl der für die Bestimmung eines jeden g_i benutzten Glieder der Reihe u, so können wir auch abgekürzt schreiben: $g_i = F(u_{i,k})$. Hierbei kann k, je nach der benutzten Methode, eine beliebige positive ganze Zahl zwischen 1 und N bedeuten. Ist nun $F(u_{i,k})$ linear, so hat man mit Berücksichtigung von System (1):

$$g_i = F(u_{i,k}) = F[(G_i - \epsilon_i), k] = F(G_{i,k}) - F(\epsilon_{i,k}).$$

Und System (3) verwandelt sich dann in:

$$\left.\begin{array}{l} \lambda_1 = \epsilon_1 - F(\epsilon_{1,k}) - [G_1 - F(G_{1,k})] \\ \lambda_2 = \epsilon_2 - F(\epsilon_{2,k}) - [G_2 - F(G_{2,k})] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_N = \epsilon_N - F(\epsilon_{N,k}) - [G_N - F(G_{N,k})] \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots (4).$$

Damit nun ein beliebiges λ_i dem *wahren Fehler* ϵ_i gleich sei, muß zwangsläufig bei beliebigem i die Bedingung

$$F(\epsilon_{i,k}) = F(G_{i,k}) - G_i$$

erfüllt sein. Dieselbe führt aber entweder zu der sinnwidrigen Annahme, daß eine lineare Funktion der offenbar als mehr oder weniger "zufällig" angenommenen wahren Fehler ϵ durch eine fast identische lineare Funktion der wahren "glatten" Größen G auszudrücken sei; oder aber muß man zugeben, daß dieselbe Funktion F zu gleicher Zeit für die wahren Fehler genau 0 und für die Größen G_i ebenso genau dieselben G_i ergibt, was ebenfalls einen nicht weniger sinnwidrigen Zusammenhang zwischen G_i und ϵ_i ergeben würde. *Folglich, können die Gleichungen des Systems* (4) *allenfalls nur im Durchschnitt, nur in mathematischer Erwartung annehmbar sein.* Ist die Ausgleichungsformel richtig deduziert und korrekt angewandt, so kann man noch zugeben, daß allenfalls, *bei größerem* ϵ_i, die Differenz $F(G_{i,k}) - G_i$ im Vergleich zu diesem ϵ_i so klein sei, daß sie praktisch vernachlässigt werden könnte, für die Funktion $F(\epsilon_{i,k})$ ist aber auch das nicht gegeben. Meines Erachtens, ergeben daher die gebräuchlichen Ausgleichungsmethoden *im besten Falle bloß* $\epsilon_1 - F(\epsilon_{1,k})$, $\epsilon_2 - F(\epsilon_{2,k})$, … $\epsilon_N - F(\epsilon_{N,k})$, *keinesfalls aber die gesuchte Reihe* $\epsilon_1, \epsilon_2, \ldots \epsilon_N$*. Und kann auch, wie es nicht selten möglich ist, bewiesen werden, daß die mathematische Erwartung von $\epsilon_i - F(\epsilon_{i,k})$ gleich der mathematischen Erwartung von ϵ_i sei, so folgt daraus noch durchaus nicht, daß die mathematischen Erwartungen der Quadrate dieser Größen ebenfalls einander wenn auch nur ungefähr gleich seien. Und kann sogar letzteres bewiesen werden, so ergibt sich hieraus noch keinesfalls, daß die mathematische Erwartung des Produktes $[\epsilon_i - F(\epsilon_{i,k})][\epsilon_j - F(\epsilon_{j,k})]$ auch nur angenähert der mathematischen Erwartung des Produktes $\epsilon_i \epsilon_j$ gleich kommen muß.

Wir gelangen also zu folgendem vielleicht etwas sensationell anmutendem aber durchaus logisch zwingendem Schluß: *insofern die üblichen Zerlegungsmethoden statistischer Reihen Funktionen (zum Beispiel, Mittlere Fehler, Momente, Korrelationskoeffizienten, u.s.w.) der berechneten "restlichen Komponenten" (also scheinbarer Fehler)* $\lambda_1, \lambda_2, \ldots \lambda_N$ *direkt für entsprechende Funktionen der wahren "restlichen Komponenten" (wahren Fehler)* $\epsilon_1, \epsilon_2, \ldots \epsilon_N$ *ansahen, sind diese Methoden nicht korrekt und können zu Trugschlüssen führen.* Alle mit ihrer Benutzung

* Die Trennung der Funktion $G_i - F(G_{i,k})$ von der Funktion $\epsilon_i - F(\epsilon_{i,k})$ kann, im Allgemeinen, nur bei linearen und einigen wenigen nicht linearen Formen von F gelingen. Das von Prof. W. M. Persons in seinem "Correlation of Time Series" (*Journ. of the Amer. Stat. Assoc.*, June 1923, p. 713 und ff.) vorgeschlagene Ausgleichungsverfahren ist weder linear, noch liefert es eine für Korrektionsberechnungen handliche Form von $F(\epsilon_{i,k})$. Solange die Fehlergrenzen dieser Methode nicht bestimmt sind, dürfte sie daher nur als *graphischen Ausgleichungsmethoden analog* zu betrachten sein.

vorgenommenen Untersuchungen sind zu revidieren und an den berechneten Koeffizienten Korrektionen anzubringen, die im Wesentlichen durch die entsprechende Form von $F(\epsilon_{i,k})$ bestimmt sind. Inwiefern diese Korrektionen die früher gefundenen Resultate umändern werden, hängt sowohl von der Formel des gebrauchten Ausgleichungsverfahrens, als auch, besonders, von den benutzten Funktionen von λ ab. In einer Reihe von Fällen dürften die früheren Forschungsergebnisse im Wesentlichen bestehen bleiben, in andern ist es dagegen zu befürchten, daß sie eine beträchtliche Modifikation erleiden werden. Besonders bezieht sich letzteres auf feinere Untersuchungen, welche ja nur dann logischen Sinn haben können, wenn die zufälligen Fehlergrenzen der gefundenen Werte im Verhältnis zu diesen klein sind.

(4) Im Rahmen dieser Veröffentlichung vermögen wir, natürlich, nicht, die verschiedenen Ausgleichungsmethoden von unserem Standpunkte aus zu sichten und deren Korrektionen zu bestimmen. Die jetzt folgenden zwei Beispiele dienen daher nur zum besseren Verständnis der bisherigen Ausführungen.

Auf S. 295—296 von E. T. Whittaker's und G. Robinson's *The Calculus of Observations* (London, 1924) finden wir die Sheppard'schen Ausgleichungsformeln für Parabeln bis zur 5. Ordnung und für Anzahl der bei der Ausgleichung benutzten Glieder von 3 bis 21 (das heißt, bis $n=10$). Nehmen wir an: Ausgleichungsparabel 2ter Ordnung, $n=2$. Dann ist, in den Bezeichnungen des zitierten Buches, (u_0-u_0'), die Differenz zwischen der beobachteten Größe u_0 und deren nach der Sheppard'schen Methode ausgeglichenem Werte u_0', durch folgenden Ausdruck gegeben:

$$\begin{aligned} u_0-u_0' &= u_0-\tfrac{1}{35}\{17u_0+12(u_1+u_{-1})-3(u_2+u_{-2})\} \\ &= \tfrac{3}{35}\{+u_{-2}-4u_{-1}+6u_0-4u_1+u_2\}=\tfrac{3}{35}\,\delta^4 u_0, \end{aligned}$$

wenn das Symbol $\delta^4 u_0$ die vierte endliche *zentrale* Differenz der Größe u_0 bedeutet. Wird also angenommen, daß im Ausdruck für den scheinbaren Fehler $-\lambda_0=u_0-u_0'$ (s. oben, System (4)), kein "glattes" Element G_0 mehr vorhanden ist, so bedeutet das, daß die Differenz $F(G_{0,k})-G_0$ praktisch gleich Null zu nehmen ist und daß, folglich,

$$\lambda_0=\epsilon_0-F(\epsilon_{0,k})=-\tfrac{3}{35}\,\delta^4\epsilon_0.$$

Es erweist sich also hier, daß λ_0 nicht ϵ_0, sondern einem Bruchteil der negativgenommenen 4ten endlichen Differenz der wahren Fehler ϵ gleichzusetzen ist.

Ein anderes Beispiel. Es liege eine Beobachtungsreihe $u_1, u_2, u_3, \ldots u_N$, vor, wobei N ungerade ist. Ein jedes u_i bestehe aus einem "glatten" Element G_i und einem "restlichen" η_i, welches durch die Gleichung gegeben sei: $\eta_i=-\epsilon_i=u_i-G_i$. Es sei angenommen, daß die "glatte" Reihe $G_1, G_2, \ldots$ durch eine *Gerade*, deren Konstanten nach der Methode der kleinsten Quadrate bestimmt werden, *genau* auszudrücken sei. Letztere ergebe die Werte: $u_1', u_2', \ldots u_N'$.

Dann erhält man, nach den bekannten Regeln der Methode der kleinsten Quadrate, für u_i-u_i', das heißt für die *scheinbare* "restliche" Komponente, schließlich

folgenden Ausdruck:

$$u_i - u_i' = \eta_i - \left\{ \frac{\sum_{i=1}^{N} \eta_i}{N} + \frac{3(N-2i+1)}{(N-1)N(N+1)} \left[(N-1)(\eta_1 - \eta_N) + (N-3)(\eta_2 - \eta_{N-1}) + \ldots + 1 \cdot \left(\eta_{\frac{N-1}{2}} - \eta_{\frac{N+3}{2}} \right) \right] \right\}.$$

Kann die Reihe $\eta_1, \eta_2, \ldots \eta_N$ als eine Reihe von empirischen Größen, die eine zufällige Variable mit einem konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen annimmt, angesehen werden, so ist die mathematische Erwartung* von u_i' gleich der mathematischen Erwartung von u_i, die mathematische Erwartung von $(u_i - u_i')$ also gleich Null. Was aber die mathematische Erwartung von $\frac{\sum_{i=1}^{N} (u_i - u_i')^2}{N}$ anbetrifft, so ergibt sich für diese nicht der Wert μ_2, sondern $\frac{N-2}{N}\mu_2$, wenn μ_2 die mathematische Erwartung des Quadrates der Abweichung der Größe η_i von ihrer mathematischen Erwartung bedeutet. Kann ferner die Komponente $G_1, G_2, \ldots G_N$ nicht genau durch eine Gerade ausgedrückt werden oder sind die einzelnen η_i nicht vollkommen von einander unabhängig, so werden in der mathematischen Erwartung von $(u_i - u_i')^2$ alle Produkte $\eta_i\eta_j$ von $i = 1$ bis $i = N$ auftreten. Eventuell könnte dann die Korrektion des Ausdruckes $\frac{\sum_{i=1}^{N} (u_i - u_i')^2}{N}$ dieselbe Größenordnung, wie er selbst, erlangen. Geht man vollends von der Berechnung einer Geraden zur Ausgleichung nach Parabeln höherer Ordnung über, so wird, wenn diese Ordnung n ist, die mathematische Erwartung von $\frac{\sum_{i=1}^{N} (u_i - u_i')^2}{N}$ gleich $\frac{N-n-1}{N}\mu_2$*, und die Korrektionsformel für $(u_i - u_i')$ mit jeder höheren Ordnung noch komplizierter. Ihre geringe Handlichkeit und die weiten Grenzen,

* In der vorliegenden Arbeit bediene ich mich durchwegs der in England bis jetzt wenig populären "russischen" Methode der mathematischen Erwartungen. Einige Hinweise auf die Eigenschaften und Vorzüge dieser Methode findet der englische Leser im Artikel meines Lehrers Prof. Al. A. Tchouproff's, "On the Mathematical Expectation of the Moments of Frequency Distributions" (*Biometrika*, Vol. XII. November 1918, pp. 140—142), eine vollständigere Darstellung deren Lehrsätze aber im 2. Kapitel von Prof. L. Bortkiewicz's *Iterationen* (Prof. Dr L. v. Bortkiewicz, *Die Iterationen. Ein Beitrag zur Wahrscheinlichkeitstheorie*, Berlin, 1917, S. 30—69).

Die Beschränktheit des der vorliegenden Arbeit zugewiesenen Raumes, läßt es, selbstverständlich, nicht zu, den Rechenweg, auf welchem die meisten der unten angeführten Formeln abgeleitet wurden, auch nur anzudeuten. Dies dürfte aber geschehen, weil deren Bestimmung, mit den wenigsten Ausnahmen, eben kein Problem, sondern nur eine Rechenaufgabe darstellt, die, bei genügender Akkuratesse, von einem jeden erledigt werden kann, der über gewisse Kenntnisse in der Theorie der mathematischen Erwartungen sowie in der Kombinatorik verfügt.

† Dieser Satz ist eigentlich schon von Gauß in seiner *Theoria combinationis*, Art. 38, bewiesen worden. Vergl. Czuber: *Wahrscheinlichkeitsrechnung*, 3. Aufl., 1914, Bd. I. S. 347—349.

in denen sich deren Größe bewegen kann, erschweren die Anwendung der Methode der kleinsten Quadrate hier außerordentlich *.

(5) Fassen wir jetzt dasselbe Problem von einer anderen Seite an. Wir haben eben gesehen, daß zwischen der Differenzenmethode und den verschiedenen Reihen-Ausgleichungsmethoden insofern kein Unterschied besteht, als wie die eine so auch die andern die "restliche" Komponente nicht unmittelbar bestimmen, sondern nur deren gewisse lineare Funktionen ergeben können. Es liegt daher der Gedanke nahe, zu untersuchen, ob die Differenzenmethode nicht als ein regelrechtes Reihen-Ausgleichungsverfahren ausgebaut werden könne †.

Es sei wieder $u_1 = G_1 + \eta_1$, $u_2 = G_2 + \eta_2$, ... $u_N = G_N + \eta_N$, eine statistische Reihe, die aus einem "glatten" Element G und einer "restlichen" Komponente η besteht. Nehmen wir nun an, daß die Reihe $G_1, G_2, \ldots G_N$ sich genau durch eine Parabel $(2k-1)$-ter Ordnung darstellen läßt. Dann wird in der Reihe der $2k$-ten endlichen Differenzen unserer u-Reihe der letzte Rest der Komponente G verschwunden sein und, folglich, wenn wir δ^{2k} als Symbol für die $2k$-te endliche *zentrale* Differenz einführen, wird

$$\delta^{2k} u_i = \delta^{2k} G_i + \delta^{2k} \eta_i = \delta^{2k} \eta_i.$$

* Wohl der bedeutendste Schüler Prof. Al. A. Tchouproff's, Prof. N. S. Tchetverikoff, hat in einer Reihe von Aufsätzen (siehe die in Moskau erscheinenden russischen Zeitschriften "Fragen der Konjunktur," "Statistische Nachrichten," u.s.w.) die Stabilitätsverhältnisse der russischen Ernten und die Zusammenhänge zwischen Ernte und Kornpreis einer genauen Untersuchung unterworfen. Es dürfte nicht leicht sein, in der statistischen Litteratur anderer Länder gleichwertiges an Scharfsinn, Feinheit der angewandten Methoden und an Fleiß zu finden. Und dennoch befürchte ich sehr, daß die Vorliebe Tchetverikoff's für Parabelanpassung nach der Methode der kleinsten Quadrate ihn in einigen Fällen zu Trugschlüssen geführt hat und daß wenigstens einige von seinen "rätselhaften" Korrelationskoeffizienten eben nur "spurious" sind.

Höher in dieser Hinsicht steht die Arbeit einer anderen Schülerin Tchouproff's, des leider zu früh verstorbenen Fräuleins M. Winogradowa: "Der Alkoholkonsum in Rußland und die Ernte," Petrograd, 1916, erschienen in den *Proceedings* (*Students Section*) *of the Econom. Departm. of the Petrograd Polytechnic Institute* (russisch).

Frl. M. Winogradowa gebrauchte erste Differenzen.

† Die Grundidee der von "Student," K. Pearson, E. S. Pearson, E. M. Elderton, B. M. Cave, A. Ritchie-Scott, J. Henderson, mir und andern in der *Biometrika* seit 1914 entwickelten Differenzenmethode nehme ich hier als bekannt an. Prof. Truman L. Kelley (*Statistical Method*, New York, 1923, pp. 272, 273) formuliert sie, nach "Student" und K. Pearson, folgendermaßen: "Given two series, $x_1, x_2, \ldots x_n$ and $y_1, y_2, \ldots y_n$ between which there is an organic correlation, R, and a spurious correlation due to a time or location factor such that the two phenomena together result in an apparent, i.e. an obtained correlation, of r. The problem is to determine R. "Student" (1914) has shown that if

$$x_1 = X_1 + bt_1 + ct_1^2 + dt_1^3 + \text{etc.}$$
$$x_2 = X_2 + bt_2 + ct_2^2 + dt_2^3 + \text{etc.}$$
etc.

and if
$$y_1 = Y_1 + Bt_1 + Ct_1^2 + Dt_1^3 + \text{etc.}$$
$$y_2 = Y_2 + Bt_2 + Ct_2^2 + Dt_2^3 + \text{etc.}$$
etc.

in which X_1, X_2, etc., Y_1, Y_2, etc., are independent of time or location, then, if the parabolic equations in t terminate with some power t^n, the correlation r_{XY} is given by the correlation between Δ_n and δ_n, the two series of nth order differences, Δ_1 standing for the measures $(x_1 - x_2)$, $(x_2 - x_3) \ldots (x_{n-1} - x_n)$; Δ_2 for the measures $[(x_1 - x_2) - (x_2 - x_3)]$, $[(x_2 - x_3) - (x_3 - x_4)]$, ... $[(x_{n-2} - x_{n-1}) - (x_{n-1} - x_n)]$; and similarly Δ_3 for third order differences; Δ_4 for fourth order differences, etc.; the δ's having comparable meanings in the case of the y-series."

Nehmen wir jetzt an, unser Ziel bestehe darin, eine solche lineare Funktion $F(u_{i,2k+1})$ zu finden, daß genau $F(G_{i,2k+1}) = G_i$. Da nun

$$u_i - F(u_{i,2k+1}) = G_i - F(G_{i,2k+1}) + \eta_i - F(\eta_{i,2k+1}),$$

so führt obige Annahme zwangsläufig zur Gleichung:

$$u_i - F(u_{i,2k+1}) = \eta_i - F(\eta_{i,2k+1}),$$

wo im rechten Teil die Komponente G gänzlich fehlt.

Nun haben wir eben gesehen, daß unter den eingangs gemachten Bedingungen gerade die $2k$-ten endlichen Differenzen der Reihe u die von uns hier benötigten Eigenschaften besitzen: (1) lineare Funktionen von u zu sein und (2) keinen Rest der Komponente G mehr aufzuweisen. Das führt uns zum naheliegenden Gedanken, $u_i - F(u_{i,2k+1})$ gleich $Z\delta^{2k}u_i$ zu setzen, wo Z einen Multiplikator bedeutet, dessen "vorteilhaftester" Wert noch gefunden werden muß. Aus der gemachten Annahme folgt unmittelbar:

$$F(u_{i,2k+1}) = u_i - Z\delta^{2k}u_i.$$

Das wäre der gesuchte Ausdruck für F, den wir auch in folgender Form schreiben können:

$$F(u_{i,2k+1}) = G_i + \eta_i - Z\delta^{2k}\eta_i.$$

Es bleibt nur noch, den "vorteilhaftesten" Wert für Z zu bestimmen. In Anlehnung an das bei der Methode der kleinsten Quadrate übliche Verfahren, wollen wir Z der Bedingung: $E\sum\limits_{i=1}^{N}[\eta_i - Z\delta^{2k}\eta_i]^2 = minimum$, unterwerfen, wo das Symbol E die mathematische Erwartung bedeutet. Und daraus:

$$E\sum_{i=1}^{N}\eta_i^2 - 2ZE\sum_{i=1}^{N}\eta_i\,\delta^{2k}\eta_i + Z^2 E\sum_{i=1}^{N}(\delta^{2k}\eta_i)^2 = minimum.$$

Wenn man die erste Ableitung davon nach Z gleich O setzt, so erhält man für Z den Ausdruck:

$$Z = \frac{E\sum\limits_{i=1}^{N}\eta_i\,\delta^{2k}\eta_i}{E\sum\limits_{i=1}^{N}(\delta^{2k}\eta_i)^2}.$$

Da die zweite Ableitung positiv ist, so ist der gefundene Wert von Z ein Minimum. Um für Z eine einfache Form zu gewinnen, müssen wir bei der Komponente η gewisse Eigenschaften voraussetzen. Es sei wiederum angenommen, daß die Reihe $\eta_1, \eta_2, \ldots \eta_N$, empirische Größen darstellt, welche eine Variable mit einem konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen ergeben hat. Die Form dieses Verteilungsgesetzes bleibe aber beliebig. Dann ist $E\eta_1^m = E\eta_2^m = \ldots E\eta_i^m$ bei beliebigem m und ferner $E(\eta_i\eta_j) = E\eta_i\,.\,E\eta_j$, wenn $i \neq j$. Nach einigen Umformungen gelangen wir dann zum Ausdruck:

$$Z = \frac{C_{2k}^{k}}{C_{4k}^{2k}} = \frac{[(2k)!]^3}{(4k)!\,k!\,k!}.$$

Derselbe Ausdruck würde sich auch ergeben, wenn wir als Bedingungsgleichung für Z einfach $E[\eta_i - Z\delta^{2k}\eta_i]^2 = minimum$, nehmen würden. Der von

uns eingeschlagene Weg dürfte aber die Anlehnung an die Methode der kleinsten Quadrate klarer hervortreten lassen.

Setzt man den gefundenen Wert von Z in die Formel für $F(u_{i,2k+1})$ ein, so erhält man die endgiltige Formel des der Differenzenmethode entsprechenden Ausgleichungsverfahrens:

$$F(u_{i,2k+1}) = u_i' = u_i - \frac{[(2k)!]^2}{(4k)!\,k!\,k!}\,\delta^{2k} u_i \text{ ....(5).}$$

Gebraucht man in dieser Formel O, statt i, als Subscriptum, und gibt man k sukzessive die Werte 1, 2, 3, 4, ... so erhält man unmittelbar:

bei $k = 1$, $u_0' = \frac{1}{3}[u_0 + (u_1 + u_{-1})]$,

„ $k = 2$, $u_0' = \frac{1}{35}[17u_0 + 12(u_1 + u_{-1}) - 3(u_2 + u_{-2})]$,

„ $k = 3$, $u_0' = \frac{1}{231}[131u_0 + 75(u_1 + u_{-1}) - 30(u_2 + u_{-2}) + 5(u_3 + u_{-3})]$,

etc.

Also genau dieselben Koeffizienten, die die Sheppard'sche Ausgleichung in dem Falle ergibt, wenn man sein n der Ordnung seiner Parabel (oder der Hälfte der Ordnung unserer Differenz) gleich setzt! (vergl. Whittaker and Robinson, *loc. cit.* pp. 295—296).

Will man also hier, bei Sheppard, die "restliche Komponente" für einen Annäherungswert der wahren Komponente η ansehen, so darf man der Differenzenmethode auch nicht vorwerfen, daß sie nur lineare Funktionen der Komponente η ergeben könne.

Gibt man, im Gegenteil, zu, daß die nach Sheppard berechnete restliche Komponente $u_i - u_i'$ nicht gleich η_i sei (und wir glauben, daß man gezwungen ist, dies zuzugeben), so muß man die Differenzenmethode, als ein in gewissen Hinsichten ebenbürtiges Mitglied, in die Reihe der anderen Ausgleichungsmethoden aufnehmen. Meines Erachtens, ist eben das letztere geboten.

(6) Um die weiteren Ausführungen der vorliegenden Schrift verständlich zu machen, ist es notwendig ihnen eine Darstellung der wichtigsten Formeln der Differenzenmethode vorausgehen zu lassen. Einige derselben werden hierbei zum ersten Mal der Öffentlichkeit übergeben.

Eine gegebene statistische Reihe $u_1, u_2, \ldots u_N$ bestehe aus einer "glatten" Komponente G und einer "restlichen" s, so daß $u_i = G_i + s_i$. Es sei ferner angenommen, daß durch die k-te endliche Differenz erstere Komponente gänzlich vertilgt werde und daß, folglich, $\Delta^k u_i = \Delta^k G_i + \Delta^k s_i = \Delta^k s_i$*. Wir besitzen dann

* Begnügt man sich damit, daß $\Delta^k G_i$, ohne ganz zu verschwinden, nur einen gewissen, für die Praxis zu vernachlässigenden, Teil von $\Delta^k u_i$ ausmachen könne, so werden, außer den durch eine ganze rationale algebraische Funktion (Parabel) bis $(k-1)$-ter Ordnung darstellbaren Reihen, auch verschiedene andere, durch andere algebraische oder transzendente Funktionen darstellbare Reihen, ungefähr zum selben Resultat führen. So auch Yule's Sinus-Reihen mit der Formel $u_i = A \sin\left(2\pi\frac{t+\tau+ih}{T}\right)$ im Intervall $O \leqq h \leqq \frac{1}{4}T$ und $T \geqq h \geqq \frac{3}{4}T$. (Vergl. den Artikel von Yule im *Journ. Roy. Stat. Soc.*, Vol. LXXXIV. p. 497 et seq.)

Weiter unten werden wir noch die allgemeinen Bedingungsgleichungen feststellen, welche die Eigenschaften der beim endlichen Differenzieren allmählich an Bedeutung verlierenden Reihen der "Gruppe G" umschreiben. (Vergl. § 10.)

$(N-k)$ lineare Funktionen der N Größen $s_1, s_2, \dots s_N$, von denen eine jede von der Form

$$\Delta^k s_i = s_i - C_k^1 s_{i+1} + C_k^2 s_{i+2} - C_k^3 s_{i+3} + \dots + (-1)^k C_k^k s_{i+k} \quad \dots\dots\dots (6$$

ist, wenn

$$\left(C_k^j = \frac{k!}{j!\,(k-j)!} \dots\right)^* \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (6a).$$

$N-k$ Gleichungen genügen aber nicht zur Bestimmung von N Unbekannten, und folglich, können die Größen $s_1, s_2, \dots s_N$, keinesfalls allein aus der Reihe der Funktionen $\Delta^k s$ berechnet werden. Diese komplizierten Funktionen können an und für sich auch durchaus nicht als Repräsentanten oder angenäherte Werte der Reihe s gelten. In dieser Hinsicht haben G. U. Yule und W. M. Person ganz recht†. Sie sollen es aber auch garnicht! Die einzige Bedeutung dieser Funktionen besteht eben darin, daß man aus ihnen, bei Einführung gewisser Hypothesen über die Beschaffenheit der ursprünglichen Reihe s, verschiedene Charakteristiken derselben berechnen kann. *Nimmt man an, letztere könne angesehen werden, als eine Reihe von empirischen Größen, die eine zufällige Variable mit einem konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen annimmt, so gestalten sich die Formeln für die gesuchten Charakteristiken besonders einfach.*

Als erste dieser Formeln sei der mittlere Fehler (Standard Deviation) σ bestimmt.

Bedeutet das Symbol E wiederum die mathematische Erwartung, und gebraucht man die Bezeichnungen Prof. Al. A. Tchouproff's‡:

$$m_r = Es^r, \quad \mu_r = E(s-m_1)^r \quad \dots\dots\dots\dots\dots\dots\dots (7),$$

so ist, bekanntlich, der mittlere Fehler (Standard Deviation):

$$\sigma = \sqrt{\mu_2} = \sqrt{E(s-m_1)^2} = \sqrt{m_2 - m_1^2} \quad \dots\dots\dots\dots\dots (8).$$

* In der Theorie der endlichen Differenzen steht gewöhnlich $\Delta' u_1$ für $u_2 - u_1$ und, überhaupt, $\Delta^k u_i$ für $\Delta^{k-1} u_{i+1} - \Delta^{k-1} u_i$. Das macht wohl die Analogie mit der Differentialrechnung evidenter, die im Texte von uns gebrauchte Bezeichnung ist jedoch für den praktischen Gebrauch entschieden bequemer, da wenn die Reihe in einer vertikalen Kolonne angeordnet ist, man aus dem oberen das ihm unmittelbar folgende untere Glied subtrahiert und nicht umgekehrt. Letzteres könnte eine unnütze Fehlerquelle abgeben. Dieselben praktischen Erwägungen veranlaßten mich auch, vom Gebrauche der sogenannten *zentralen Differenzen* hier abzustehen, obgleich einige Formeln durch Einführung der letzteren ein eleganteres Aussehen erhalten hätten.

An diesem Orte sei noch besonders auf die unbedingte Notwendigkeit *der Kontrollen* bei Berechnung der endl. Differenzen hingewiesen. Am bequemsten ist die Benutzung der Kontrollformel

$$\sum_{i=1}^{N-k} \Delta^k s_i = \Delta^{k-1} s_1 - \Delta^{k-1} s_{N-k+1}.$$

† Es bedarf garnicht der ziemlich weitschweifigen Ausführungen W. M. Person's, um die "theory of the tendency of signs of terms of higher differences to alternate" zu beweisen. (Vergl. sein "On the variate difference correlation method and curve fitting" in *Quarterly Publications of the American Statistical Association*, New Series, No. 118 (Vol. xv.), June 1917, p. 612.) Es leuchtet ohne weiteres ein, daß wenn auch die einzelnen s_i von einander vollkommen unabhängig sind, schon die *erste Differenz* $s_i - s_{i+1}$ mit der ihr folgenden ersten Differenz $s_{i+1} - s_{i+2}$ *negativ* korreliert sein muß, und daß $\Delta^k s_{i+1}$, in welchem k von $k+1$ einzelnen s dieselben sind, wie im Ausdruck für $\Delta^k s_i$, mit diesem desto stärker *negativ* korreliert ist, je größer k genommen wird.

Die Werte dieser Korrelationen sind weiter unten durch Formel (19) gegeben.

‡ *Loc. cit.*, p. 148.

Die Größe $\sigma^2 = \mu_2$ werde ich hier apriorische Streuung (Variance) nennen.

Dem *apriorischen* mittleren Fehler σ sind bei der oben gemachten Annahme die mathematischen Erwartungen zweier Formen des *empirischen* mittleren Fehlers σ' gleich, die wir durch ein Subskriptum 0 von einander unterscheiden wollen:

$$\sigma_0' = \sqrt{\frac{\sum_{i=1}^{N} (s_i - m_1)^2}{N}} \quad \text{...(9),}$$

und

$$\sigma' = \sqrt{\frac{\sum_{i=1}^{N} (s_i - s_{(N)})^2}{N-1}}, \text{ wenn } s_{(N)} = \frac{\sum_{i=1}^{N} s_i}{N} \quad \text{..................(10).}$$

Es ist also dann $\qquad E\sigma_0' = E\sigma' = \sigma = \sqrt{\mu_2}.$

Führt man die Bezeichnung ein:

$$\sigma_k' = \sqrt{\frac{1}{C_{2k}^k} \cdot \frac{\sum_{i=1}^{N-k} (\Delta^k s_i)^2}{N-k}} \quad \text{....................................(11),}$$

wo $\Delta^k s_i$ durch die Formel (6) gegeben ist und C_{2k}^k (vergl. Form. 6a) gleich $\frac{2k\,!}{k!\,k!}$, so kann es—immer unter derselben Annahme einer zufälligen Variablen mit konstantem Verteilungsgesetz und N unabhängigen Versuchen—ganz strenge bewiesen werden, daß

$$E\sigma_k'^2 = \sigma^2 = \mu_2.$$

Diese Gleichung bleibt hier genau, unbeschadet, ob $N-k$ gleich 1 oder einer Million sei*.

Ist k so groß, daß man für seine Faktorial-Funktionen die Stirling'sche Formel anwenden darf, so kann (11) auch in folgender Form geschrieben werden:

$$\sigma_k'^2 = \frac{\sqrt{\pi k}}{4^k} \cdot \frac{\sum_{i=1}^{N-k} \Delta^k s_i^2}{N-k} \quad \text{...........................(11}a\text{).}$$

Schon bei $k=6$ beträgt der Fehler, den man bei Anwendung dieser Annäherungsformel begeht, kaum 2,1 % des wahren Wertes, und bei $k=9$—bloß 1,4 %.

* Formel (11) dient nur zur Bestimmung der uns einzig interessierenden Größe μ_2. Es hat daher keinen Sinn, ihr eine Form zu verleihen, die der Formel (10) analog wäre, also etwa

$$\frac{1}{C_{2k}^k (N-k)} \cdot \sum_{i=1}^{N-k} \left(\Delta^k x_i - \frac{1}{N-k} \sum_{i=1}^{N-k} \Delta^k x_i \right)^2.$$

Dadurch wird nur der Rechenweg komplizierter, aber der wahrscheinliche Fehler der Bestimmung nicht kleiner. Ebenso wenig Sinn hat es, zur Bestimmung von $\sigma_k'^2$ nicht gleich alle erhaltbaren Werte von s hinzuzuziehen, sondern, um die Zahl der $\Delta^k s$ immer konstant bleiben zu lassen, mit jeder neuen endlichen Differenz je ein neues Glied s der Reihe hinzuzufügen. Der Gewinn (Vereinfachung der Formeln) ist gering und die mittleren Fehler für die ersten Differenzen werden nur unnützerweise vergrößert. Es ist ziemlich dasselbe, als wenn wir bei Anwendung der Methode der kleinsten Quadrate einen Teil der möglichen Normalgleichungen unberücksichtigt ließen.

Eine ganz andere Frage ist es, natürlich, wie weit die empirische Größe $\sigma_k'^2$ sich in der Praxis von seiner mathematischen Erwartung μ_2 entfernen könne. Zur Lösung dieser Frage ist die Bestimmung des mittleren Fehlers von $\sigma_k'^2$ erforderlich, und hier spielt, selbstredend, die absolute Größe von $N-k$ eine ausschlaggebende Rolle.

$\sigma^2_{(\sigma^2, k)}$, die apriorische Streuung (Variance) von $\sigma_k'^2$, bestimmt sich aus der Formel

$$\sigma^2_{(\sigma^2, k)} = E\,(\sigma_k'^2 - \mu_2)^2.$$

Nach ziemlich langwierigen Umformungen ergibt sich hieraus (immer nur unter denselben Annahmen über die Natur der Reihe s), *wenn* $k \leqq \frac{N}{2}$:

$$\sigma^2_{(\sigma^2, k)} = \frac{\mu_4 - 3\mu_2^2}{N-k}\left[1 - \frac{2S_{(k)}}{(C_{2k}^k)^2\,(N-k)}\right] + \frac{2\mu_2^2}{N-k}\left[\frac{C_{4k}^{2k}}{(C_{2k}^k)^2} - \frac{k}{2\,(N-k)}\right] \ldots (12),$$

wobei

$$S_{(k)} = \sum_{i=0}^{k-1}(C_k^i\,C_k^{i+1})^2 + 2\sum_{i=1}^{k-2}(C_k^i\,C_k^{i+2})^2 + 3\sum_{i=1}^{k-3}(C_k^i\,C_k^{i+3})^2 + \ldots + k\,(C_k^0\,C_k^k)^2 \ldots *.$$

Da es bewiesen werden kann, daß jedenfalls

$$\frac{k}{2\,(N-k)} \geqq \frac{2S_{(k)}}{(C_{2k}^k)^2\,(N-k)},$$

so erhält man, wenn man für größere $(N-k)$ die Ausdrücke von der Ordnung $\frac{1}{(N-k)^2}$ unterdrückt, angenähert:

$$\sigma^2_{(\sigma^2, k)} = \frac{\mu_4 - 3\mu_2^2}{N-k} + \frac{C_{4k}^{2k}}{(C_{2k}^k)^2} \cdot \frac{2\mu_2^2}{N-k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Ist schließlich k so groß, daß für seine Faktorial-Funktionen die Stirling'sche Formel angewandt zu werden vermag, so kann man auch annäherungsweise schreiben:

$$\sigma^2_{(\sigma^2, k)} = \frac{\mu_4 - 3\mu_2^2}{N-k} + \frac{\mu_2^2\sqrt{2k\pi}}{N-k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13a).$$

Schon bei $k=6$ beträgt der Gesamtfehler bei Anwendung dieser Formel bloß etwa 3 %.

Formeln (12), (13) und (13*a*) gelten für ein beliebiges konstantes Verteilungsgesetz der Variablen. Ist es jedoch "normal," so besteht, bekanntlich, die Beziehung $\mu_4 = 3\mu_2^2$, und in obigen Formeln werden überall die ersten Glieder der rechten Seite verschwinden. Aus (13*a*) erhalten wir dann, zum Beispiel:

$$\sigma^2_{(\sigma^2, k)} = \frac{\mu_2^2\sqrt{2k\pi}}{N-k}.$$

Denkt man sich, also, k genügend und N beträchtlich groß und das Verteilungsgesetz "normal," so werden die mathematischen Erwartungen der empirischen

* Es ist also $S_{(1)}=1$; $S_{(2)}=10$; $S_{(3)}=138$; $S_{(4)}=1940$; $S_{(5)}=28180$; $S_{(6)}=414372$, u.s.w.

Streuungen der Größen $\sigma_k'^2$, σ'^2_{k+1}, σ'^2_{k+2}, ... sich etwa so zu einander verhalten, wie $\sqrt{k} : \sqrt{k+1} : \sqrt{k+2} : \ldots.$

In Tabelle I finden sich die bis zur 3[tten] Dezimalstelle genauen Werte, welche Formel (12) für die ersten 6 endlichen Differenzen ergibt.

TABELLE I.

Ordnung der Differenz k	$\sigma^2_{(\sigma^2,\,k)}$ für ein beliebiges konstantes Verteilungsgesetz der zufälligen Variablen S	$\sigma^2_{(\sigma^2,\,k)}$ für ein "normales" Verteilungsgesetz der zufälligen Variablen S
$k=0$	$\dfrac{\mu_4-\mu_2^2}{N}$	$\dfrac{2\mu_2^2}{N}$
$k=1$	$\dfrac{(N-1{,}500)\,\mu_4+0{,}500\,\mu_2^2}{(N-1)^2}$	$\dfrac{(3{,}000\,N-4{,}000)\,\mu_2^2}{(N-1)^2}$
$k=2$	$\dfrac{(N-2{,}556)\,\mu_4+(0{,}889\,N-2{,}111)\,\mu_2^2}{(N-2)^2}$	$\dfrac{(3{,}889\,N-9{,}778)\,\mu_2^2}{(N-2)^2}$
$k=3$	$\dfrac{(N-3{,}690)\,\mu_4+(1{,}620\,N-5{,}790)\,\mu_2^2}{(N-3)^2}$	$\dfrac{(4{,}620\,N-16{,}860)\,\mu_2^2}{(N-3)^2}$
$k=4$	$\dfrac{(N-4{,}792)\,\mu_4+(2{,}253\,N-10{,}633)\,\mu_2^2}{(N-4)^2}$	$\dfrac{(5{,}253\,N-25{,}012)\,\mu_2^2}{(N-4)^2}$
$k=5$	$\dfrac{(N-5{,}887)\,\mu_4+(2{,}819\,N-16{,}436)\,\mu_2^2}{(N-5)^2}$	$\dfrac{(5{,}819\,N-34{,}094)\,\mu_2^2}{(N-5)^2}$
$k=6$	$\dfrac{(N-6{,}971)\,\mu_4+(3{,}335\,N-23{,}096)\,\mu_2^2}{(N-6)^2}$	$\dfrac{(6{,}335\,N-44{,}008)\,\mu_2^2}{(N-6)^2}$

Aus Tabelle I erhalten wir die apriorischen *Streuungen* der Größen $\sigma_k'^2$. Um deren *mittlere Fehler* zu bestimmen, müssen aus den für erstere berechneten Zahlenwerten Quadratwurzeln gezogen werden. Die Größen μ_4 und μ_2 sind apriorisch und müssen beim Gebrauch der Tabelle in der Praxis durch empirische Annäherungswerte ersetzt werden. Dadurch entstehen Schwankungen, welche eigentlich ihrerseits mit Hilfe der mittleren Fehler der Streuungen $\sigma^2_{(\sigma^2,\,k)}$ gemessen werden sollten. Die Verfolgung dieses Weges würde uns aber hier zu weit führen.

(7) Formel (12) ergibt nur die apriorische Streuung (oder auch den mittleren Fehler) eines jeden $\sigma_k'^2$ gegenüber dem apriorischen μ_2. Sie besagt noch nichts über das gegenseitige Verhältnis der Größen $\sigma_k'^2$, σ'^2_{k+1}, σ'^2_{k+2}, ... welche, wie leicht ersichtlich, stark positiv miteinander korreliert sein müssen. Ihre mathematischen Erwartungen sind wohl alle einander gleich und diejenigen ihrer Differenzen, folglich, gleich Null.

In der Praxis werden jedoch die Differenzen dieser *empirischen* Größen von O mehr oder weniger beträchtlich abweichen. Um ein Maß dieses Abweichens zu erhalten, wollen wir die apriorischen Streuungen der Differenzen $\sigma'^2_{k+1} - \sigma_k'^2$ bestimmen:

$$E\left[(\sigma'^2_{k+1} - \sigma_k'^2) - O\right]^2 = E\left[(\sigma'^2_{k+1} - \mu_2) - (\sigma_k'^2 - \mu_2)^2\right]^2$$
$$= E\,(\sigma'^2_{k+1} - \mu_2)^2 + E\,(\sigma_k'^2 - \mu_2)^2 - 2E\,(\sigma_k'^2 - \mu_2)(\sigma'^2_{k+1} - \mu_2) \;\ldots\ldots(14).$$

Die beiden ersten Glieder der rechten Seite ergeben sich direkt aus Formel (12), für das dritte erhalten wir, wenn $k \leqq \frac{N}{2}$, folgenden Ausdruck:

$$E(\sigma_k'^2 - \mu_2)(\sigma'^2_{k+1} - \mu_2) = \frac{\mu_4 - 3\mu_2^2}{(N-k)}\left[1 - \frac{2S'_{(k)}}{C_{2k}^{k}\, C_{2k+2}^{k+1}(N-k-1)}\right]$$
$$+ \frac{2\mu_2^2}{N-k}\left[\frac{C_{4k+1}^{2k}}{C_{2k}^{k}\, C_{2k+2}^{k+1}} \cdot \frac{2N-2k-1}{N-k-1} - \frac{k+1}{2(N-k-1)}\right] \quad \text{......(15)},$$

wobei

$$S'_{(k)} = \sum_{i=0}^{k-1}(C_k^i\, C_{k+1}^{i+2})^2 + 2\sum_{i=0}^{k-2}(C_k^i\, C_{k+1}^{i+3})^2 + 3\sum_{i=0}^{k-3}(C_k^i\, C_{k+1}^{i+4})^2 + \ldots + k(C_k^0\, C_{k+1}^{k+1})^2,$$

und jedenfalls: $$\frac{k+1}{2(N-k-1)} \geqq \frac{2S'_{(k)}}{C_{2k}^{k}\, C_{2k+2}^{k+1}(N-k-1)} \; * \dagger.$$

Setzt man die entsprechenden Ausdrücke aus (12) und (15) in (14) ein, so erhält man, wenn man alle Größen von der Ordnung $\frac{1}{N^3}$ unterdrückt, für die ersten 6 Differenzen folgende Werte (s. Tabelle II).

TABELLE II‡.

Ordnung der Streuung	Formel für die apriorische Streuung, bis zur Ordnung $\frac{1}{N^2}$, für ein beliebiges konstantes Verteilungsgesetz der zufälligen Variablen S	Dieselbe, bis zur Ordnung $\frac{1}{N}$ abgerundet
$E(\sigma_1'^2 - \sigma_0'^2)^2 =$	$\frac{0{,}5000\,(\mu_4 - 3\mu_2^2)}{N(N-1)} + \frac{(N+1)\,\mu_2^2}{N(N-1)}$	$\frac{\mu_2^2}{N-1}$
$E(\sigma_2'^2 - \sigma_1'^2)^2 =$	$\frac{0{,}2778\,(\mu_4 - 3\mu_2^2)}{(N-1)(N-2)} + \frac{(0{,}2222\,N + 1{,}1111)\,\mu_2^2}{(N-1)(N-2)}$	$\frac{0{,}2222\,\mu_2^2}{N-2}$
$E(\sigma_3'^2 - \sigma_2'^2)^2 =$	$\frac{0{,}2544\,(\mu_4 - 3\mu_2^2)}{(N-2)(N-3)} + \frac{(0{,}1089\,N + 1{,}0933)\,\mu_2^2}{(N-2)(N-3)}$	$\frac{0{,}1089\,\mu_2^2}{N-3}$
$E(\sigma_4'^2 - \sigma_3'^2)^2 =$	$\frac{0{,}2096\,(\mu_4 - 3\mu_2^2)}{(N-3)(N-4)} + \frac{(0{,}0673\,N + 1{,}0808)\,\mu_2^2}{(N-3)(N-4)}$	$\frac{0{,}0673\,\mu_2^2}{N-4}$
$E(\sigma_5'^2 - \sigma_4'^2)^2 =$	$\frac{0{,}1873\,(\mu_4 - 3\mu_2^2)}{(N-4)(N-5)} + \frac{(0{,}0468\,N + 1{,}0721)\,\mu_2^2}{(N-4)(N-5)}$	$\frac{0{,}0468\,\mu_2^2}{N-5}$
$E(\sigma_6'^2 - \sigma_5'^2)^2 =$	$\frac{0{,}1693\,(\mu_4 - 3\mu_2^2)}{(N-5)(N-6)} + \frac{(0{,}0350\,N + 1{,}0656)\,\mu_2^2}{(N-5)(N-6)}$	$\frac{0{,}0350\,\mu_2^2}{N-6}$

Es ergibt sich also, daß, im Allgemeinen, je größer die Ordnung der endlichen Differenz k, desto geringer die mathematische Erwartung des Quadrates der

* Es ist also: $S_1' = 1$; $S_2' = 15$; $S_3' = 242$; $S_4' = 3815$; $S_5' = 59724$; etc.

† Formel (15) erlaubt es auch, den Korrelationskoeffizienten zwischen $\sigma_k'^2$ und σ'^2_{k+1} genau zu bestimmen.

‡ Für ein "normales" Verteilungsgesetz verschwinden alle Glieder, welche $\mu_4 - 3\mu_2^2$ enthalten, und die einzelnen Größen der Tabelle unterscheiden sich von denjenigen in Formel (28) meiner Monographie "Über ein neues Verfahren bei Anwendung der 'Variate-Difference'-Methode" (*Biometrika*, Vol. xv., 1923, p. 145) nur durch den Faktor $\frac{1}{(2k+1)^2}$. Die Formel (21) für $\sigma^2{}_0D^2{}_{k+1}$ ist nämlich nicht

Differenz $\sigma'^2_{k+1} - \sigma_k'^2$ ausfällt. Das ist auch ganz natürlich, denn obgleich ihr möglicher minimaler Wert immer derselbe bleibt ($-\sigma_k'^2$ bei $\sigma'^2_{k+1} = 0$), ist ihr Maximum durch $+\frac{\sigma_k'^2}{2k+1}$ gegeben, verringert sich also mit wachsendem k*. Der mittlere Fehler der Differenz $\sigma'^2_{k+1} - \sigma_k'^2$ bleibt jedoch endlich, solange μ_2 und N endlich bleiben.

Ferner ist zu bemerken, daß die Differenz $\mu_4 - 3\mu_2^2$ sich hier bloß von der Ordnung $\frac{1}{N^2}$ erweist. Können bei beträchtlichem N auch alle Größen dieser Ordnung unterdrückt werden, so erhalten wir die Werte der letzten Kolonne der Tabelle II, welche, bis zur Größenordnug $\frac{1}{N}$, für ein beliebiges konstantes Verteilungsgesetz der Variablen gelten. Sie bestimmen sich, ganz allgemein, nach der Formel:

$$E(\sigma'^2_{k+1} - \sigma_k'^2)^2 = \frac{3k+1}{(2k+1)^3} \cdot \frac{C_{4k}^{2k}}{(C_{2k}^{k})^2} \cdot \frac{\mu_2^2}{N-k-1} \quad \text{.............}(16).$$

Ist k so groß, daß man die Stirling'sche Formel anwenden kann, so erhält man hieraus:

$$E(\sigma'^2_{k+1} - \sigma_k'^2)^2 = \frac{(3k+1)\,\mu_2^2\sqrt{2k\pi}}{2\,(2k+1)^3\,(N-k-1)} \quad \text{.............}(16a).$$

Schon bei $k = 6$ ergeben sich ganz annehmbare Resultate.

Die Formeln der Tabellen I und II vervollständigen nur einander, können sich aber gegenseitig nicht ersetzen. Will man feststellen, ob eine für die Theorie genügende Stabilität der Reihe $\sigma_k'^2, \sigma'^2_{k+1}, \sigma'^2_{k+2}, \dots$ erreicht ist, berechnet man die ersten Differenzen derselben und gebraucht die Formeln der Tabelle II. Ist es aber notwendig, die Grenzen der möglichen Abweichung des gefundenen *stabilen* $\sigma_k'^2$ von seiner mathematischen Erwartung—sagen wir, mit Hilfe des bekannten Tschebyscheff'schen Theorems—zu beurteilen, so wird man sich der Größen der Tabelle I bedienen müssen.

Ein anderes, in manchen Fällen interessantes, Charakteristikum derselben Reihe S bilden die Koeffizienten vom Typus:

$$R'_{j,k} = \frac{\dfrac{1}{(N-k-j)} \sum\limits_{i=1}^{N-k-j} \Delta^k s_i \, \Delta^k s_{i+j}}{\dfrac{1}{(N-k)} \sum\limits_{i=1}^{N-k} (\Delta^k s_i)^2} \quad \text{....................}(17),$$

wobei j eine beliebige ganze positive Zahl zwischen O und $(N-k-1)$ sein kann.

$E\,\{_0D_{k+1} - E\,_0D_{k+1}\}^2$ gleich, wie es dort fälschlich angenommen wurde, sondern der Größe:

$$\frac{(2k+1)^2\,E\,\{\sigma'^2_{k+1} - \sigma_k'^2\}^2}{\mu_2^2},$$

und zeigt, nach Bestimmung seiner Quadratwurzel, welch einen Teil der Größe μ_2 der mittlere Fehler der Differenz $(2k+1)\,\sigma'^2_{k+1} - (2k+1)\sigma_k'^2$ ausmachen kann. Nur in diesem Sinne darf sie angewandt werden. Vergl. darüber: O. Anderson, "Variate Difference Method," in den *Mélanges Pierre Struve, Recueil des Ecrits, présentés à M. Pierre Struve le* 30. *Janvier*, 1925, Prague, pp. 26, 27 (russisch).

* Durch die Formel dieses Maximums erklärt sich auch, weshalb, bei konkreten Messungen, die Reihe der Differenzen $\sigma_2'^2 - \sigma_1'^2, \sigma_3'^2 - \sigma_2'^2, \dots \sigma'^2_{k+1} - \sigma_k'^2$ verhältnißmäßig oft auf dem Diagramm ein ganz hyperbelmäßiges Aussehen erhält.

Es kann nämlich bewiesen werden, daß $ER'_{j,k}$ desto näher zu

$$\frac{\frac{1}{(N-k-j)} E \sum_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s_{i+j}}{\frac{1}{(N-k)} E \sum_{i=1}^{N-k} (\Delta^k s_i)^2}$$

herankommt, je größer N ist, um jedenfalls im Grenzfall, bei sehr großem N, mit diesem Ausdruck identisch zu werden*. Nun ist aber

$$\frac{1}{N-k-j} E \sum_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s_{i+j} = (-1)^j C_{2k}^{k+j} \mu_2 \text{ und } \frac{1}{N-k} E \sum_{i=1}^{N-k} (\Delta^k s_i)^2 = C_{2k}^k \mu_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

Daher erhalten wir für $ER'_{j,k}$ den Grenzwert:

$$ER'_{j,k} = (-1)^j \frac{C_{2k}^{k+j}}{C_{2k}^k} = (-1)^j \frac{k!\,k!}{(k-j)!\,(k+j)!} \quad \ldots\ldots\ldots\ldots\ldots (19),$$

* Der empirische Korrelationskoeffizient und die meisten ihm ähnlichen Gebilde besitzen die Eigenschaft, daß ihre Nenner und Zähler verschiedene Funktionen *derselben gegebenen Werte* $Z_1, Z_2, \ldots Z_N$ und $Z_1', Z_2', \ldots Z'_N$, sind. Die Wahrscheinlichkeit des Auftretens einer bestimmten i-ten Kombination von N zusammengehörigen Paaren Z_1 und Z_1', Z_2 und Z_2', … Z_N und Z'_N, eines, sagen wir, Korrelationskoeffizienten sei p_i. Und es seien nur n verschiedene Kombinationen dieser Paare, zu N in jeder, möglich, so daß $\sum_{i=1}^{n} p_i = 1$.

Bezeichnen wir einen Koeffizienten, entstanden aus der i-ten Kombination der Paare, durch $r_i = \frac{x_i}{y_i}$, so ist es ersichtlich, daß unter den gemachten Annahmen die Zähler x nur n verschiedene Werte: $x_1, x_2, \ldots x_n$ mit den entsprechenden Wahrscheinlichkeiten $p_1, p_2, \ldots p_n$ erhalten können; ist aber x_i gegeben so ist nur ein ganz bestimmtes einziges y_i möglich, dessen bedingte Wahrscheinlichkeit also gleich 1 zu setzen ist, so daß, folglich, auch die Zähler y ebenfalls nur n Werte: $y_1, y_2, \ldots y_N$, *mit denselben Wahrscheinlichkeiten* $p_1, p_2, \ldots p_n$ annehmen können. Dieselben Wahrscheinlichkeiten bestehen schließlich auch für die r_i selbst, so daß, also:

$$(1)\ Er = E\frac{x}{y} = p_1\frac{x_1}{y_1} + p_2\frac{x_2}{y_2} + \ldots + p_n\frac{x_n}{y_n}, \text{ und } (2)\ \frac{Ex}{Ey} = \frac{p_1x_1 + p_2x_2 + \ldots + p_nx_n}{p_1y_1 + p_2y_2 + \ldots + p_ny_n}.$$

Setzt man $\frac{x_i}{y_i} - Er = \epsilon_i$, oder $x_i = y_i Er + y_i\epsilon_i$, so verwandeln sich obige Ausdrücke in folgende:

$$(1)\quad E\frac{x}{y} = Er\,.\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} p_i\epsilon_i = E\frac{x}{y} + \sum_{i=1}^{n} p_i\epsilon_i \text{ (und folglich } \sum_{i=1}^{n} p_i\epsilon = 0),$$

und

$$(2)\quad \frac{Ex}{Ey} = \frac{Er\,.\sum_{i=1}^{n} p_iy_i + \sum_{i=1}^{n} y_ip_i\epsilon_i}{Ey} = E\frac{x}{y} + \frac{\sum_{i=1}^{n} y_ip_i\epsilon_i}{Ey}.$$

Wir sehen also, daß, damit $E\frac{x}{y} = \frac{Ex}{Ey}$, notwendigerweise auch $\sum_{i=1}^{n} y_ip_i\epsilon_i$ gleich Null sein muß, wenn man, natürlich, vom Falle $Ey=0$ absieht.

Wenn die Größen $y_1, y_2, \ldots y_n$ mit wachsendem N demselben Grenzwert (sagen wir, Ey) zustreben, so können hier die einzelnen y schließlich als einander gleich angesehen und vor das Summenzeichen genommen werden. Dann erhält man auch vermöge der Beziehung $\sum_{i=1}^{n} p_i\epsilon_i = 0$:

$$\sum_{i=1}^{n} y_ip_i\epsilon_i = Ey\,.\sum_{i=1}^{n} p_i\epsilon_i = 0.$$

Mit wachsendem N kommen also dann die Ausdrücke $E\frac{x}{y}$ und $\frac{Ex}{Ey}$ sich immer näher, um im Grenzfalle einander gleich zu werden.

und folglich:

$$ER'_{1,k} = -\frac{k}{k+1}, \quad ER'_{2,k} = +\frac{k(k-1)}{(k+1)(k+2)}, \quad ER'_{3,k} = -\frac{k(k-1)(k-2)}{(k+1)(k+2)(k+3)} \text{ u.s.w.*} \quad .(19a).$$

Es ist möglich, zu beweisen†, daß wenn $ER'_{1,k}$ den eben angegebenen Wert besitzt, auch unbedingt $E\sigma'^2_{k+1} = E\sigma_k'^2$; wenn, ferner, $ER'_{2,k}$, $ER'_{3,k}, \ldots ER'_{j,k}$ die in Form. (19) angegebenen Werte annehmen, so ist auch, ganz allgemein,

$$E\sigma'^2_{k+2} = E\sigma'^2_{k+3} = \ldots = E\sigma'^2_{k+j}.$$

Folglich, können die Koeffizienten $R'_{i,k}$, wenn ein entsprechendes Kriteriensystem aufgebaut ist, die schwierige Berechnung der höheren endlichen Differenzen in einigen Fällen ersetzen und dadurch die notwendige Rechenarbeit erleichtern‡.

(8) Außer der statistischen Reihe $u_1, u_2, \ldots u_N$, sei noch eine andere Reihe: $v_1, v_2, \ldots v_N$, gegeben. Sie bestehe ebenfalls aus einer "glatten" Komponente G' und einer "restlichen" s', so daß $v_i = G_i' + s_i'$. Es sei festgestellt, daß in der k-ten endlichen Differenz keine Reste von G' mehr verblieben seien, so daß $\Delta^k v_i = \Delta^k s_i'$. Außerdem sei angenommen, daß auch die Reihe s' als eine solche Reihe angesehen werden könne, welche eine zufällige Variable mit einem konstanten Verteilungsgesetze bei N von einander unabhängigen Versuchen ergibt.

Führt man, in Anlehnung an Prof. Tchouproff §, die Bezeichnungen ein:

$$\left.\begin{aligned}
m_{g/h} &= Es_i^g s_i'^h \text{ (und folglich: } m_{g/0} = Es^g,\ m_{0/h} = Es'^h)\\
m^{(j)}_{g/h} &= Es_i^g s'^h_{i+j}\\
\mu_{g/h} &= E(s_i - m_{1/0})^g (s_i' - m_{0/1})^h \text{ (und folglich:}\\
&\qquad \mu_{g/0} = E(s - m_{1/0})^g,\ \mu_{0/h} = E(s' - m_{0/1})^h)\\
\mu^{(j)}_{g/h} &= E(s_i - m_{1/0})^g (s'_{i+j} - m_{0/1})^h
\end{aligned}\right\} \ldots\ldots(20),$$

* Um $\frac{1}{N-k-j} E \sum_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s_{i+j}$ zu bestimmen, ist es hauptsächlich notwendig, den Satz: $\sum_{i=0}^{k-j} C_k^i C_k^{i+j} = C_{2k}^{k+j}$ zu beweisen. Dieser Satz ist nicht neu (es dürfte überhaupt nicht leicht sein, in der Theorie der Kombinatorik etwas neues zu sagen) und findet sich, zum Beispiel, in Netto's *Lehrbuch der Kombinatorik*. Am leichtesten ist er aus der Betrachtung der Identität $(1+t)^k \left(1+\frac{1}{t}\right)^k = \frac{(1+t)^{2k}}{t^k}$ zu beweisen. Entwickelt man nämlich die linke Seite, so ist: $\sum_{i=0}^{k-j} C_k^i C_k^{i+j}$ der Koeffizient vor t^j; entwickelt man die rechte Seite, so steht vor t^j der Koeffizient: C_{2k}^{k+j}.

† Vergl. O. Anderson, *Über ein neues Verfahren; etc.*, S. 138—140.

‡ Näheres darüber siehe in der eben zitierten Schrift. Die dort benutzten Kriterien $\frac{{}_0D_i}{2i-1}$ und $\frac{{}_kD_{i-k}}{2i-1}$ bestimmen sich aus $\frac{\sigma_i'^2 - \sigma'^2_{i-1}}{\sigma'^2_{i-1}}$.

§ Al. A. Tchouproff, "Aufgaben und Voraussetzungen der Korrelationsrechnung" (*Nordisk Statistisk Tidskrift*, 1923, Bd. 2, Häft 1), S. 52, Anhang.

ferner die Bezeichnungen:

$$\left.\begin{aligned} p_k' &= \frac{1}{C^k_{2k}} \cdot \frac{\sum\limits_{i=1}^{N-k} \Delta^k s_i \Delta^k s_i'}{N-k} \\ \text{und} \qquad p'_{j(k)} &= \frac{1}{C^k_{2k}} \cdot \frac{\sum\limits_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s'_{i+j}}{N-k-j} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

so kann es leicht bewiesen werden, daß:

$$Ep_k' = \mu_{1/1},$$

wenn nur s_i und s'_{i+j}, bei jedem $j \neq 0$, vollkommen unabhängig von einander sind und folglich $\mu^{(j)}_{1/1} = 0$ bei $j \neq 0$.

Unter denselben Bedingungen wird auch:

$$\sigma^2_{(p,k)} = E(p_k' - \mu_{1/1})^2 = \frac{\mu_{2/2} - \mu_{2/0}\mu_{0/2} - 2\mu^2_{1/1}}{N-k}\left[1 - \frac{2S_k}{(C^k_{2k})^2(N-k)}\right]$$
$$+ \frac{(\mu^2_{1/1} + \mu_{2/0}\mu_{0/2})}{N-k}\left[\frac{C^{2k}_{4k}}{(C^k_{2k})^2} - \frac{k}{2(N-k)}\right] \quad \ldots\ldots\ldots\ldots (22).$$

Der Aufbau dieser Formel ist genau derselbe, wie bei (12). Nur muß man $\frac{\mu^2_{1/1} + \mu_{2/0}\mu_{0/2}}{2}$, statt μ_2^2, und $(\mu_{2/2} - \mu_{2/0}\mu_{0/2} - 2\mu^2_{1/1})$, statt $(\mu_4 - 3\mu_2^2)$ setzen. Letzterer Ausdruck verschwindet ebenfalls, wie $(\mu_4 - 3\mu_2^2)$, im Falle einer "normalen" Korrelationsfläche, wo die bekannte Relation $r_{2/2} = 1 + 2r^2_{1/1}$ besteht*. Mit den soeben erwähnten Substitutionen können also auch hier die Formeln der Tabelle I benutzt werden.

Mit denselben Substitutionen sind ebenfalls die Formeln der Tabelle II auf die mathem. Erwartungen der Quadrate der Differenzen: $(p_k' - p'_{k-1})$ anwendbar.

Desgleichen erhalten wir ferner:

$$Ep_k' = Ep'_{k+1} = Ep'_{k+2} = \ldots = Ep'_{k+j},$$

wenn:

$$\frac{\mu^{(-1)}_{1/1} + \mu^{(+1)}_{1/1}}{2\mu_{1/1}} = -\frac{k}{k+1}, \quad \frac{\mu^{(-2)}_{1/1} + \mu^{(+2)}_{1/1}}{2\mu_{1/1}} = \frac{k(k-1)}{(k+1)(k+2)}, \ldots \frac{\mu^{(-j)}_{1/1} + \mu^{(+j)}_{1/1}}{2\mu_{1/1}}$$
$$= (-1)^j \frac{k!\,k!}{(k-j)!\,(k+j)!}\dagger \quad \ldots\ldots\ldots\ldots (23).$$

* S. unten: Formel (24). Bei Anwendung letzterer Formel kann man (22) auch folgendermaßen darstellen:

$$\sigma^2_{(p,k)} = \frac{\mu_{2/0}\mu_{0/2}(r_{2/2} - 1 - 2r^2_{1/1})}{N-k}\left[1 - \frac{2S_k}{(C^k_{2k})^2(N-k)}\right] + \frac{\mu_{2/0}\mu_{0/2}(1 + r^2_{1/1})}{N-k}\left[\frac{C^{2k}_{4k}}{(C^k_{2k})^2} - \frac{k}{2(N-k)}\right] \ldots (22a).$$

† O. Anderson, *Über ein neues Verfahren, etc.*, S. 138.

Definiert man:

$$r_{g/h} = \frac{\mu_{g/h}}{[\mu_{2/0}]^{\frac{1}{2}g} \cdot [\mu_{0/2}]^{\frac{1}{2}h}} \left(\text{und folglich: } r_{1/1} = \frac{\mu_{1/1}}{\sqrt{\mu_{2/0}\,\mu_{0/2}}}\right) \quad \ldots\ldots\ldots(24),$$

$$r^{(j)}_{g/h} = \frac{\mu^{(j)}_{g/h}}{[\mu_{2/0}]^{\frac{1}{2}g} \cdot [\mu_{0/2}]^{\frac{1}{2}h}}; \quad {}_0\rho_j = \frac{\frac{1}{N-j}\sum\limits_{i=1}^{N-j}(s_i - m_{1/0})(s'_{i+j} - m_{0/1})}{\sqrt{\frac{1}{N}\sum\limits_{i=1}^{N}(s_i - m_{1/0})^2 \cdot \frac{1}{N}\sum\limits_{i=1}^{N}(s_i' - m_{0/1})^2}}$$

und

$$\rho_j = \frac{\frac{1}{N-j}\sum\limits_{i=1}^{N-j}(s_i - s_{(N)})(s'_{i+j} - s'_{(N)})}{\sqrt{\frac{1}{N-1}\sum\limits_{i=1}^{N}(s_i - s_{(N)})^2 \cdot \frac{1}{N-1}\sum\limits_{i=1}^{N}(s_i' - s'_{(N)})^2}} \quad \ldots\ldots(25),$$

wenn $s_{(N)} = \frac{\sum\limits_{i=1}^{N} s_i}{N}$ *, und ferner:

$$\rho_{j(k)} = \frac{p'_{j(k)}}{\sqrt{{}_1\sigma_k'^2 \cdot {}_2\sigma_k'^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

(wobei ${}_1\sigma_k'^2$ bei Anwendung der Formel (11) auf die Reihe $s_1, s_2, \ldots$, und ${}_2\sigma_k'^2$ bei Anwendung derselben Formel auf die Reihe $s_1', s_2', \ldots$ entsteht),—so kann man leicht beweisen, daß

$$E\rho_{0(k)} = E\frac{p_k'}{\sqrt{{}_1\sigma_k'^2 \cdot {}_2\sigma_k'^2}} = r_{1/1},$$

wenn man nur davon absieht, daß $E\rho_{0(k)}$ bloß in erster Annäherung dem Ausdruck $\frac{Ep_k'}{\sqrt{E_1\sigma_k'^2 \cdot E_2\sigma_k'^2}}$ gleich gesetzt werden kann†.

Was die Größe $\sigma^2_{(r,k)} = E(\rho_{0(k)} - r_{1/1})^2$ anbetrifft, so ergibt sie sich—in erster Annäherung, für eine "normale" Verteilung und mit allen Reserven, die durch den neuesten Stand der Forschung geboten sind,—aus der Formel:

$$\sigma^2_{(r,k)} = \frac{(1 - r^2_{1/1})^2}{N-k}\left[\frac{C^{2k}_{4k}}{(C^k_{2k})^2} - \frac{k}{2(N-k)}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(27).$$

Auch hier können, also, die Formeln der letzten Kolonne der Tabelle I benutzt werden, wenn man dort $\mu_2{}^2$ durch $\frac{(1 - r^2_{1/1})^2}{2}$ ersetzt‡.

* Wenn j negativ ist, so muß, statt $\frac{1}{N-j}$, überall $\frac{1}{N+j}$ genommen werden.

† Vergleich hierüber besonders die neuesten (russischen) Arbeiten Prof. A. Tchouproff's: *Über die mathematische Erwartung des Quotienten zweier gegenseitig nicht unabhängigen zufälligen Variablen*, und *Das Ausgangsproblem der mathematischen Theorie der verschiedenen Verfahren für die statistische Untersuchung der Zusammenhänge zwischen zwei zufälligen Variablen.* Ferner E. Slutsky, *Über einige Korrelationsschemen und über den systematischen Fehler des empirischen Korrelationskoeffizienten.*

‡ Werden demzufolge die vor N stehenden Koeffizienten dieser Kolonne durch 2 dividiert, so erhält man ganz richtig die von A. Ritchie-Scott im XI. Bande der *Biometrika* ("Note on the probable error of the Coefficient of Correlation in the Var. Diff. Corr. Method") berechneten Funktionen: $2\phi(m) - 1$.

Ist schließlich N und k genügend groß, so kommt man auch zur Formel:

$$\sigma^2_{(r,k)} = \frac{(1-r^2_{1/1})^2}{N-k}\sqrt{\frac{k\pi}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(27a).$$

Die Annahme der gegenseitigen vollkommenen Unabhängigkeit aller s_i und s_j', ausgenommen den Fall $i=j$, kann aber lange nicht immer gerechtfertigt werden. Es ist sehr wohl möglich, daß, obgleich jede der beiden zufälligen Variablen ein konstantes Verteilungsgesetz aufweist und bei jeder die einzelnen Versuche von einander vollkommen unabhängig sind, es dennoch zwischen verschiedenen s_i und s'_{i+j} eine gewisse Korrelation bestehen könne. So vermag man, zum Beispiel, aus der Reihe s_1, s_2, s_3, folgende zwei Reihen zu bilden:

$$\begin{aligned} x_1 &= s_1 + s_2 + s_3, & y_1 &= s_1 + s_5 + s_9, \\ x_2 &= s_4 + s_5 + s_6, & y_2 &= s_4 + s_8 + s_{12}, \\ x_3 &= s_7 + s_8 + s_9, \quad \text{und} & y_3 &= s_7 + s_{11} + s_{15}, \\ x_4 &= s_{10} + s_{11} + s_{12}, & y_4 &= s_{10} + s_{14} + s_{18}, \end{aligned}$$

Wenn die s von einander unabhängig sind, so sind auch alle x von einander unabhängig, und ebenso alle y. Und trotzdem hat jedes y_i je eine gemeinsame Komponente nicht nur mit jedem x_i, sondern auch mit x_{i+1} und x_{i+2}.

In solchen Fällen (und sie brauchen nicht selten vorzukommen) wird also weder die Reihe der Koeffizienten $E\rho_{0(k)}, E\rho_{0(k+1)}, \ldots$ konstant, noch wird einer derselben $r_{1/1}$ ergeben.

Um hier die Differenzenmethode anzuwenden, muß man entweder zu E. S. Pearson's Funktionen $\phi(n, \rho_0, \rho)$* oder, was auf dasselbe hinausläuft, zu den von mir im Artikel "Über ein neues Verfahren," etc.†, entwickelten Formeln greifen. Der Gebrauch letzterer wird weiter unten an einigen konkreten Beispielen illustriert. Hier seien sie nur des Zusammenhanges wegen angeführt.

Bei größeren N und unter den Voraussetzungen des § 10 (siehe dort) kann mit ganz guter Annäherung gesetzt werden:

$$p_j'{}_{(k)} = \frac{(-1)^k}{C^k_{2k}} \cdot {}_1\sigma' \, {}_2\sigma' \left\{C^{2k}_{2k}\rho_{k+j} - C^{2k-1}_{2k}\rho_{k+j-1} + C^{2k-2}_{2k}\rho_{k+j-2} - \ldots + C^0_{2k}\rho_{j-k}\right\} \ldots(28),$$

$$\sigma_k'^2 = \frac{1}{C^k_{2k}} \cdot \sigma'^2 \cdot \left\{C^k_{2k} - 2C^{k+1}_{2k}R_1' + 2C^{k+2}_{2k}R_2' - \ldots \pm 2C^{2k}_{2k}R_k'\right\} \quad \ldots\ldots\ldots(29),$$

wenn

$$R_j' = \frac{\frac{1}{N-j}\sum_{i=1}^{N-j}(s_i - s_{(N)})(s_{i+j} - s_{(N)})}{\frac{1}{N-1}\sum_{i=1}^{N}(s_i - s_{(N)})^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(29a),$$

und ${}_1\sigma'$ und ${}_2\sigma'$ die auf die Reihen s und s' angewandte Formel (10) bedeuten.

* K. Pearson and E. M. Elderton, "On the Variate-Difference Method" (*Biometrika*, Vol. XIV. March 1923, p. 294), Form. (xviii) und ff.

† *Loc. cit.*, pp. 186, 187.

(9) Gehen wir nun weiter. Angenommen, die empirischen Formeln für

$$\sigma_k'^2,\ \sigma'^2_{k+1}, \ldots,\ \sigma'^2_{k+1} - \sigma_k'^2,\ \sigma'^2_{k+2} - \sigma'^2_{k+1}, \ldots,\ R'_{1,k},\ R'_{2,k}, \ldots,$$
$$p_k',\ p'_{k+1}, \ldots,\ \rho_{0(k)},\ \rho_{0(k+1)}, \ldots, \text{ etc.}$$

entsprechen "m. B.," das heißt: *modo Bernoulliano**, dem Schema einer zufälligen Variablen mit einem konstanten Verteilungsgesetz und N von einander unabhängigen Versuchen. Ist dadurch schon bewiesen, daß die ursprüngliche Reihe $u_1, u_2, \ldots u_N$ eine solche Komponente wirklich besitzt? Ist es überhaupt sicher, daß die gefundenen Werte, wenn auch ebenfalls nur "modo Bernoulliano," die gesuchten sind? Und wenn nicht, so was haben wir denn eigentlich erreicht?

Die erste und zweite Frage sind entschieden zu verneinen. Es wäre naiv zu glauben, daß die Differenzenmethode *oder eine beliebige andere Methode* imstande wären, das Kunststück zu vollbringen,—ohne irgendwelche apriorische Kenntnisse über die zu zerlegende Reihe zu besitzen (über solche Kenntnisse verfügt man in der Praxis nur in den seltensten Fällen) und ohne mehr oder weniger gewagte Hypothesen aufzustellen,—aus N gegebenen Gleichungen vom Typus $u_i = G_i + s_i$ ganze $2N$ Unbekannte ("G" und "s") zu bestimmen†. Ohne derartige Hypothesen kann man eben beim Zerlegen statistischer Reihen nicht auskommen. Aber gerade in dieser Beziehung ermöglicht die Anwendung der Differenzenmethode, einen Schritt weiter zu tun, als es sonst in der Regel üblich ist, denn bei ihr wird das evident gehalten, was bei anderen Methoden nur im stillen und unbewußt hineingeschmuggelt wird.

(1) Eine jede gegebene statistische Reihe von N Gliedern kann durch eine Parabel, nicht höher als $(N-1)$ Ordnung genau wiedergegeben werden. (2) Eine jede gegebene statistische Reihe kann durch eine Fourier-Serie oder durch eine andere trigonometrische Funktion genau dargestellt werden. Je höher die Ordnung der Parabel, je größer die Zahl der Parameter in einer trigonometrischen Funktion, desto genauer wird sich, im Allgemeinen, die berechnete "ausgeglichene" Kurve den Gliedern einer gegebenen Reihe anpassen und desto geringere Werte werden die *scheinbaren* "restlichen Komponenten" aufweisen, bis sie, endlich, ganz verschwinden. Wo hat hier der Forscher Halt zu machen? Wo, bei welcher Potenz der Parabel, bei welcher Parameter-Zahl der trigonometrischen Formel hat er Grund festzustellen: dies hier ist ungefähr die "glatte" Komponente und das dort dürfte nur eine Funktion der wahren "restlichen Komponente" ausmachen? So naheliegend eigentlich diese Frage ist, kann man doch nicht behaupten, daß

* Ein Ausdruck Prof. W. Romanowsky's, der in der russischen statistischen Litteratur sich einzubürgern scheint.

† Wie es die neueren Untersuchungen Prof. Al. A. Tchouproff's und eines seiner Schüler (J. Morduch) erwiesen haben, kann man, überhaupt—sogar aus dem Studium einer einfachen (nicht zusammengesetzten) statistischen Reihe heraus—lange nicht alle Eigenschaften der ihr zu Grunde liegenden zufälligen Variablen absolut sicher bestimmen. So, zum Beispiel, ist es direkt unmöglich, den Fall der gegenseitigen Unabhängigkeit der an einer zufälligen Variablen angestellten Versuche von einigen anderen Fällen, etwa vom Falle einer "uniformen Verbundenheit" der Versuche, zu unterscheiden. (Vergl. Al. A. Tchouproff, "Ist die normale Stabilität empirisch nachweisbar?" in *Nordisk Statistisk Tidskrift*, Band 1, 1922, Häf. 3, 4.) Es liegt hier sogar kein bescheidenes "ignoramus," sondern geradezu ein anmaßendes "ignorabimus" vor.

sie bis jetzt eine eindeutige Antwort erhalten hat. Viele Forscher machen eben gerade dort halt, wo es in ihre Theorien am besten paßt, oder wo ihnen die Rechenarbeit beschwerlich zu werden beginnt. Und da die menschliche Geduld eine stark veränderliche Größe ist, so bleibt der eine schon bei einer Geraden stehen und geht der andere bis in die hohen Parabelordnungen hinein.

Die Differenzenmethode erlaubt hier eine objektivere Stellung einzunehmen. Sich ihrer bedienend, kann der Forscher etwa folgendes feststellen. Von einer gewissen k-ten Differenz angefangen, sind bestimmte Koeffizienten für eine oder für mehrere statistische Reihen ungefähr stabil geworden. Das *könnte* bedeuten, daß die "glatte" Komponente angenähert durch eine Parabel $(k-1)$-ter Ordnung darstellbar sei, und das *könnte* ferner bedeuten, daß, die "restliche" Komponente als eine zufällige Variable mit konstantem Fehlergesetz und gegenseitiger Unabhängigkeit der einzelnen Versuche angesehen werden könne. Und je länger die Reihe der stabilen Koeffizienten ("modo Bernoulliano," natürlich),—desto wahrscheinlicher der Schluß. Oder: die Reihe der Koeffizienten wird nicht stabil; es ist also *unwahrscheinlich*, daß die "restliche" Komponente eine derartige zufällige Variable sei; wahrscheinlicher sei, daß sie diese oder jene anderen Eigenschaften besitze. Oder: die mittleren Fehler sind zu groß und das Material läßt verschiedene Deutungen zu; es ist, folglich, für eine Zerlegung in einzelne Komponenten überhaupt wenig geeignet, u.s.w.*

Haben solche Feststellungen theoretischen Wert? In manchen Fällen doch wohl! Können sie dadurch widerlegt werden, daß man etwa beweist, daß kurzperiodische Sinus-Komponenten unerklärbaren Ursprunges die zufällige Komponente in gewissen Fällen darstellen oder verdecken können? Ich glaube nicht. Denn, erstens, ist die Annahme der Existenz einer zufälligen Beobachtungskomponente mit gegenseitiger Unabhängigkeit der Glieder doch schließlich diejenige Hypothese, auf welcher bis jetzt, trotz aller Kritik, der größte Teil der Fehlertheorie aufgebaut ist; und, zweitens, ist die vorgeschlagene Hypothese jedenfalls nicht besser begründet. Kein Forscher verwirft aber eine alte Hypothese, die einen gegebenen Erscheinungskomplex befriedigend erklärt, nur deshalb, weil derselbe auf eine andere, nicht weniger künstliche, Weise *auch* erklärt werden könnte†.

* Die Frage, ob das gegebene Material, bei Annahme einer bestimmten Formel für die "glatte" Komponente, noch die Hypothese der Anwesenheit einer "zufälligen" restlichen Komponente überhaupt zuläßt, wurde bis jetzt meistenteils doch wohl nur von denjenigen Forschern untersucht, welche bei Sterblichkeitstafel-Ausgleichungen das Lexis'sche Kriterium Q^2 benutzten. So, zum Beispiel, in Rußland von Herrn B. Jastremsky (seine Methode zur "Auffindung des sich verändernden Niveau's einer statistischen Reihe" läuft letzten Endes auch auf die Berechnung von Q^2 hinaus). Es fragt sich nur ob der Lexis'sche Divergenzkoeffizient für einen solchen Gebrauch wirklich geeignet ist (vergl. unten § 12).

† Und vollkommends geht es nicht an, nur auf Grund der theoretischen *Möglichkeit* eines störenden Einflusses von, sagen wir, Yules kurzperiodischen Sinus-Komponenten, behaupten zu wollen, daß diese auch wirklich stören. Ihre Existenz müßte zu allererst nachgewiesen werden, was bis jetzt einwandfrei noch nicht gelungen ist. Ein drastischer Vergleich. Wenn ein Meteorstein auf eine Lokomotive herabfällt, so dürfte es voraussichtlich ein Eisenbahnunglück geben. Folgt aber daraus, daß man in der Eisenbahn nicht fahren soll, *weil* Meteorsteine auf Lokomotiven fallen?

(10) Zur Kontrolle, untersuchen wir jetzt die Frage, was man *mit Sicherheit* aus einem bestimmten Verhalten der Größen $\sigma_k'^2$, σ'^2_{k+1} u.s.w. über die Beschaffenheit der restlichen Komponente folgern kann. Wie ein jedes "umgekehrtes" Problem, ist diese Frage schwieriger zu behandeln, als die "direkte."

Das denkbar allgemeinste stochastische Schema zur Darstellung der restlichen Komponente dürfte wie folgt lauten: "An N zufälligen Variablen—$S_1, S_2, \ldots S_N$—werden N Versuche vorgenommen—je ein Versuch an jeder derselben. Die Variablen mögen beliebigen Verteilungsgesetzen folgen und unter einander in beliebigen Abhängigkeitsbeziehungen stehen*." Wie es aber Professor A. Tchouproff selbst gezeigt hat†, kommt aus einem derartigen Schema nichts rechtes heraus, da, ohne bestimmte Annahmen über die Verteilungsgesetze und die gegenseitige Verbundenheit der Versuche, die Formeln immer beträchtlich mehr Unbekannte aufweisen werden, als es verschiedene Gleichungen aufgestellt werden können. Führen wir deshalb noch 2 beschränkende Bedingungen ein. Erstens, mögen die mathematischen Erwartungen beliebiger Potenzen aller Variablen konstant sein:

$$Es_1^h = Es_2^h = \ldots Es_N^h, \text{ bei } h = 1, 2, 3, \ldots$$

und zweitens, möge die Differenz

$$Es_i s_{i+j} - Es_i Es_{i+j} = E(s_i - Es)(s_{i+j} - Es),$$

von i unabhängig und bloß von j abhängig sein‡.

* Vergl. Al. A. Tchouproff, "Zur Theorie der Stabilität statistischer Reihen," *Skandinavisk Aktuarietidskrift*, 1919, S. 203.

† Al. A. Tchouproff, "On the mathematical expectation of the moments of frequency distributions in the case of correlated observations," *Metron*, Vol. II. N. 3—4, 1923, pp. 7, 8.

‡ Vergl. Tchouproff, "Zur Theorie der Stabilität, etc.," S. 209. O. Anderson, "Über ein neues Verfahren, etc.," S. 137. (Die beschränkenden Bedingungen sind in der letzten Schrift etwas allgemeiner gehalten, als oben. Der Gewinn ist aber nicht besonders groß und verdunkelt nur die Formeln.) Czuber regt ähnliches an. Auch bei A. A. Markoff finden sich ähnliche Gedankengänge. Dadurch, daß nicht alle $Es_is_{i+j} - Es_iEs_{i+j}$ bei beliebigen j einander gleich gesetzt werden (was zu einer "uniform-verbundenen Reihe" führt), sondern daß "Ketten" gebildet werden:

$$Es_1s_2 - Es_1Es_2 = Es_2s_3 - Es_2Es_3 = Es_3s_4 - Es_3Es_4 = \ldots = Es_{N-1}s_N - Es_{N-1}Es_N,$$
$$Es_1s_3 - Es_1Es_3 = Es_2s_4 - Es_2Es_4 = Es_3s_5 - Es_3Es_5 = \ldots = Es_{N-2}s_N - Es_{N-2}Es_N,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Es_1s_{1+j} - Es_1Es_{1+j} = Es_2s_{2+j} - Es_2Es_{2+j} = \ldots = Es_{N-j}s_N - Es_{N-j}Es_N,$$
$$Es_1s_{N-1} - Es_1Es_{N-1} = Es_2s_N - Es_2Es_N,$$

gewinnt man einen höheren Grad der Allgemeinheit und nähert sich den in der Praxis zu beobachtenden Verhältnissen. Man denke, zum Beispiel, an Messungen, die an einer Reihe von Generationen unternommen werden: Vater—Sohn—Enkel—Urenkel, u.s.w.; andererseits—an Temperaturkurven eines Malaria-Kranken, an Auftretungsdichtigkeiten gewisser Insekten mit einer mehrjährigen Verwandlungsdauer, wie, zum Beispiel, Maikäfer (*Melolontha vulgaris*), u.s.w.

In der Theorie der Stabilität statistischer Reihen werden gewöhnlich ganz andere beschränkende Bedingungen eingeführt. Man nimmt, nämlich, in der Regel an, daß an jeder der Variablen S je k Versuche vorgenommen werden, wobei die Reihenfolge der Variablen durch das Los bestimmt wird. Auf eine in der Zeit sich verändernde statistische Reihe angewandt, bedeutet das die Annahme einer serienweisen Veränderung der Grundwahrscheinlichkeiten, also von Beobachtungsreihe zu Beobachtungsreihe, so daß während einer solchen eine *konstante* Wahrscheinlichkeit herrscht (vergl. Czuber,

Bezeichnet man noch

$$r_j = \frac{E(s_i - Es)(s_{i+j} - Es)}{E(s_i - Es)^2}, \text{ (folglich, ist } r_0 = 1) \quad \ldots\ldots\ldots\ldots(30),$$

und denkt sich diese Koeffizienten in der Reihenfolge

$$r_{k+j},\ r_{k+j-1},\ r_{k+j-2}, \ldots\ r_j,\ r_{j-1},\ r_{j-2}, \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(31),$$

geordnet, so erhält man leicht die beiden apriorischen Formeln, welche bei den gegebenen Annahmen genau sind, den empirischen Annäherungs-Formeln (28) und (29) zu Grunde liegen und, schließlich, diejenigen Bedingungsgleichungen ergeben, welchen die Koeffizienten r_j genügen müssen, damit genau

$$E\sigma_i'^2 = E\sigma'^2_{i+1} = \ldots :$$

$$\frac{E\sum_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s_{i+j}}{N-k-j} = \mu_2 (-1)^k \Delta^{2k} r_{k+j} \text{ und } \frac{E\sum_{i=1}^{N-k} \Delta^k s_i^2}{N-k} = \mu_2 (-1)^k \Delta^{2k} r_k \ldots * \quad \ldots\ldots\ldots\ldots\ldots\ldots(32).$$

Die genannten Bedingungsgleichungen können in zwei verschiedenen Systemen dargestellt werden, die auf einander unschwer zurückzuführen sind:

$$\left.\begin{array}{ll}
\text{bei } i = 1, & -r_1 + r_2 = 0 \\
\text{,, } i = 2, & -5r_1 + 8r_2 - 3r_3 = 0 \\
\text{,, } i = 3, & -7r_1 + 14r_2 - 9r_3 + 2r_4 = 0 \\
\text{,, } i = 4, & -42r_1 + 96r_2 - 81r_3 + 32r_4 - 5r_5 = 0 \\
\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\text{,, } i = k-1, & -1^2 C_{2k}^{k+1} r_1 + 2^2 C_{2k}^{k+2} r_2 - 3^2 C_{2k}^{k+3} r_3 \\
& \quad + 4^2 C_{2k}^{k+4} r_4 - \ldots + (-1)^k k^2 C_{2k}^{2k} r_k = 0
\end{array}\right\} \ldots\ldots(33),$$

oder:

$$\left.\begin{array}{ll}
\text{bei } i = 1, & -\nabla' r_1 = 0 \\
\text{,, } i = 2, & -2\nabla' r_1 - 3\nabla'' r_1 = 0 \\
\text{,, } i = 3, & -2\nabla' r_1 - 3\nabla'' r_1 - 2\nabla''' r_1 = 0 \\
\text{,, } i = 4, & -10\nabla' r_1 - 15\nabla'' r_1 - 12\nabla''' r_1 - 5\nabla'''' r_1 = 0 \\
\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\text{,, } i = k, & -1.2\, C_{2k-2}^{k-1} \nabla' r_1 - 2.3\, C_{2k-3}^{k-1} \nabla'' r_1 - 3.4\, C_{2k-4}^{k-1} \nabla''' r_1 \\
& \quad - 4.5\, C_{2k-5}^{k-1} \nabla'''' r_1 - \ldots - k(k+1)\, C_{k-1}^{k-1} \nabla^{(k)} r_1 = 0
\end{array}\right\} \ldots(33a).$$

Das umgekehrte Symbol ∇ wurde deshalb eingeführt, weil die Reihe (31) hier als in umgekehrter Richtung (von rechts nach links) differenziert gedacht

loc. cit., II. S. 42); demgegenüber setzt das von uns hier angewandte Schema eben eine stetige und sozusagen gesetzmäßige Veränderlichkeit der Grundwahrscheinlichkeiten voraus.

Es läßt sich natürlich darüber streiten, welches der beiden stochastischen Schemen allgemeiner und weniger gezwungen ist. Wir, wenigstens, glauben, daß sie als ziemlich ebenbürtig einander gegenübergestellt werden können. Jedenfalls führt unser Schema, auf den Lexis'schen Divergenzkoeffizienten angewandt, zu einigen ganz interessanten und vielleicht auch ziemlich unerwarteten Resultaten. Vergl. hierüber unten § (12).

* O. Anderson, *Über ein neues Verfahren, etc.*, S. 142 und 143.

wird; es ist also $\nabla' r_1$ gleich $r_1 - r_2$ und nicht $r_1 - r_0$, $\nabla'' r_1$ gleich $r_1 - 2r_2 + r_3$ und nicht $r_1 - 2r_0 + r_{-1}$, $\nabla''' r_1$ gleich $r_1 - 3r_2 + 3r_3 - r_4$ und nicht $r_1 - 3r_0 + 3r_{-1} - r_{-2}$, u.s.w. Diese Bezeichnung ist mit der von den Aktuarien gewöhnlich gebrauchten nicht zu verwechseln. Es ist noch zu beachten, daß die unterste Formel in (33) dem Falle $i = k - 1$ und in (33a) dem Falle $i = k$ entspricht.

Ist schon $E\sigma_1'^2 = E\sigma_2'^2 = E\sigma_3'^2 = \ldots = E\sigma_i'^2$, so entspricht das dem Falle $i = 1$, und alle Bedingungsgleichungen (33) und (33a) müssen erfüllt sein. Ist erst, zum Beispiel, $E\sigma_3'^2 = E\sigma_4'^2 = \ldots = E\sigma_i'^2$, so entspricht das dem Falle $i = 3$ und alle Gleichungen des Systems (33) müssen erfüllt sein, *ausgenommen die beiden ersten*, wo die linken Seiten nicht Null zu sein brauchen. Ebenso im System (33a). Andererseits, sind die Bedingungsgleichungen von einem gewissen i abwärts erfüllt, so müssen auch die Beziehungen bestehen: $\sigma_i'^2 = \sigma'^2_{i+1} = \sigma'^2_{i+2} = \ldots.$ Reihen, für welche solche Beziehungen gelten, bezeichne ich als zur "Gruppe R" gehörig.

Der Inhalt der Bedingungsgleichungen (33) und (33a) ist nicht so leicht zu erschöpfen, und wir werden uns hier nur mit dem Nötigsten befassen.

Ist i gleich 1, so verfügen wir über $k-1$ verschiedene Gleichungen zur Bestimmung von k Unbekannten: $r_1, r_2, \ldots r_k$. Wir erhalten aus ihnen leicht:

$$r_1 = r_2 = r_3 = \ldots = r_k,$$

das heißt, die Bedingungen einer "uniformen Reihe" Tchouproff's.

Ist i gleich 2, so besitzen wir nur $(k-2)$ Gleichungen für k Unbekannte und kommen zum System:

$$r_3 - r_4 = \frac{7}{3}(r_1 - r_2); \; r_4 - r_5 = \frac{9}{3}(r_1 - r_2);$$
$$r_5 - r_6 = \frac{11}{3}(r_1 - r_2); \ldots r_{k-1} - r_k = \frac{(2k-1)}{3}(r_1 - r_2).$$

Die Reihe $r_1, r_2, r_3, \ldots r_k$ wird, folglich, durch gewisse Parabeln 2-ter Ordnung genau ausgedrückt. Wie man sieht, werden dabei die Differenzen $r_i - r_{i+1}$, welche durchwegs dasselbe Vorzeichen behalten, mit wachsendem i immer größer. Ein derartiges Verhalten der Reihe r entspricht so wenig demjenigen, welches wir von der Wirklichkeit erwarten können, daß es wohl bei weitem wahrscheinlicher bleibt, daß $r_1 - r_2 = 0$. Wir kehren, folglich, auch hier zum System der "uniformen" Verbundenheit der Versuche zurück.

Ist, überhaupt, i gleich j, so verfügt man über nicht mehr als $k-j$ Gleichungen zur Bestimmung von k Unbekannten. Es können, folglich, nur je $(k-j)$ Unbekannte durch gewisse lineare Funktionen der übrigen j ausgedrückt werden. Es ist aber möglich, ganz allgemein zu beweisen, daß auch in diesem Falle die Reihe (31) durch Parabeln jedenfalls nicht höher, als $(2j-1)$-ter Ordnung darstellbar sein müßte*. Je größer die letzte erreichte Differenz k im Vergleich zu $(2j-1)$ ist, desto weniger wahrscheinlich wird ein so gesetzmäßiges Verhalten

* Ebenda, S. 146 Anmerkung.

der Reihe der r, und desto wahrscheinlicher, also, die frühere Annahme einer "uniformen" Reihe*.

Wenden wir uns jetzt den Bedingungen zu, unter welchen die Reihe $E\sigma_1'^2$, $E\sigma_2'^2$, $E\sigma_3'^2, \ldots$ *nicht konstant werden kann.*

Es kann bewiesen werden†, daß wenn, von einem gewissen i angefangen, in System (33) und (33a) die linken Seiten aller Gleichungen > 0 werden, auch $E\sigma_i'^2 < E\sigma'^2_{i+1} < E\sigma'^2_{i+2} < \ldots$ sein müssen. Reihen, für welche solche Beziehungen bestehen, gehören zur "Z-Gruppe."

Und umgekehrt, wenn diese linken Seiten in (33) und (33a), von i angefangen, alle < 0 sind, so haben wir, im Gegenteil, $E\sigma_i'^2 > E\sigma'^2_{i+1} > E\sigma'^2_{i+2} > \ldots$ Reihen mit solchen Beziehungen zwischen ihren Streuungen gehören zur "G-Gruppe."

Betrachten wir anfangs den zweiten Fall und begnügen wir uns mit der Untersuchung nur derjenigen Reihen, für welche *von Anfang an*

$$E\sigma_0'^2 \ddagger > E\sigma_1'^2 > E\sigma_2'^2 > E\sigma_3'^2 > \ldots.$$

(Im Verhältnis zu den Streuungen der "R-Gruppe" werden also die Streuungen dieser Gruppe mit jeder Differenz immer geringer.) Dann können wir zum System (33a) noch die fehlende Ungleichung $-r_1 < 0$ hinzufügen und erhalten dann: $r_1 > 0$; $r_1 - r_2 > 0$; $r_1 - r_2 > \frac{3}{2}(r_1 - 2r_2 + r_3)$, u.s.w. Die Ungleichungen werden mit wachsenden i immer komplizierter und weniger durchsichtig. Um zu einfacheren Formeln zu gelangen, wollen wir jetzt noch eine weitere begrenzende Bedingung einführen: die linken Seiten des Systems (33a) werden nämlich *a fortiori* < 0 sein, wenn alle $\nabla^{(i)} r_1 > 0$ (bei $i = 1, 2, 3, \ldots k$). Und dann kommen wir zum System:

$$r_1 > 0;\; r_1 > r_2;\; \nabla' r_1 > \nabla' r_2;\; \nabla'' r_1 > \nabla'' r_2, \ldots \nabla^{k-1} r_1 > \nabla^{k-1} r_2.$$

Dieses System ist viel handlicher und kann durch verschiedene "Modelle" dargestellt werden. Am einfachsten würde wohl die Annahme sein: $r_1 = p$, $r_2 = p^2$, $r_3 = p^3, \ldots r_i = p^i$. In diesem Falle ergiebt es sich nämlich, daß $\nabla^i r_i = p(1-p)^i$, und dieser Ausdruck bleibt immer positiv, solange $1 > p > 0$.

* Besäßen wir Kenntnis über den genauen Wert von Es, so könnten wir leicht, falls schon $\frac{E\sum_{i=1}^{N}(s_i - Es)^2}{N} = \frac{E\sum_{i=1}^{N-1}(\Delta' s)^2}{2(N-1)}$, die fehlende Gleichung $r_1 = 0$ erhalten und das System (33) eindeutig auflösen. Uber eine solche Kenntnis verfügen wir aber nur in den seltensten Fällen. Aus der Gleichung $\frac{E\sum_{i=1}^{N}(s_i - s_{(N)})^2}{N-1} = \frac{E\sum_{i=1}^{N-1}(\Delta' x_i)^2}{2(N-1)}$, wo $s_{(N)} = \frac{1}{N}\sum_{i=1}^{N} s_i$, erhalten wir die fehlende Gleichung *nicht*, da in ihr, dank der Einführung von $s_{(N)}$, alle r_i von r_1 bis r_{N-1} auftreten werden. Und wenn man sogar bis zur $(N-1)$-ten Differenz vorgeht, so erweist es sich, daß diese Gleichung schon im vollen System (33) enthalten ist, also nichts neues bietet.

Direkt ist kein einziges r_i zu bestimmen, da es keine einzige empirische Formel gibt und *geben kann*, deren mathematische Erwartung genau r_i ausmachen würde. Vergl. hierüber noch unten, eine Anmerkung in § (12).

† Ebenda, S. 142—143.

‡ Siehe Form. (9).

Andererseits, würden wir auch in dem Falle zu einfachen Formeln gelangen, wenn wir wieder annehmen, daß die Reihe r durch eine gewisse Parabel i-ter Ordnung dargestellt werden könne. Dann würden alle endlichen Differenzen von der Ordnung $(i+1)$ aufwärts gleich 0 sein. Verschwinden, zum Beispiel, schon die zweiten Differenzen, so erhalten wir folgendes leicht faßliches System:

$$r_1 - r_2 = r_2 - r_3 = r_3 - r_4 = \ldots = r_{k-1} - r_k > 0.$$

Verschwinden erst die 3-tten Differenzen, so kommen wir zum System:

$$\frac{r_1 + r_3}{2} - r_2 = \frac{r_2 + r_4}{2} - r_3 = \ldots = \frac{r_{k-2} + r_k}{2} - r_{k-1} > 0, \text{ u.s.w.}$$

Solch ein Verhalten würde für eine "zufällige" Variable, als welche wir die "restliche Komponente" ansehen, wenig wahrscheinlich sein, *könnte aber sehr wohl ein Gegenstück für die Hypothese einer parabolischen Beschaffenheit der "glatten Komponente" abgeben.*

Schließlich sei noch bemerkt, daß auch Yule's Sinus-Reihen mit der Formel $u_i = A \sin\left(2\pi \frac{t + \tau + ih}{T}\right)$ im Zwischenraum $0 \leqq h \leqq \frac{1}{4}T$ und $T \geqq h \geqq \frac{3}{4}T$ ebenfalls zur "G-Gruppe" gehören.

Was nun den 3-tten Fall anbetrifft, das heißt, die Bedingungen für eine solche "restliche Komponente," welche beim endlichen Differenzieren rascher wächst, als es eine Reihe mit gegenseitig unabhängigen oder uniform verbundenen Versuchen zuläßt ("Z-Gruppe"), so können, wie gesagt, zu ihrer Untersuchung ebenfalls die Formeln des Systems (33a) zugezogen werden, indem hier vor 0 überall das Zeichen > gesetzt wird. Und ebenfalls *a fortiori* kann man auch folgende Bedingungen aufstellen:

$$r_1 < 0,\ r_1 < r_2,\ \nabla' r_1 < \nabla' r_2,\ \nabla'' r_1 < \nabla'' r_2, \ldots \nabla^{k-1} r_1 < \nabla^{k-1} r_2.$$

Diesem System würde, zum Beispiel, eine jede Reihe s genügen, bei welcher alle r_i bei ungeradem i negativ und bei geradem i positiv sein würden. Zum Beispiel:

(1) $r_1 = p,\ r_2 = p^2,\ r_3 = p^3, \ldots r_i = p^i$, wenn $0 > p \geqq -1$; oder:

(2) $r_1 = A - a,\ r_2 = A + a, \ldots r_i = A + (-1)^i a$, wenn $a > |A|$ und $|A| + a < 1$; oder:

(3) $r_i = A + (-1)^i (a + ih)$, wobei jedoch $|kh| < a > 0$ sein muß, wenn h negativ ist, und außerdem $|A| + a + |kh| < 1$, u.s.w.

In diese Klasse gehören auch Yule's kurzperiodische Sinus-Reihen.

(11) Wenn wir nun von diesen wahrscheinlichkeitstheoretischen Schemata zur statistischen Wirklichkeit zurückkehren, wo, in der Regel, weder r_i noch $E\sigma_i'^2$ gegeben sind, so ist es am vorteilhaftesten, von der Betrachtung der Differenzen $\sigma'^2_{i+1} - \sigma_i'^2,\ \sigma'^2_{i+2} - \sigma'^2_{i+1}, \ldots\ \sigma_k'^2 - \sigma'^2_{k-1}$, auszugehen. Können wir an Hand der Formeln der Tabelle II feststellen, daß diese Differenzen relativ noch so klein sind, daß man *modo Bernoulliano* das Bestehen der Gleichungen $E\sigma_i'^2 = E\sigma'^2_{i+1} = \ldots$ annehmen darf, so kann man, auf Grund des im vorhergehenden Paragraphen gesagten,—mit einer Wahrscheinlichkeit, die desto größer

wird, je weiter der Zwischenraum zwischen i und k,—annehmen, daß die i-ten endl. Differenzen der Reihe $u_1, u_2, \dots u_N$ nur lineare Funktionen einer zur "R-Gruppe" gehörigen Komponente s enthalten. Man kann dabei mit einer ziemlichen Sicherheit, wenigstens für größere N, erwarten, daß die Glieder derselben entweder unabhängig von einander oder, im schlimmsten Fall, miteinander nur uniform verbunden sind. Bleiben die Differenzen der Streuungen bis $\sigma_k'^2 - \sigma'^2_{k-1}$ im Verhältniss zu ihren mittleren Fehlern groß und dabei negativ, so kann man noch mit höheren Differenzen sein Glück versuchen. Sind sie, im Gegenteil, groß und positiv, so bedeutet das die wahrscheinliche Anwesenheit einer Komponente von der "Z-Gruppe."

Es muß hierbei noch folgendes bemerkt werden. Gehört die Reihe s zur "R-Gruppe," so besitzt die Reihe $\Delta's_1, \Delta's_2, \dots$ schon die Eigenschaften einer "Z-Gruppe," und je höher k, die Ordnung der Differenz, desto mehr werden diese "Z"-Eigenschaften hervortreten. Besteht nun jedes Glied der Reihe: $s_1, s_2, s_3, \dots$ selbst aus einer Summe von 2 Komponenten: einer "R"-artigen und einer "Z"-artigen, so kann die "schädliche" Wirkung der letzteren in den meisten Fällen sich nur in den wenigen ersten Differenzen bemerkbar machen. Das Maximum der "Schädlichkeit" erreicht eine Komponente der "Z-Gruppe" bei

$$r_1 = -1,\ r_2 = +1, \dots r_i = (-1)^i.$$

Nun ist für eine solche Reihe

$$\frac{E \sum\limits_{i=1}^{N-k} (\Delta_i^k)^2}{N-k} = 4^k \mu_2,$$

während eine von der "R-Gruppe" bloß $\frac{2k!}{k!\,k!}\mu_2$ ergeben würde, oder, angenähert, $\frac{4^k}{\sqrt{\pi k}}\mu_2$ (vergl. Form. 11a). Für $k = 6$ würde das eine $4\frac{100}{231} \times$ fache Vergrößerung des "relativen Einflusses" der *maximalen* "Z"-Komponente bedeuten. Eine weitere Verdoppelung desselben tritt aber nur bei $k = 24$ ein. Ist daher, zum Beispiel, an der Differenz $\sigma_6'^2 - \sigma_5'^2$ die Wahrscheinlichkeit des Vorhandenseins einer "Z-Gruppen"-Komponente nicht festzustellen, so ist es auch verhältnismäßig wenig wahrscheinlich, daß die weiteren Differenzen eine solche Komponente zum Vorschein treten lassen könnten (wenn, natürlich, in der 6-ten Differenz die Komponente "G" schon eliminiert ist). Diese "Z"-Komponente existiert, offenbar, überhaupt nicht oder ist im Verhältnis zur "R"-Komponente nicht bedeutend.

Es entsteht nun die Frage: sind die möglichen Ergebnisse einer Untersuchung der Streuungen der Differenzen einer gegebenen Reihe auch genügend wichtig, um die hierbei erforderliche Rechenarbeit zu rechtfertigen? hat die Unterscheidung der Gruppen "Z," "R" und "G" irgendeinen Erkenntniswert auch außerhalb der Anwendung der Differenzenmethode?

(*Ein Schlußartikel folgt.*)

ON THE MEANS AND SQUARED STANDARD-DEVIATIONS OF SMALL SAMPLES FROM ANY POPULATION.

By A. E. R. CHURCH, M.A., B.Sc.

Part I. Introductory.

The theory of sampling, both as to mean and standard-deviation, has been known with completeness for some considerable time, provided the sampled population follows the *normal* law. That the distribution of means follows the normal law whatever the size of the sample, provided the sampling is from normal material, has been known from the early days of probability theory and the moment relations of the distribution of means of such samples were determined by Tchebycheff many years ago. The distribution of standard-deviations of samples from normal material has also been fully worked out both for small and large samples, and actual experimental verification of the theoretical formulae obtained was provided by "Student" in *Biometrika*, Vol. VI. pp. 1—25.

While the work indicated above has rounded off the case of large or small samples from a normal population, both from the point of view of their means and their standard-deviations, it gives us no ability to deal with the case of samples from material which is definitely skew in character.

However, relations of the type:

$$ {}_1M_1' = \bar{\mu}_1' \qquad \text{(1)}, \qquad\qquad {}_1B_1 = \frac{\bar{\beta}_1}{N} \qquad \text{(5)},$$

$$ {}_1M_2 = \frac{\bar{\mu}_2}{N} \qquad \text{(2)}, \qquad\qquad {}_1B_2 - 3 = \frac{\bar{\beta}_2 - 3}{N} \qquad \text{(6)},$$

$$ {}_1M_3 = \frac{\bar{\mu}_3}{N^2} \qquad \text{(3)}, \qquad\qquad \Sigma_1 = \frac{\bar{\sigma}}{N^{\frac{1}{2}}} \qquad \text{(7)},$$

$$ {}_1M_4 = \frac{\bar{\mu}_4}{N^3} + \frac{3(N-1)}{N^3} \cdot \bar{\mu}_2^2 \qquad \text{(4)},$$

where ${}_1M_1'$, ${}_1M_2$, ${}_1M_3$, ${}_1M_4$, ${}_1B_1$, ${}_1B_2$ and Σ_1 are the mean, second, third and fourth moment-coefficients, first two betas and standard-deviation respectively of the distribution of means of samples of N, whilst $\bar{\mu}_1'$, $\bar{\mu}_2$, $\bar{\mu}_3$, $\bar{\mu}_4$, $\bar{\beta}_1$, $\bar{\beta}_2$ and $\bar{\sigma}$ are the corresponding constants of the sampled population, follow from Tchebycheff's work and enable us to obtain some appreciation of the distribution of means of samples of any size from any population. It is clear from them that, if the deviation of the sampled population from normality be not very great, the distribution of means of samples of N itself will not diverge widely from normality. In fact for $\beta_1 \leqslant 0.2$ and $\bar{\beta}_2 - 3 \leqslant 0.4$ it is to be expected that, even for $N = 10$, a normal curve will give a really excellent fit to an experimental distribution of means.

As regards theoretical distributions of means of samples of N obtained directly without the use of moments from a given sampled population, when this population is a Type III Pearson Curve the corresponding distribution of means has been deduced in Part II by direct integration, the sampled population being considered "infinite" in all its categories *.

The deduction of β-relations for the distribution of σ^2 of samples drawn from any sampled population infinite in all its categories is due to Tchouproff †. At first sight these relations place the distributions of means and σ^2 of samples from non-normal material on the same footing; that is the constants of these distributions can be computed and frequency curves to represent them can be determined. But as far as calculation is concerned, the two cases are on somewhat different footings. Tchouproff's formulae are very lengthy, one of them having 37 terms in it; but, what is much more troublesome in practice, the evaluation of the formula for the 4th moment-coefficient of the distribution of σ^2 of samples involves a computation of the moments of the sampled population up to and including the 8th. Again Tchouproff's formulae are absolute, not approximate, but it would be wrong to suppose that, for small and moderately small samples, the terms in the higher inverse powers of N can be neglected summarily in comparison with the terms in the lower inverse powers of N. For such values of the moment-coefficients as obtain for moderate skewness of the sampled population and moderately small samples, I find there is not such a rapid convergence of the Tchouproff terms as to warrant the general neglect of the terms in the higher inverse powers of N. This is a practical illustration of the theorem developed by Isserlis, namely that the moment-coefficients of the distribution of σ^2 of samples from an infinite population do obey, when the sample is large enough, the normal law, but the sample must be very large before we can look upon the distribution of even the 4th moment-coefficient as normal.

Now the computation of the 8th moment-coefficient of the sampled population, as required by Tchouproff's formulae, can be carried out in a special experimental investigation like this, but can hardly be demanded in everyday statistical practice. Accordingly I looked round for a method of determining the higher betas without direct computation and it seemed desirable to ascertain whether, assuming the applicability of Pearson's general frequency curves which provide finite difference relations ‡ between successive betas, it is possible to determine the higher betas of the sampled population from the lower ones by these formulae with sufficient accuracy to obtain good fits to the experimental distributions of σ^2 of samples obtained from the skew populations employed.

* The term "infinite sampled population" has a quite simple and definite meaning for the statistician, although it seems to have troubled some mathematicians recently (see Burnside: *Camb. Phil. Soc. Proc.* Vol. XXII, pp. 726—7). I indicate its significance below.

† *Biometrika*, Vol. XII. pp. 193—194. For the correction of Tchouproff's formulae in the case of the 4th moment-coefficient by the present writer, see *Biometrika*, Vol. XVII. pp. 79—83.

‡ Tables of β_3, β_4, β_5 and β_6 in terms of β_1 and β_2 are given in *Tables for Statisticians*, pp. 78—79, but as some errors occur they have been recalculated by Professor K. Yasukawa: see pp. 268—275 of the present issue of this Journal.

It is well at this point to draw attention to an important matter. The population from which the samples are taken will be in general itself a sample for a much larger population, and hence its higher moment-coefficients may differ from their "true" values by very considerable variations, by even 40 or 50 per cent. Accordingly on this ground alone we should not be justified—except for the special purpose of testing our formulae experimentally—in laying any marked stress on the exact values of these higher betas of the sampled population.

This method of obtaining rapidly the higher betas of the sampled population will be shown to be successful and thus, expressing Tchouproff's formulae in terms of the betas of the sampled population, we can use for the higher betas their values in terms of the lower ones as obtained by the difference relations.

While formulae for the moment-coefficients of the distributions of means and σ^2 of samples of N from any sampled population may now be said to be established and confirmed experimentally, it must yet be remarked that they are dependent on the sampled population being indefinitely large in all its categories compared with the size of the sample. In practice this means that when any individual is drawn in actual sampling, the chance that the next one drawn will belong to a definite category of the sampled population must be quite unaffected by this previous "draw," or otherwise individuals must be drawn singly and replaced before the next is drawn. This method is generally referred to as sampling from an "infinite" population. If the whole sample be drawn before replacement, the frequency of draws from a single category of the sampled population will no longer follow a binomial but a hypergeometrical distribution. This method is generally referred to as sampling from a "finite" population.

Turning now to the consideration of sampling from a finite population, the moment-coefficients of the distribution of means of samples of N have been known for some time, although it is not clear from whom they originated, whilst Dr J. Splawa-Neyman* has given the mean and 2nd moment-coefficient of the distribution of σ^2 of such samples. However, he did not give formulae for the 3rd and 4th moment-coefficients of the distribution of σ^2 of samples, which moment-coefficients are required if the distribution is to be represented by a Pearson or other frequency curve. These formulae proved very laborious and difficult to obtain, and when obtained were found to be extremely unwieldy and complicated, but they are given together with an outline of the method of obtaining them in the last part of this paper.

This very considerable piece of work has apparently placed the discussion of the distributions of means and σ^2 of samples of N from a finite population on the same basis as that given for the corresponding distributions obtained from an infinite population. This is more or less true for the distribution of means of samples, for like the corresponding distribution from an infinite population, it can be represented by a Pearson curve obtained from the values of the moment-coefficients

* *Biometrika*, Vol. XVII. p. 472. In this paper Dr Neyman also gives the moment-coefficients of the distribution of means of samples.

given by their formulae, and moreover, even when the sampled population is very skew and as small as 36, the distribution shows a remarkable approach to normality.

In considering the distribution of σ^2 of samples of N from a finite population, the formulae for the moment-coefficients being so unwieldy, it is found that even moderate-sized sampled populations must be treated as infinite; that is, Tchouproff's formulae must be used in lieu of the proper formulae for these moment-coefficients, if the labour involved in the representation of the distribution is to be kept within limits that are reasonable even for an investigation like this. In practical statistics the labour involved would undoubtedly prohibit in all cases the use of these proper formulae for the moment-coefficients.

These formulae could be used for very small sampled populations like the binomial considered below, but judging from this case, it is likely that the resulting frequency curve would in no way represent even the general trend of the frequency groups of an actual distribution of σ^2 of samples obtained from the population. This distribution of σ^2 of samples in the case considered proved to be a discrete frequency with widely varying intervals between the separate frequency groups, and was moreover tri-modal. This rather unprecedented and startling result, coupled with the unwieldiness that is certain to appear in any attempt to find formulae for the higher moment-coefficients of the distribution of σ^2 of samples, means that, at present, the representation of such a distribution by a smooth frequency curve is practically impossible, if the sampled population be very small.

For the true application of all this theory, too much emphasis cannot be laid on the fact that the samples in any actual case must be truly "random" in character. The recording of the individual, its return to the "bag" and the thorough shuffling between each draw involve, in the case of 10,000 or 20,000 draws, a very great expenditure of time and labour. At the same time, in my experience and that of other workers in the Biometric Laboratory, it is impossible with many hundreds of marked tickets to ensure the returned ticket being equally likely to be anywhere in the bag or bowl, and so equally likely to be drawn next time as any other ticket. In fact, working with many tickets, perfect shuffling seems a practical impossibility.

However, when the sampling is conducted on the basis of an infinite sampled population, the most satisfactory method seems to be to procure one good random sample, such as that obtained by Mr L. H. C. Tippett, and to apply this to all samplings of this character. This method, which is described later, has also the great advantage of comparative speed and ease of operation, and I wish at this stage to thank Mr Tippett for the loan of his set of random numbers, for it is undoubtedly a boon when sampling from an infinite population.

When the sampling is made on the basis of a finite sampled population, however, such a set of random numbers is at present of no avail. Consequently, whilst discussing the distributions of means and σ^2 of samples of N from such populations, material other than tickets was used in order to find a way of minimising the difficulties of shuffling.

When the sampled population, though finite, is large, quite approximately symmetrical and homogeneous material like small commercial coloured glass beads has been found to overcome the difficulties of shuffling to the extent necessary to produce good agreement between theory and practice. When however the sampled population is quite small, satisfactory results can only be obtained by using material like coloured marbles, which approximate very closely indeed to the obvious ideal of complete symmetry of shape and homogeneity of size and weight.

Turning now to the consideration of the experimental material used, it should be remarked that the number of significant figures given for the moment-coefficients is only adopted in order to obtain the higher betas with the accuracy needful for their use in Tchouproff's formulae*. They can, of course, be dropped as soon as the two betas of the distribution of means or of second moments of samples, as the case may be, have been reached. The two sampled populations used in the discussion of samples from an infinite population are the following:

(A) *Barometric Readings taken over a period of eleven Years at Laudale in Scotland.*

Total Frequency 4011.

Central Bar. Height in ins.	*27·9*	*28·0*	*28·1*	*28·2*	*28·3*	*28·4*	*28·5*	*28·6*	*28·7*	*28·8*	*28·9*	*29·0*	*29·1*	*29·2*	*29·3*
Frequency	1	1	0	0	0	2	2	4	8	12	32	47	72	87	125
Central Bar. Height in ins.	*29·4*	*29·5*	*29·6*	*29·7*	*29·8*	*29·9*	*30·0*	*30·1*	*30·2*	*30·3*	*30·4*	*30·5*	*30·6*	*30·7*	*30·8*
Frequency	187	255	304	346	394	380	400	361	376	266	182	94	50	21	2

The Constants for this distribution were found to be as follows:

[Working Origin 29·7 ins. of Bar. Height. Working Unit 0·1 in. of Bar. Height.]

$$\begin{array}{llll}
\nu_1' = & 1\cdot580{,}155, & \nu_5' = & 1{,}416\cdot561{,}456, \\
\nu_2' = & 17\cdot255{,}547, & \nu_6' = & 57{,}788\cdot534{,}530, \\
\nu_3' = & 49\cdot231{,}613, & \nu_7' = - & 155{,}697\cdot948{,}891, \\
\nu_4' = & 759\cdot882{,}324, & \nu_8' = & 7{,}868{,}534\cdot826{,}228.
\end{array}$$

Mean = 29·858,0155 ins. of Bar. Height.

$$\begin{array}{llll}
\nu_2 = & 14\cdot758{,}657, & \nu_6 = & 70{,}469\cdot386{,}836, \\
\nu_3 = - & 24\cdot676{,}758, & \nu_7 = - & 818{,}237\cdot881{,}078, \\
\nu_4 = & 688\cdot517{,}798, & \nu_8 = & 13{,}875{,}641\cdot654{,}318, \\
\nu_5 = - & 3{,}999\cdot253{,}754, & \sigma = & 0\cdot384{,}1700 \text{ in. of Bar. Height.}
\end{array}$$

$$\beta_1 = 0\cdot189{,}424, \quad \beta_2 = 3\cdot160{,}978, \quad \beta_3 = 2\cdot080{,}078,$$

$$\beta_4 = 21\cdot920{,}980, \quad \beta_6 = 292\cdot459{,}489.$$

The Skewness is ·237.

* Some of the powers and products of moment-coefficients have to be multiplied by numerical coefficients of the order of 1800.

(B) *A Graduated Frequency Distribution obtained from a smooth Frequency Curve given in* Biometrika, Vol. XI. p. 400.

Total Frequency 10,000.

Central Abscissa of Group	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2
Frequency	1	2	4	8	15	27	47	78	125	191	282	399	541
Central Abscissa of Group	−1	0	1	2	3	4	5	6	7	8	9	10	11
Frequency	700	861	1005	1101	1127	1064	910	691	451	241	98	27	4

The Constants for this distribution were found to be:

$$\nu_1' = 1{\cdot}8518, \qquad \nu_5' = 2{,}274{\cdot}2528,$$
$$\nu_2' = 16{\cdot}1684, \qquad \nu_6' = 35{,}855{\cdot}6624,$$
$$\nu_3' = 55{\cdot}8272, \qquad \nu_7' = 95{,}052{\cdot}5432,$$
$$\nu_4' = 628{\cdot}5896, \qquad \nu_8' = 2{,}676{,}103{\cdot}2296.$$

$$\text{Mean} = 1{\cdot}8518.$$

$$\nu_2 = 12{\cdot}739{,}237, \qquad \nu_6 = 38{,}480{\cdot}027{,}968,$$
$$\nu_3 = -\ 21{\cdot}294{,}481, \qquad \nu_7 = -\ 329{,}631{\cdot}122{,}867,$$
$$\nu_4 = 512{\cdot}453{,}377, \qquad \nu_8 = 4{,}368{,}582{\cdot}223{,}120,$$
$$\nu_5 = -\ 2{,}571{\cdot}063{,}549, \qquad \sigma = 3{\cdot}569{,}207.$$
$$\beta_1 = 0{\cdot}219{,}333. \qquad \beta_2 = 3{\cdot}157{,}676, \qquad \beta_3 = 2{\cdot}078{,}769,$$
$$\beta_4 = 18{\cdot}612{,}518, \qquad \beta_6 = 165{\cdot}869{,}619.$$

The Skewness is ·263.

Since for the purpose of sampling the material must be arranged in groups and the individuals of one group all given the *same* class-index, it is not only unnecessary, but even erroneous, to apply Sheppard's grouping corrections to the moment-coefficients, before using them to determine the distributions of means and squared standard-deviations of samples. Consequently in the preceding values, and throughout this paper, the corrections for grouping have not been used, the betas being calculated directly from the raw moments (ν's) transferred to the mean.

The sets of samples taken from these two populations (A) and (B) are indicated in the following, the constants of the corresponding distributions of means and squared standard-deviations (second moments) of samples being given at the same time.

I. 1000 Samples of 10 from Population (A), the sampling being on the basis of an "infinite" population.

Method of Sampling, that of drawing single tickets with return from a bag.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	29·275–	29·325–	29·375–	29·425–	29·475–	29·525–	29·575–	29·625–	29·675–	29·725–	29·775–
Frequency	1	1	0	7	9	18	22	52	83	125	132
Bar. Height in ins.	29·825–	29·875–	29·925–	29·975–	30·025–	30·075–	30·125–	30·175–	30·225–	30·275–	30·325–
Frequency	158	145	101	63	37	29	13	3	0	0	1

The Constants were found to be:

[Working Origin = 29·8 ins. of Bar. Height. Working Unit = 0·05 in. of Bar. Height.]

$\nu_1' = 0·741$, Mean = 29·837,050 ins. of Bar. Height.
$\nu_2' = 8·149$, $\nu_2 = 7·599,919$, $\sigma = 0·137,840$ in. of Bar. Height,
$\nu_3' = 14·367$, $\nu_3 = -2·934,489$, $\beta_1 = 0·019,617$,
$\nu_4' = 213·217$, $\nu_4 = 196·575,508$, $\beta_2 = 3·403,388$.

The Skewness is ·057.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	1·5–	2·5–	3·5–	4·5–	5·5–	6·5–	7·5–	8·5–	9·5–	10·5–	11·5–	12·5–	13·5–
Frequency	1	3	9	18	25	39	48	60	56	64	51	71	52
Value of σ^2	14·5	15·5–	16·5–	17·5–	18·5–	19·5–	20·5–	21·5	22·5–	23·5–	24·5–	25·5–	26·5–
Frequency	65	47	53	46	32	40	43	19	25	16	16	14	16
Value of σ^2	27·5–	28·5–	29·5–	30·5–	31·5–	32·5–	33·5–	34·5–	35·5–	36·5–	37·5–	38·5	39·5–
Frequency	13	5	9	7	3	2	5	3	0	4	2	2	1
Value of σ^2	40·5–	41·5–	42·5–	43·5–	44·5–	45·5–	46·5–	47·5–	48·5–	49·5–	50·5–	51·5–	52·5–
Frequency	1	1	1	1	0	0	2	1	1	2	2	0	3

The Constants were found to be:

[Working Origin = 14.]

$\nu_1' = 1·848$, Mean = 15·848, $\beta_1 = 2·149,615$,
$\nu_2' = 64·578$, $\nu_2 = 61·162,896$, $\beta_2 = 6·560,572$,
$\nu_3' = 1,046·712$, $\nu_3 = 701·313,792$, $\sigma = 7·820,671$.
$\nu_4' = 30,991·482$, $\nu_4 = 24,542·441,607$,

The Skewness is ·643.

II. 1000 Samples of 10 from Population (A), the sampling being on the basis of a "finite" population.

Method of Sampling, that of drawing tickets from a bag, complete samples being drawn before return to bag.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	29·445–	29·485–	29·525–	29·565–	29·605–	29·645–	29·685–	29·725–	29·765–	29·805–
Frequency	1	4	11	16	21	37	69	94	128	120
Bar. Height in ins.	29·845–	29·885–	29·925–	29·965–	30·005–	30·045–	30·085–	30·125–	30·165–	30·205–
Frequency	115	129	89	77	45	26	11	5	1	1

The Constants were found to be:

[Working Origin = 29·825 ins. of Bar. Height. Working Unit = 0·04 in. of Bar. Height.]

$\nu_1' = 0·475$, Mean = 29·844,000 ins. of Bar. Height, $\beta_1 = 0·013,218$,
$\nu_2' = 9·459$, $\nu_2 = 9·233,375$, $\beta_2 = 2·907,286$,
$\nu_3' = 10·039$, $\nu_3 = -3·225,731$, $\sigma = 0·121,546$ in. of Bar. Height.
$\nu_4' = 254·283$, $\nu_4 = 247·861,301$,

The Skewness is ·063.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	0·5–	1·5–	2·5–	3·5–	4·5–	5·5–	6·5–	7·5–	8·5–	9·5–	10·5–	11·5–
Frequency	2	4	5	19	24	35	47	56	83	58	82	65
Value of σ^2	12·5–	13·5–	14·5–	15·5–	16·5–	17·5–	18·5–	19·5–	20·5–	21·5–	22·5–	23·5–
Frequency	54	67	57	63	44	34	33	19	26	17	29	11
Value of σ^2	24·5–	25·5–	26·5–	27·5–	28·5–	29·5–	30·5–	31·5–	32·5–	33·5–	34·5–	35·5–
Frequency	10	7	9	7	6	2	4	6	3	1	1	2
Value of σ^2	36·5–	37·5–	38·5–	39·5–	40·5–	41·5–	42·5–	43·5–	44·5–			
Frequency	1	3	1	1	1	0	0	0	1			

The Constants were found to be:

[Working Origin = 12.]

$\nu_1' = 1{\cdot}915$,	Mean $= 13{\cdot}915$,	$\beta_1 = 1{\cdot}113{,}857$,
$\nu_2' = 46{\cdot}419$,	$\nu_2 = 42{\cdot}751{,}775$,	$\beta_2 = 4{\cdot}684{,}768$,
$\nu_3' = 550{\cdot}903$,	$\nu_3 = 298{\cdot}271{,}317$,	$\sigma = 6{\cdot}538{,}484$.
$\nu_4' = 11{,}801{\cdot}307$,	$\nu_4 = 8{,}562{\cdot}417{,}905$,	

The Skewness is ·524.

III. 1000 Samples of 10 from Population (A), the sampling being on the basis of an "infinite" population.

Method of Sampling, that by aid of Tippett's Table of Random Numbers*.

(a) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	*29·425–*	*29·475–*	*29·525–*	*29·575–*	*29·625–*	*29·675–*	*29·725–*	*29·775–*
Frequency	2	3	7	20	44	67	109	158
Bar. Height in ins.	*29·825–*	*29·875–*	*29·925–*	*29·975–*	*30·025–*	*30·075–*	*30·125–*	*30·175–*
Frequency	146	163	123	85	44	20	8	1

The Constants were found to be:

[Working Origin = 29·85 ins. of Bar. Height. Working Unit = 0·05 in. of Bar. Height.]

$\nu_1' = 0{\cdot}063$,	Mean $= 29{\cdot}853{,}150$ ins. of Bar. Height,	$\beta_1 = 0{\cdot}019{,}694$,
$\nu_2' = 5{\cdot}889$,	$\nu_2 = 5{\cdot}885{,}031$,	$\beta_2 = 2{\cdot}929{,}922$,
$\nu_3' = -\ 0{\cdot}891$,	$\nu_3 = -\ 2{\cdot}003{,}521$,	$\sigma = 0{\cdot}121{,}295$ in.
$\nu_4' = 101{\cdot}109$,	$\nu_4 = 101{\cdot}473{,}726$,	of Bar. Height.

The Skewness is ·075.

* A description of this method is given at the end of this part.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	1·5–	2·5–	3·5–	4·5–	5·5–	6·5–	7·5–	8·5–	9·5–	10·5–	11·5–	12·5–	13·5–	14·5–
Frequency	2	10	17	27	45	57	55	58	75	73	79	63	48	62
Value of σ^2	15·5–	16·5–	17·5–	18·5–	19·5–	20·5–	21·5–	22·5–	23·5–	24·5–	25·5–	26·5–	27·5–	28·5–
Frequency	59	42	43	29	24	27	15	13	15	13	11	6	4	4
Value of σ^2	29·5–	30·5–	31·5–	32·5–	33·5–	34·5–	35·5–	36·5–	37·5–	38·5–	39·5–	40·5–	41·5–	42·5–
Frequency	5	1	3	0	0	2	2	1	1	2	1	0	1	0
Value of σ^2	43·5–	44·5–	45·5–	46·5–	47·5–	48·5–	49·5–	50·5–	51·5–	52·5	53·5–	54·5–	55·5–	
Frequency	0	0	0	0	1	0	0	1	0	2	0	0	1	

The Constants were found to be:

[Working Origin = 12.]

$\nu_1' = 1{\cdot}725$, Mean $= 13{\cdot}725$, $\beta_1 = 2{\cdot}659{,}598$,

$\nu_2' = 48{\cdot}805$, $\nu_2 = 45{\cdot}829{,}375$, $\beta_2 = 8{\cdot}520{,}217$,

$\nu_3' = 748{\cdot}269$, $\nu_3 = 505{\cdot}969{,}031$, $\sigma = 6{\cdot}769{,}732$.

$\nu_4' = 22{,}213{\cdot}549$, $\nu_4 = 17{,}895{\cdot}282{,}136$,

The Skewness is ·532.

IV. 1000 Samples of 10 from Population (B), the sampling being on the basis of an "infinite" population.

Method of Sampling, that by aid of Tippett's Table of Random Numbers.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Value of the Mean	−2·49–	−1·99–	−1·49–	−0·99–	−0·49–	0·01–	0·51–	1·01–	1·51–
Frequency	1	2	6	16	44	53	129	152	188
Value of the Mean	2·01-	2·51–	3·01–	3·51–	4·01–	4·51–	5·01–	5·51–	
Frequency	157	121	78	33	15	4	0	1	

The Constants were found to be:

[Working Origin = 1·76. Working Unit = 0·5.]

$\nu_1' = -\ 0{\cdot}011$,	Mean $= 1{\cdot}7545$,	$\beta_1 = 0{\cdot}010{,}318$,
$\nu_2' = 5{\cdot}081$,	$\nu_2 = 5{\cdot}080{,}879$,	$\beta_2 = 3{\cdot}095{,}297$,
$\nu_3' = -\ 1{\cdot}331$,	$\nu_3 = -\ 1{\cdot}163{,}329$,	$\sigma = 1{\cdot}127{,}041$.
$\nu_4' = 79{\cdot}961$,	$\nu_4 = 79{\cdot}906{,}126$,	

The Skewness is ·048.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	*0·5–*	*1·5–*	*2·5–*	*3·5–*	*4·5–*	*5·5–*	*6·5–*	*7·5–*	*8·5–*	*9·5–*	*10·5–*	*11·5–*	*12·5–*	*13·5–*
Frequency	1	1	18	34	66	67	74	80	62	85	71	61	67	51
Value of σ^2	*14·5–*	*15·5–*	*16·5–*	*17·5–*	*18·5–*	*19·5–*	*20·5–*	*21·5–*	*22·5–*	*23·5–*	*24·5–*	*25·5–*	*26·5–*	*27·5–*
Frequency	48	34	33	34	26	22	8	14	7	7	5	4	2	5
Value of σ^2	*28·5–*	*29·5–*	*30·5–*	*31·5–*	*32·5–*	*33·5–*	*34·5–*	*35·5–*	*36·5–*	*37·5–*	*38·5–*	*39·5–*	*40·5–*	
Frequency	0	5	1	2	2	1	1	0	0	0	0	0	1	

The Constants were found to be:

[Working Origin = 10.]

$\nu_1' = 1{\cdot}609$,	Mean $= 11{\cdot}609$,	$\beta_1 = 1{\cdot}136{,}733$,
$\nu_2' = 34{\cdot}793$,	$\nu_2 = 32{\cdot}204{,}119$,	$\beta_2 = 4{\cdot}683{,}644$,
$\nu_3' = 354{\cdot}463$,	$\nu_3 = 194{\cdot}848{,}209$,	$\sigma = 5{\cdot}674{,}867$.
$\nu_4' = 6{,}618{\cdot}413$,	$\nu_4 = 4{,}857{\cdot}431{,}837$,	

The Skewness is ·539.

V. 1000 Samples of 10 from Population (A), the sampling being on the basis of a "finite" population.

Method of Sampling, that of drawing coloured beads from a bowl, complete samples being drawn before return to bowl.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	*29·425–*	*29·475–*	*29·525–*	*29·575–*	*29·625–*	*29·675–*	*29·725–*	*29·775–*
Frequency	2	2	7	25	34	67	92	136
Bar. Height in ins.	*29·825–*	*29·875–*	*29·925–*	*29·975–*	*30·025–*	*30·075–*	*30·125–*	over *30·175*
Frequency	170	154	127	84	54	29	13	4

The Constants were found to be:

[Working Origin = 29·85 ins. of Bar. Height. Working Unit = 0·05 in. of Bar. Height.]

$\nu_1' = 0\cdot274$,	Mean =	29·863,700 ins. of Bar. Height,		
$\nu_2' = 6\cdot440$,	ν_2 =	6·364,924,	$\sigma = 0\cdot126{,}144$ in. of Bar. Height,
$\nu_3' = 3\cdot664$,	ν_3 =	− 1·588,538,	$\beta_1 = 0\cdot009{,}786$,
$\nu_4' = 122\cdot516$,	ν_4 =	121·384,282,	$\beta_2 = 2\cdot996{,}236$.

The Skewness is ·050.

The distributions of means and σ^2 of samples from the third population (C) are given, with that population, in Parts IV and V.

Tippett's Random Sample.

Tippett's Table of Random Numbers consists of 10,000 random numbers, the method of using them as an aid to sampling being as follows. The numbers range from 0000 to 9999 and, before using the table, each individual of the population is given one of these numbers. The population therefore must not be greater than 10,000 but generally it will be considerably less and consequently only certain of these numbers will be required. Suppose one category of the sampled population consists of 55 individuals, then these 55 individuals may be given the numbers say 0010—0064, and on traversing the table in any systematic manner (down columns or across rows for example) whenever one of these numbers is met an individual of this category is considered drawn. All categories of the original population are similarly dealt with and the table is traversed until the requisite number of "draws" has been made. For example, if 500 samples of 5 are required, 2500 "draws" must be made and these put into groups of 5 in any systematic manner. It may be, as in the present work, that the number of "draws" required to complete the sampling has not been attained when the table has been completely traversed. In this case, the requisite number of extra "draws" must be obtained, either by giving the individuals of the sampled population a new set of numbers out of the 10,000 and traversing the table as before, or by retaining the numbers already given and re-traversing the table in a different order.

The advantages of Tippett's Random Sample are firstly, that its randomness is reasonably established and consequently any set of samples obtained by it may be considered a random set, secondly that it is very much speedier in application than the ordinary method of drawing tickets or other material from a bag, bowl, or urn.

Its disadvantage is that it necessitates the individuals of the sampled population being drawn one at a time, that is the sampling must be conducted on the basis of an infinite sampled population.

When it is definitely desired to draw the whole sample at a time, i.e. a sample on the basis of a "finite" sampled population, the method of drawing objects from a bag or an urn must be employed. The difficulties of shuffling when the objects are tickets is well known, so in this work samplings made on this basis have been taken both by drawing tickets from a bag and also by drawing more symmetrical and easily shuffled material. This material was, in one case a large number of small commercial coloured beads whose homogeneity of size and shape was rough, and in the other case a small number of coloured marbles which approximated much more to the ideal of complete homogeneity. This ideal of complete homogeneity in the individuals representing the sampled population is well known to be necessary in theory and desirable in practice; but in the past, it has scarcely been recognised how necessary as well as desirable it is in practice, especially when sampling from small populations.

PART II. THE DISTRIBUTION OF MEANS OF SAMPLES TAKEN FROM AN "INFINITE" POPULATION.

(*a*) As mentioned in the introduction the relations known to exist between the moment-coefficients of the Distribution of Means of Samples and the corresponding constants of the Sampled Population indicate that, when N, the number in the sample, is large, the Distribution of Means should be well represented by a Normal Frequency Distribution. When $N = 10$, although a similar result may be expected, it is not so clear that a close approach to normality will be obtained in practice. Consequently it was considered of importance to fit a normal curve to the three experimental distributions (I (*a*), III (*a*) and IV (*a*)) employed in this portion of the work.

The results were as follows:

Distribution I (a).

Bar. Height in ins.	Normal Frequency	Actual Frequency in I (*a*)	Bar. Height in ins.	Normal Frequency	Actual Frequency in I (*a*)
29·275—	0·0730	1	*29·825*—	144·0537	158
29·325—	0·2813	1	*29·875*—	130·4000	145
29·375—	0·9491	0	*29·925*—	103·4294	101
29·425—	2·8079	7	*29·975*—	71·9655	63
29·475—	7·2799	9	*30·025*—	43·8715	37
29·525—	16·6427	18	*30·075*—	23·4632	29
29·575—	32·8911	22	*30·125*—	10·9927	13
29·625—	56·6393	52	*30·175*—	4·5189	3
29·675—	89·2672	83	*30·225*—	1·6269	0
29·725—	118·5678	125	*30·275*—	0·5140	0
29·775—	139·5584	132	*30·325*—	0·1423	1

Testing for Goodness of Fit with 17 Groups, this gave quite a fair fit with $\chi^2 = 20{\cdot}41$ and $P = {\cdot}2031$.

Distribution III (a).

Bar. Height in ins.	Normal Frequency	Actual Frequency in III (*a*)	Bar. Height in ins.	Normal Frequency	Actual Frequency in III (*a*)
29·425—	0·9037	2	*29·825*—	163·2387	146
29·475—	2·5000	3	*29·875*—	151·7139	163
29·525—	7·5084	7	*29·925*—	119·2535	123
29·575—	19·0701	20	*29·975*—	79·2800	85
29·625—	40·9641	44	*30·025*—	44·5745	44
29·675—	74·4157	67	*30·075*—	21·1937	20
29·725—	114·3247	109	*30·125*—	8·5221	8
29·775—	148·5447	158	*30·175*—	2·8981	1

Testing for Goodness of Fit with 15 Groups, this gave an excellent fit with $\chi^2 = 7·33$ and $P = ·9197$.

Distribution IV (a).

Value of Mean	Normal Frequency	Actual Frequency in IV (*a*)	Value of Mean	Normal Frequency	Actual Frequency in IV (*a*)
−2·49—	0·3633	1	*2·01*—	159·0085	157
−1·99—	1·5499	2	*2·51*—	118·6741	121
−1·49—	5·4475	6	*3·01*—	72·9834	78
−0·99—	15·7701	16	*3·51*—	36·9808	33
−0·49—	37·6160	44	*4·01*—	15·4375	15
0·01—	73·9223	53	*4·51*—	5·3100	4
0·51—	119·6895	129	*5·01*—	1·5044	0
1·01—	159·6869	152	*5·51*—	0·3512	1
1·51—	175·5419	188			

Testing for Goodness of Fit with 14 Groups, this gave quite a good fit with $\chi^2 = 11·42$ and $P = ·5758$.

It is apparent from these three values of P that the Distribution of Means of Samples approaches normality in practice very closely indeed, even when N is as low as 10, provided the original population is not of an extremely skew character. In fact, for practical applications, the assumption that the Distribution of Means is a Normal Distribution would be quite justified and would lead to quite satisfactory results, except possibly when N was very small indeed, say 2 or 3, or if the sampled population was extremely skew, N being still rather small.

(*b*) However useful the assumption of normality may be when dealing with the distribution of means of samples in purely practical cases, it is highly desirable that a curve for this distribution should be obtained which shows its skewness, slight as this may perhaps be, and which represents generally the distribution from a theoretical point of view.

To this end it was hoped that, commencing with the various Pearson Frequency Curves, the equations of the corresponding Curves of Means of Samples of N might be obtained directly without the use of the method of moments, but success resulted only in the case when the sampled population was of Type III.

The solution of this is now given, but as Populations (A) and (B) are both represented by Type I curves, the result could not be applied to either of them with the expectation of any reasonable accuracy of representation. In point of fact, as the betas of (A) and (B) are such that the points representing them in the usual diagram, although distinctly within the Type I area, are in the neighbourhood of the Type III line, this application was actually attempted by treating the populations as if they were of Type III. As was to be expected however, the Distributions of Means of Samples so deduced showed no agreement whatever with Distributions I (a) and III (a) or IV (a), and consequently the method is not considered worthy of further consideration here.

The Theoretical Curve of Frequency of the Distribution of Means of Samples of N drawn from an "infinite" sampled population which is represented by a Type III Pearson Curve.

The equation $y = y_0 e^{-\gamma x}\left(1+\frac{x}{a}\right)^{\gamma a}$ of the Type III curve can be written $y = y_0' e^{-px} x^p$, where $p = \gamma a$, by transforming the variate to $x = 1 + \frac{x}{a}$. Here the origin has been changed so that the variate is always positive and the unit in which it is measured is $\frac{1}{a}$ times the original unit.

Now the probability that the values $x_1 x_2 \dots x_N$ of the variate x, forming a sample of N from the above population, lie between x_1 and $x_1 + dx_1$, x_2 and $x_2 + dx_2, \dots,$ x_N and $x_N + dx_N$, is

$$\text{Constant } (x_1 x_2 \dots x_N)^p e^{-p(x_1 + x_2 + \dots + x_N)} dx_1 dx_2 \dots dx_N.$$

Changing the variables to

$$\begin{aligned} X_1 &= x_1 + x_2 + \dots + x_N, \\ X_2 &= x_2, \\ X_3 &= x_3, \\ &\\ X_N &= x_N, \end{aligned}$$

we have $J = 1$, and this expression becomes

$$\text{Constant } (X_1 - X_2 - X_3 - \dots - X_N)^p (X_2 X_3 \dots X_N)^p e^{-pX_1} dX_1 dX_2 \dots dX_N.$$

Putting $C_2 = X_1 - X_3 - \dots - X_N$ and integrating for X_2 between 0 and C_2 the expression becomes

$$\text{Constant } e^{-pX_1} (X_3 X_4 \dots X_N)^p dX_1 dX_3 \dots dX_N \int_0^{C_2} (C_2 - X_2)^p X_2^p dX_2,$$

which, putting $X_2 = C_2 Z_2$, becomes

$$\text{Constant } e^{-pX_1} (X_3 X_4 \dots X_N)^p dX_1 dX_3 \dots dX_N . C_2^{2p+1} . B(p+1, p+1),$$

that is

$$\text{Constant } (X_1 - X_3 - \dots - X_N)^{2p+1} (X_3 X_4 \dots X_N)^p . e^{-pX_1} dX_1 dX_3 \dots dX_N.$$

Thus the result of this integration with regard to X_2 is to leave the form of the expression unaltered except that the power of the difference factor has been raised by $p + 1$.

Therefore repeating the process $N-2$ times to eliminate all the X's except X_1, the expression becomes

$$\text{Constant } X_1^{Np+N-1} e^{-pX_1} dX_1.$$

But if m is the Mean of the Sample, $Nm = X_1$, therefore the probability that the value of the Mean of a Sample will lie between m and $m + dm$ is

$$\text{Constant } m^{N(p+1)-1} e^{-Npm} dm,$$

and the Frequency Curve of Means of Samples is of the form

$$y = y_0 m^{N(p+1)-1} e^{-Npm}.$$

To determine y_0 let n be the total frequency of this Distribution of Means of Samples, that is let n be the number of samples taken.

Then
$$n = \int_0^\infty y\,dm$$

$$= y_0 \int_0^\infty m^{N(p+1)-1} e^{-Npm} dm,$$

and putting $Npm = z$,

$$n = \frac{y_0}{(Np)^{N(p+1)}} \int_0^\infty e^{-z} z^{N(p+1)-1} dz$$

$$= \frac{y_0}{(Np)^{N(p+1)}} \Gamma\{N(p+1)\},$$

therefore
$$y_0 = \frac{n(Np)^{N(p+1)}}{\Gamma\{N(p+1)\}}.$$

Thus the curve of frequency of Means of Samples of N drawn from a sampled population represented by a Type III curve is

$$y = \frac{n(Np)^{N(p+1)}}{\Gamma\{N(p+1)\}} m^{N(p+1)-1} e^{-Npm},$$

where n is the total frequency of the distribution.

We now turn to the constants of this Distribution of Means of Samples.

With the notation employed previously for the constants of the Distribution of Means of Samples of N, and n being the total frequency, we have

$$n \,.\, {}_1M_1' = y_0 \int_0^\infty e^{-Npm} m^{N(p+1)} dm$$

$$= y_0 \int_0^\infty \frac{e^{-z} z^{N(p+1)}}{(Np)^{N(p+1)+1}} dz, \text{ where } z = Npm$$

$$= y_0 \frac{\Gamma\{N(p+1)+1\}}{(Np)^{N(p+1)+1}},$$

but
$$y_0 = \frac{n(Np)^{N(p+1)}}{\Gamma\{N(p+1)\}}.$$

Therefore
$${}_1M_1' = \frac{1}{Np} \frac{\Gamma\{N(p+1)+1\}}{\Gamma\{N(p+1)\}} = \frac{p+1}{p}.$$

Thus in the original unit

$$_1M_1=\frac{(p+1)\,a}{p}=\frac{p+1}{\gamma},$$

which shows that here, as is necessary, the mean of this distribution is the mean of the original population.

Again $$n\,.\,_1M_2'=y_0\int_0^\infty e^{-Npm}\,m^{N(p+1)+1}\,dm$$

$$=y_0\int_0^\infty\frac{e^{-z}\,z^{N(p+1)+1}}{(Np)^{N(p+1)+2}}\,dz$$

$$=y_0\,\frac{\Gamma\{N(p+1)+2\}}{(Np)^{N(p+1)+2}}.$$

Therefore $$_1M_2'=\frac{\{N(p+1)+1\}\,\{N(p+1)\}}{N^2p^2}$$

$$=\frac{(p+1)^2}{p^2}+\frac{1}{N}\,\frac{p+1}{p^2}.$$

Therefore $$_1M_2\,(=\Sigma_1{}^2)=\frac{1}{N}\,\frac{p+1}{p^2},$$

i.e. $$_1M_2=\frac{1}{N}\,.\,\frac{p+1}{\gamma^2}\ \text{in the original unit}$$

$$=\frac{1}{N}\ \text{in terms of the standard-deviation of original population.}$$

Similarly

$$_1M_3'=\frac{\{N(p+1)+2\}\,\{N(p+1)+1\}\,\{N(p+1)\}}{N^3p^3}$$

$$=\frac{1}{p^3}\left\{(p+1)+\frac{2}{N}\right\}\left\{(p+1)+\frac{1}{N}\right\}\{p+1\},$$

but $$_1M_3={_1M_3'}-3\,.\,{_1M_1'}\,.\,{_1M_2'}+2\,({_1M_1'})^3.$$

Therefore $$_1M_3=\frac{2\,(p+1)}{N^2p^3},$$

i.e. $$_1M_3=\frac{2\,(p+1)}{N^2\gamma^3}\ \text{in the original unit.}$$

Again, in a like manner,

$$_1M_4'=\frac{\left\{(p+1)+\frac{3}{N}\right\}\left\{(p+1)+\frac{2}{N}\right\}\left\{(p+1)+\frac{1}{N}\right\}\{p+1\}}{N^4p^4},$$

but $$_1M_4={_1M_4'}-4\,.\,{_1M_1'}\,.\,{_1M_3'}+6\,({_1M_1'})^2\,.\,{_1M_2'}-3\,({_1M_1'})^4.$$

Therefore $$_1M_4=\frac{1}{p^4}\left\{\frac{3\,(p+1)^2}{N^2}+\frac{6\,(p+1)}{N^3}\right\},$$

i.e. $$_1M_4=\frac{1}{\gamma^4}\left\{\frac{3\,(p+1)^2}{N^2}+\frac{6\,(p+1)}{N^3}\right\}\ \text{in the original unit.}$$

Again $\quad {}_1B_1 = \dfrac{({}_1M_3)^2}{({}_1M_2)^3} = \dfrac{4}{N(p+1)},$

and $\quad {}_1B_2 = \dfrac{{}_1M_4}{({}_1M_2)^2} = 3 + \dfrac{6}{N(p+1)};$

and thus $\quad 2\,{}_1B_2 = 6 + 3\,{}_1B_1,$

showing this curve to be of Type III.

Again, the betas of the sampled Type III population were

$$\bar{\beta}_1 = \frac{4}{p+1}, \quad \bar{\beta}_2 = 3 + \frac{6}{p+1},$$

and therefore from this we have

$${}_1B_1 = \frac{\bar{\beta}_1}{N}, \quad {}_1B_2 - 3 = \frac{\bar{\beta}_2 - 3}{N},$$

showing that this curve satisfies all the conditions required of a curve of frequency of means of samples.

Thus, for a sampled population of Type III, we have a complete theoretical representation of any distribution of means of samples taken from it.

(*c*) At present, however, the determination of frequency curves for the distributions of means of samples from populations of types other than Type III is restricted to the intermediary use of the method of moments.

The sampled population and its moments being known, the formulae given in Part I can be used to determine the moments of the distribution of means of samples and hence, when the frequency type is determined, the Pearson Frequency Curve representing this distribution can be obtained. To determine the frequency type speedily, consider a pair of rectangular axes $\beta_1 = 0$, $\beta_2 = 0$ (as in *Tables for Statisticians*, XXXV, p. 66) and represent any distribution by the point (β_1, β_2), whose Cartesian coordinates, with regard to the given axes, are its betas. Now considering the two points $(\bar{\beta}_1, \bar{\beta}_2)$, $({}_1B_1, {}_1B_2)$ representing sampled population and distribution of means of samples respectively we have, from Part I, the following relation between their coordinates:

$$\frac{\bar{\beta}_2 - 3}{\bar{\beta}_1} = \frac{{}_1B_2 - 3}{{}_1B_1} = \lambda,$$

where λ is independent of N the number in the sample, that is λ is a constant for all distributions of means of samples derived from the original population.

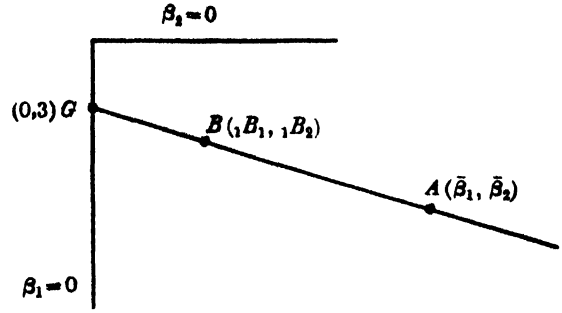
Therefore, given the point A, $(\bar{\beta}_1, \bar{\beta}_2)$, the point B, $({}_1B_1, {}_1B_2)$, lies on the line

$${}_1B_2 = \lambda \,.\, {}_1B_1 + 3,$$

which passes through the Gaussian point and the given point A, $(\bar{\beta}_1, \bar{\beta}_2)$.

Now since ${}_1B_1 = \dfrac{\bar{\beta}_1}{N}$, we have from the diagram $\dfrac{GB}{GA} = \dfrac{1}{N}$, and since N is $\geqslant 1$ necessarily, the point B, i.e. $({}_1B_1, {}_1B_2)$, lies on the line considered between G and A, and *Tables for Statisticians*, XXXV shows the frequency type immediately.

It may be noted that, if the original population be any of Types I, II, III, IV or VII and its betas are of moderate size, the frequency type of the distribution of means of samples is the same as that of the sampled population, agreeing with the result obtained in Part II (*b*).



This method was applied to Populations (A) and (B) in order to determine the degree of accuracy with which the actual Distributions I (*a*), III (*a*) and IV (*a*) are represented by the corresponding theoretical curves obtained in this manner.

Population (A) has for its constants:

$$\text{Mean} = 29{\cdot}858{,}0155 \text{ ins. of Bar. Height},$$
$$\bar{\sigma} = 0{\cdot}384{,}1700 \text{ in. of Bar. Height},$$
$$\bar{\beta}_1 = 0{\cdot}189{,}424, \quad \bar{\beta}_2 = 3{\cdot}160{,}978,$$

and its frequency type is Type I. Therefore the Distribution of Means of Samples of 10 is represented by a Type I curve and, from the equations of Part I, its constants are:

$$\text{Mean} = 29{\cdot}858{,}0155 \text{ ins. of Bar. Height},$$
$$\Sigma_1 = 0{\cdot}121{,}4852 \text{ in. of Bar. Height},$$
$${}_1B_1 = 0{\cdot}018{,}942, \quad {}_1B_2 = 3{\cdot}016{,}098.$$

With the usual notation, the fitting of these constants to a Type I curve gives

$$r = 486{\cdot}517{,}905, \quad \epsilon = 37{,}463{\cdot}612{,}112;$$

the quadratic is $\quad m'^2 - 486{\cdot}517{,}905m' + 37{,}463{\cdot}612{,}112 = 0,$

its roots are $\quad 390{\cdot}606{,}521$ and $95{\cdot}911{,}385$.

Now as the $\bar{\nu}_3$ of Population (A) is negative, the ${}_1M_3$ of this distribution is also negative and, both the roots of this quadratic being positive, m_1' is the larger root and therefore $m_1 = 389{\cdot}606{,}521$, $m_2 = 94{\cdot}911{,}385$. Using 0·1 in. of Bar. Height as working unit, $b = 67{\cdot}423{,}715$ and $a_1 = 54{\cdot}216{,}202$, $a_2 = 13{\cdot}207{,}422$.

Now $$y_0 = \frac{N}{b}\,\frac{m_1^{m_1}\,.\,m_2^{m_2}}{(m_1+m_2)^{m_1+m_2}}\,\frac{\Gamma(m_1+m_2+2)}{\Gamma(m_1+1)\,.\,\Gamma(m_2+1)},$$

therefore $$\log y_0 = 2{\cdot}516{,}5925,$$

and $$y_0 = 328{\cdot}5432.$$

The frequency curve representing the Distribution of Means of Samples of 10 *from Population* (A) *is thus*

$$y = 328{\cdot}5432\left(1 + \frac{x}{54{\cdot}216{,}202}\right)^{389{\cdot}606{,}521}\left(1 - \frac{x}{13{\cdot}207{,}422}\right)^{94{\cdot}911{,}385}.$$

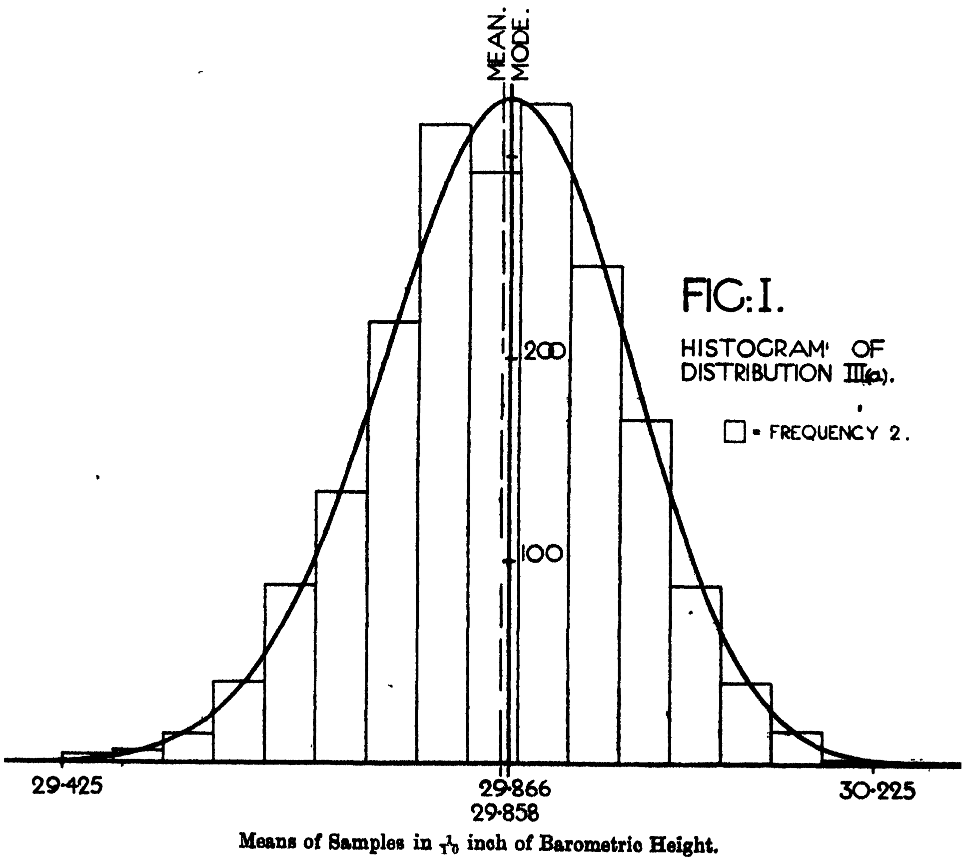
The Mean is 29·858,0155 ins. of Bar. Height and

Mean-Mode = − 0·008,4290 in. of Bar. Height.

Therefore the origin, which is the mode, is at 29·866,445 ins. of Bar. Height, and the range is from $x = -54{\cdot}216{,}202$ to $x = 13{\cdot}207{,}422$. Now the actual distributions of means of samples derived from Population (A) range from 29·275 ins. to 30·325 ins., so it is sufficient to plot this curve from $x = -5{\cdot}0$ to $x = 4{\cdot}5$.

The Plotted Ordinates of this Curve, which is shown in Fig. I, are

x	y	x	y	x	y	x	y
−5·0	0·2390	−1·5	159·736	0·125	326·837	2·5	31·2449
−4·5	0·8729	−1·0	237·122	0·25	321·595	3·0	10·2027
−4·0	2·8415	−0·75	273·029	0·50	301·400	3·5	2·5745
−3·5	8·2122	−0·50	302·381	0·75	270·046	4·0	0·4694
−3·0	20·9904	−0·25	321·724	1·0	231·044	4·5	0·0686
−2·5	47·2366	−0·125	326·809	1·5	146·257		
−2·0	93·0828	0·00	328·543	2·0	75·4552		



Means of Samples in $\frac{1}{10}$ inch of Barometric Height.

Distribution of Means in 1000 Samples of 10 from the Skew Distribution of Barometric Heights (A). Fitted Curve from theoretical values of σ, β_1 and β_2 for curve of means.

Again, Population (B) has constants

$$\text{Mean} = 1{\cdot}8518, \qquad \bar{\sigma} = 3{\cdot}569{,}207,$$
$$\bar{\beta}_1 = 0{\cdot}219{,}333, \qquad \bar{\beta}_2 = 3{\cdot}157{,}676,$$

and its frequency type is Type I. Therefore the curve representing the distribution of means of samples of 10 is of Type I, with constants

$$\text{Mean} = 1{\cdot}85818, \qquad \Sigma_1 = 1{\cdot}128{,}682,$$
$$_1B_1 = 0{\cdot}021{,}933, \qquad _1B_2 = 3{\cdot}015{,}768.$$

We have, as before,

$$r = 349{\cdot}152{,}438, \qquad \epsilon = 20{,}554{\cdot}430{,}847,$$
$$m_1 = 273{\cdot}187{,}591, \qquad m_2 = 73{\cdot}964{,}847,$$
$$b = 51{\cdot}434{,}693, \qquad a_1 = 40{\cdot}476{,}020,$$
$$a_2 = 10{\cdot}958{,}780, \qquad y_0 = 353{\cdot}5420.$$

The frequency curve representing the Distribution of Means of Samples of 10 *from Population* (B) *is thus*

$$y = 353{\cdot}5420\left(1 + \frac{x}{40{\cdot}476{,}020}\right)^{273{\cdot}187{,}591}\left(1 - \frac{x}{10{\cdot}958{,}780}\right)^{73{\cdot}964{,}847}.$$

The Mean is 1·858,18 and Mean-Mode $= -0{\cdot}084{,}541$. Therefore the origin, which is the mode, is 1·942,721, and the range is from $x = -40{\cdot}476{,}020$ to $x = 10{\cdot}958{,}780$. Now the actual distribution (IV (a)) of means of samples of 10 derived from Population (B) ranges from $-2{\cdot}5$ to 6·0 units, so it is sufficient to plot the curve from $x = -4{\cdot}5$ to $x = 4{\cdot}5$.

The Plotted Ordinates of this Curve, which is shown in Fig. II, are

x	y	x	y	x	y	x	y
−4·5	0·4137	−1·0	242·950	0·25	344·913	3·0	5·6736
−4·0	1·5782	−0·75	285·652	0·50	319·821	3·5	1·0692
−3·5	5·2649	−0·50	322·369	0·75	281·500	4·0	0·1385
−3·0	15·2917	−0·25	345·082	1·0	234·611	4·5	0·0118
−2·5	38·4765	−0·125	351·396	1·5	137·218		
−2·0	83·3704	0·00	353·542	2·0	62·7644		
−1·5	154·493	0·125	351·382	2·5	21·9485		

Tests for Goodness of Fit.

With the obvious exception of the tests for normality, the data for all tests of Goodness of Fit of the theoretical curves against the corresponding experimental distributions were obtained by first drawing the curves accurately to a large scale on prepared cardboard, and then planimetering the areas corresponding to the frequency groups of the experimental distributions. This is the most satisfactory method in practice, but is somewhat lengthy and laborious.

In this case, the tests for Goodness of Fit gave

(1) For theoretical curve of means of samples of 10 from Population (A) against experimental Distribution I (a), i.e. ticket drawing:

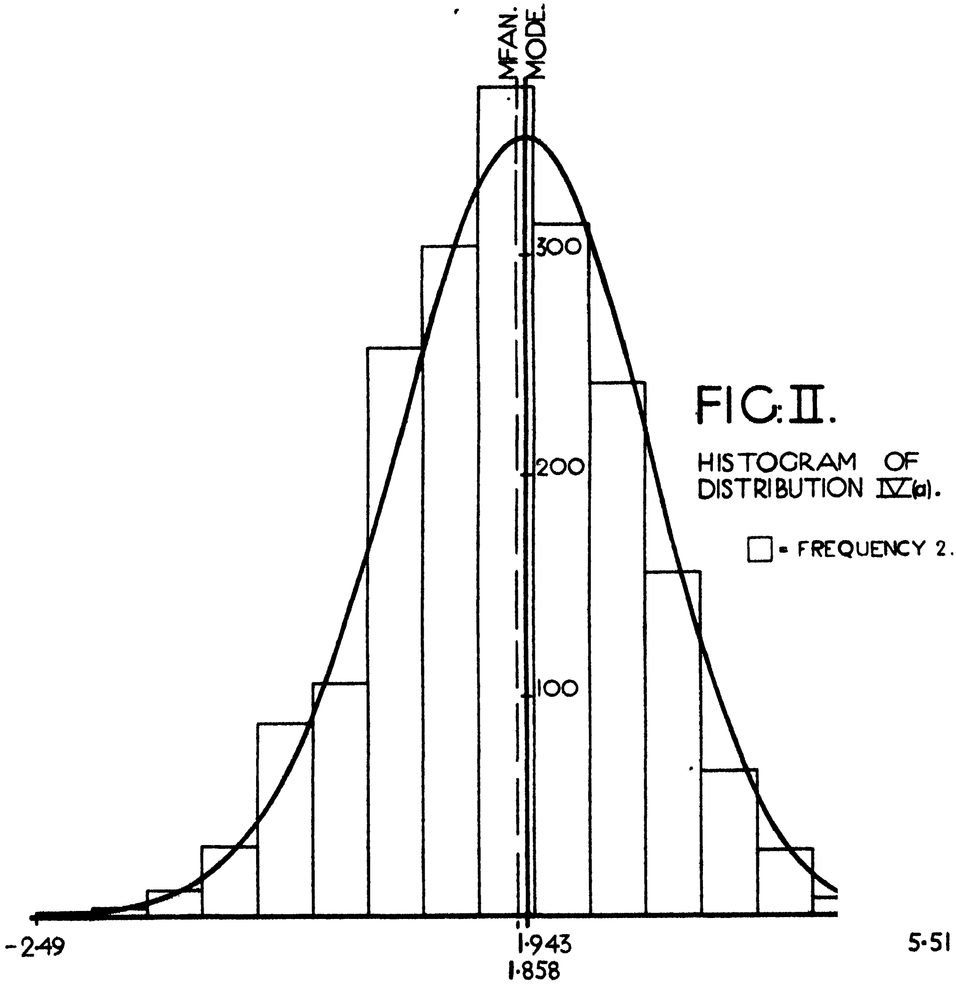
15 Groups. $\chi^2 = 84{\cdot}07$. $P = {\cdot}0000$.

(2) For theoretical curve of means of samples of 10 from Population (A) against experimental Distribution III (a):

16 Groups. $\chi^2 = 7{\cdot}01$. $P = {\cdot}9573$.

(3) For theoretical curve of means of samples of 10 from Population (B) against experimental Distribution IV (a):

16 Groups. $\chi^2 = 18{\cdot}70$. $P = {\cdot}2284$.



Means of Samples in units of original Distribution (B) with sub-range ½ unit.

Distribution of Means of 1000 Samples of 10 from the Smooth Skew Distribution (B).
Fitted Curve from theoretical values of σ, β_1 and β_2 for curve of means.

On the whole these theoretical curves of means of samples do not give as good fits to the actual distributions as did the Normal Curve. There is little in this beyond the fact that for mere purposes of representing distributions of means of small samples, the Normal Curve will give in all probability as good or better results as any other method, and therefore may be preferred often in practice on account of its relative simplicity.

In the two cases when the sampling was carried out by Tippett's Random Sample, that is when the randomness could be considered established, the fits to theory were $P = \cdot 9573$ and $\cdot 2284$; consequently we may conclude, fairly, that the Pearson Frequency Curves employed here as a result of the equations

$${}_1B_1 = \frac{\bar{\beta}_1}{N} \text{ and } {}_1B_2 - 3 = \frac{\bar{\beta}_2 - 3}{N}$$

do represent adequately actual distributions of means of samples provided the assumption of an "infinite" sampled population be made and the randomness of the sampling is reasonably certain.

It is fairly clear from this, that the poor fit of Distribution I (a) must be due to lack of effective shuffling of the tickets, although great care was employed in the attempt to ensure this shuffling was effective and the resulting samples to be of a quite random character. This difficulty is quite well known of course, but can be overcome to a great extent by the use of beads or similar material instead of the tickets. This point is dealt with in Part IV.

PART III. THE DISTRIBUTION OF SECOND MOMENTS (σ^2) OF SMALL SAMPLES FROM ANY "INFINITE" POPULATION.

(a) Commencing with the various Pearson Curves as types of sampled populations, many attempts were made to deduce without the use of moments the corresponding theoretical frequency curves of σ^2 for samples. These were not successful, so recourse was necessarily taken to the method of moments.

In *Biometrika*, Vol. XII. p. 193 (23), p. 194 (24) and (25), Professor Al. Tchouproff has given formulae for the moments of the distribution of σ^2 of samples from any "infinite" population in terms of the moments of that population. In *Biometrika*, Vol. XVII. pp. 79—83, I have obtained these formulae afresh by a shorter method and have corrected an error in the expression for the 4th moment of the distribution of σ^2 of samples. This investigation was carried out, because Professor Tchouproff's results followed as a result of some very intricate algebra, whilst on close inspection the expression for the 4th moment was seen to be in error, so it was considered of some value, not only to correct the expression for the 4th moment, but also to check the others and to deduce all of them by a simpler analysis more in conformity with English statistical methods.

The formulae*, thus corrected, for the moments of the distribution of σ^2 of samples are

$$\text{Mean } (= {}_2M_1') = \bar{\nu}_2\left(1 - \frac{1}{N}\right) \quad\text{.........................(1)},$$

$${}_2M_2 = \frac{1}{N}[\bar{\nu}_4 - \bar{\nu}_2^2] - \frac{2}{N^2}[\bar{\nu}_4 - 2\bar{\nu}_2^2] + \frac{1}{N^3}[\bar{\nu}_4 - 3\bar{\nu}_2^2] \quad\text{.........................(2)},$$

$$\begin{aligned}{}_2M_3 = &\frac{1}{N^2}[\bar{\nu}_6 - 3\bar{\nu}_4\bar{\nu}_2 - 6\bar{\nu}_3^2 + 2\bar{\nu}_2^3] \\ &- \frac{1}{N^3}[3\bar{\nu}_6 - 21\bar{\nu}_4\bar{\nu}_2 - 18\bar{\nu}_3^2 + 26\bar{\nu}_2^3] + \frac{1}{N^4}[3\bar{\nu}_6 - 33\bar{\nu}_4\bar{\nu}_2 - 22\bar{\nu}_3^2 + 54\bar{\nu}_2^3] \\ &- \frac{1}{N^5}[\bar{\nu}_6 - 15\bar{\nu}_4\bar{\nu}_2 - 10\bar{\nu}_3^2 + 30\bar{\nu}_2^3] \quad\text{.........................(3)},\end{aligned}$$

$$\begin{aligned}{}_2M_4 = &\frac{3}{N^2}[\bar{\nu}_4 - \bar{\nu}_2^2]^2 + \frac{1}{N^3}[\bar{\nu}_8 - 4\bar{\nu}_6\bar{\nu}_2 - 24\bar{\nu}_5\bar{\nu}_3 - 15\bar{\nu}_4^2 + 48\bar{\nu}_4\bar{\nu}_2^2 + 96\bar{\nu}_3^2\bar{\nu}_2 - 30\bar{\nu}_2^4] \\ &- \frac{1}{N^4}[4\bar{\nu}_8 - 40\bar{\nu}_6\bar{\nu}_2 - 96\bar{\nu}_5\bar{\nu}_3 - 54\bar{\nu}_4^2 + 336\bar{\nu}_4\bar{\nu}_2^2 + 528\bar{\nu}_3^2\bar{\nu}_2 - 306\bar{\nu}_2^4] \\ &+ \frac{1}{N^5}[6\bar{\nu}_8 - 96\bar{\nu}_6\bar{\nu}_2 - 176\bar{\nu}_5\bar{\nu}_3 - 102\bar{\nu}_4^2 + 924\bar{\nu}_4\bar{\nu}_2^2 + 1232\bar{\nu}_3^2\bar{\nu}_2 - 1044\bar{\nu}_2^4] \\ &- \frac{1}{N^6}[4\bar{\nu}_8 - 88\bar{\nu}_6\bar{\nu}_2 - 160\bar{\nu}_5\bar{\nu}_3 - 95\bar{\nu}_4^2 + 1050\bar{\nu}_4\bar{\nu}_2^2 + 1360\bar{\nu}_3^2\bar{\nu}_2 - 1395\bar{\nu}_2^4] \\ &+ \frac{1}{N^7}[\bar{\nu}_8 - 28\bar{\nu}_6\bar{\nu}_2 - 56\bar{\nu}_5\bar{\nu}_3 - 35\bar{\nu}_4^2 + 420\bar{\nu}_4\bar{\nu}_2^2 + 560\bar{\nu}_3^2\bar{\nu}_2 - 630\bar{\nu}_2^4] \\ &\text{............(4)},\end{aligned}$$

where ${}_2M_2$, ${}_2M_3$, etc. are the moments of the distribution of σ^2 about its mean ${}_2M_1'$, $\bar{\nu}_2$, $\bar{\nu}_3$, etc. are the uncorrected moments of the sampled population about its mean, and N is the number in the sample.

Tchouproff's formulae may be expressed in terms of the betas of the sampled population as follows:

$$\text{Mean} = \left(1 - \frac{1}{N}\right)\bar{\nu}_2,$$

$${}_2M_2 = \frac{\bar{\nu}_2^2}{N}\left(1 - \frac{1}{N}\right)\left\{\bar{\beta}_2 - 1 - \frac{\bar{\beta}_2 - 3}{N}\right\},$$

$$\begin{aligned}{}_2M_3 = \frac{\bar{\nu}_2^3}{N^2}\Big\{&\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2 - \frac{1}{N}(3\bar{\beta}_4 - 21\bar{\beta}_2 - 18\bar{\beta}_1 + 26) \\ &+ \frac{1}{N^2}(3\bar{\beta}_4 - 33\bar{\beta}_2 - 22\bar{\beta}_1 + 54) - \frac{1}{N^3}(\bar{\beta}_4 - 15\bar{\beta}_2 - 10\bar{\beta}_1 + 30)\Big\},\end{aligned}$$

$$\begin{aligned}{}_2M_4 = \frac{\bar{\nu}_2^4}{N^2}\Big\{&3(\bar{\beta}_2 - 1)^2 + \frac{1}{N}(\bar{\beta}_6 - 4\bar{\beta}_4 - 24\bar{\beta}_3 - 15\bar{\beta}_2^2 + 48\bar{\beta}_2 + 96\bar{\beta}_1 - 30) \\ &- \frac{1}{N^2}(4\bar{\beta}_6 - 40\bar{\beta}_4 - 96\bar{\beta}_3 - 54\bar{\beta}_2^2 + 336\bar{\beta}_2 + 528\bar{\beta}_1 - 306) \\ &+ \frac{1}{N^3}(6\bar{\beta}_6 - 96\bar{\beta}_4 - 176\bar{\beta}_3 - 102\bar{\beta}_2^2 + 924\bar{\beta}_2 + 1232\bar{\beta}_1 - 1044) \\ &- \frac{1}{N^4}(4\bar{\beta}_6 - 88\bar{\beta}_4 - 160\bar{\beta}_3 - 95\bar{\beta}_2^2 + 1050\bar{\beta}_2 + 1360\bar{\beta}_1 - 1395) \\ &+ \frac{1}{N^5}(\bar{\beta}_6 - 28\bar{\beta}_4 - 56\bar{\beta}_3 - 35\bar{\beta}_2^2 + 420\bar{\beta}_2 + 560\bar{\beta}_1 - 630)\Big\}.\end{aligned}$$

[* The formulae (1) and (2) were known before the publication of Prof. Tchouproff's paper. ED.]

These formulae, though theoretically exact, suffer from the very considerable disadvantage that they involve the first eight moments of the sampled population. In the first place the large probable errors of the higher moments of the sampled population (this is in general merely a larger sample from a still larger population) cause formulae involving their use to be of doubtful value in practical statistics, and secondly the labour involved in calculating the higher of these eight moments is too great to be undertaken in ordinary statistical practice.

(*b*) The consideration of the second of these points will be left to the next section, the purpose of the present section being to determine whether or no these formulae do give good representations of actual experimental distributions of σ^2 of samples in spite of the fact that they, the formulae, contain high moments whose exact size is questionable. To this end Tchouproff's formulae were applied directly to the betas of Populations (A) and (B) given in the Introduction.

The results obtained were:

(1) *The Distribution of* σ^2 *of Samples of* 10, *obtained by the application of Tchouproff's Formulae to the actual uncorrected Moments of Population* (A), *had for its constants:*

$$\text{Mean} = 13{\cdot}282{,}791, \qquad {}_2M_2 = 42{\cdot}047{,}408,$$
$${}_2M_3 = 360{\cdot}510{,}852, \qquad {}_2M_4 = 11{,}747{\cdot}805{,}045,$$

and Standard Deviation

$$(\Sigma_2) = 6{\cdot}484{,}397,$$

which gave betas of

$${}_2B_1 = 1{\cdot}748{,}312, \qquad {}_2B_2 = 6{\cdot}644{,}744.$$

(2) *The Distribution of* σ^2 *of Samples of* 10, *obtained by the application of Tchouproff's Formulae to the actual uncorrected Moments of Population* (B), *had for its constants:*

$$\text{Mean} = 11{\cdot}465{,}313, \qquad {}_2M_2 = 31{\cdot}284{,}573,$$
$${}_2M_3 = 179{\cdot}342{,}927, \qquad {}_2M_4 = 4{,}445{\cdot}996{,}023,$$

and Standard Deviation

$$(\Sigma_2) = 5{\cdot}593{,}261,$$

which gave betas of

$${}_2B_1 = 1{\cdot}050{,}297, \qquad {}_2B_2 = 4{\cdot}541{,}956.$$

Tables of comparison of these constants with the corresponding constants of the actual Distributions I (*b*), III (*b*) and IV (*b*) of σ^2 of 1000 samples of 10 from Populations (A) and (B) are

Table for the Distributions from Population (A).

Constant	Theoretic Value	Value from I (*b*)	Value from III (*b*)
Mean	13·282,791	15·848 ± 0·138	13·725 ± 0·138
${}_2M_2$	42·047,408	61·162,896 ± 2·130,785	45·829,375 ± 2·130,785
${}_2M_3$	360·510,852	701·313,792 ± 74·847,038	505·969,031 ± 74·847,038
${}_2M_4$	11,747·805,045	24,542·441,607 ± 5,299·341,004	17,895·282,136 ± 5,299·341,004
${}_2B_1$	1·7483	2·1496 ± 0·5379	2·6596 ± 0·5379
${}_2B_2$	6·6447	6·5606 ± 2·6844	8·5202 ± 2·6844
Σ_2	6·4844	7·8207 ± 0·0978	6·7697 ± 0·0978

Table for the Distributions from Population (B).

Constant	Theoretic Value	Value from IV (b)
Mean	11·465,313	11·609 ± 0·119
${}_2M_2$	31·284,573	32·204,119 ± 1·255,819
${}_2M_3$	179·342,927	194·848,209 ± 22·569,784
${}_2M_4$	4,445·996,023	4,857·431,837 ± 702·936,762
${}_2B_1$	1·0503	1·1367 ± 0·1788
${}_2B_2$	4·5420	4·6836 ± 0·5087
Σ_2	5·5933	5·6749 ± 0·0843

In these tables the P.E.'s of the constants of the experimental distributions were obtained (in accordance with theory) from the constants of the corresponding theoretic distributions. The higher betas ${}_2B_3$, ${}_2B_4$ and ${}_2B_6$ of the two theoretic distributions necessary to calculate the P.E.'s of ${}_2M_3$ and ${}_2M_4$ of the experimental distributions were calculated from the ${}_2B_1$ and ${}_2B_2$ of the theoretic distributions by use of the difference formulae of the next section, this being the only feasible method. This fact, together with the general theory upon which the P.E. rests, causes the values of the P.E.'s given here to be of doubtful accuracy and utility. The indication they give of the size of the variations of the constants due to sampling is probably only of a quite approximate nature.

The tables show very little beyond the large variations to be expected in the betas and higher moments of actual distributions of σ^2 of samples. The subsequent tests for Goodness of Fit show that these variations have little significance; for example, although the agreement between the constants of Distribution III (b) and the theoretic constants in the first table is strikingly less close than that between the constants of Distribution IV (b) and the theoretic constants in the second table, the fit in the first case is rather better than that in the second.

With the application of the Goodness of Fit tests in view these two theoretic distributions of σ^2 of samples of 10 from Populations (A) and (B) were now fitted with Pearson Frequency Curves.

The Constants of the first being

$$\text{Mean} = 13{\cdot}282{,}791, \qquad {}_2B_1 = 1{\cdot}748{,}312,$$
$$ {}_2B_2 = 6{\cdot}644{,}744, \qquad \Sigma_2 = 6{\cdot}484{,}397,$$

the Skewness is ·932 and it is represented by a Type IV curve with equation of the form

$$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu \tan^{-1}\frac{x}{a}},$$

where, in the usual notation,

$$r = 11{\cdot}434{,}579, \qquad m = 6{\cdot}717{,}290,$$
$$z = 1{,}926{\cdot}017{,}102, \qquad \nu = -42{\cdot}370{,}612, \qquad a = 5{\cdot}457{,}542.$$

Taking $$y_0 = \frac{N}{a}\sqrt{\frac{r}{2\pi}} \cdot \frac{e^{\frac{\cos^2\phi}{r} - \frac{1}{12r} - \phi\nu}}{(\cos\phi)^{r+1}},$$

where $$\tan\phi = \frac{\nu}{r} = -3{\cdot}705{,}4807,$$

we have $$\log y_0 = -14{\cdot}416{,}779,$$

and the curve is therefore

$$y = 10^{-14{\cdot}416{,}779}\left[1 + \frac{x^2}{(5{\cdot}457{,}542)^2}\right]^{-6{\cdot}717{,}290} e^{42{\cdot}370{,}612 \tan^{-1}\left(\frac{x}{5{\cdot}457{,}542}\right)}.$$

The Origin $= \text{mean} + \dfrac{\nu a}{r} = -6{\cdot}940{,}026.$

The Mode $= \text{mean} - \dfrac{1}{2}\dfrac{{}_2M_3}{{}_2M_2}\dfrac{r-2}{r+2} = 10{\cdot}272{,}229,$

which corresponds to $\theta = 72^\circ\ 25{\cdot}6'$.

Putting $x = a\tan\theta$ the curve becomes

$$\log y = -14{\cdot}416{,}779 + 13{\cdot}434{,}579 \log\cos\theta + 18{\cdot}401{,}324\,\theta.$$

Again, Distributions I (*b*) and III (*b*) range from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 55{\cdot}5$, so it is sufficient to plot from $\theta = 53^\circ$ to $\theta = 84^\circ$.

The Plotted Ordinates of this Curve, which is shown in Fig. III, are

θ°	x	y	θ°	x	y	θ°	x	y
53	7·2424	0·4386	68	13·5079	49·3761	74·5	19·6793	64·6459
55	7·7942	1·0093	68·5	13·8548	53·2682	75	20·3678	60·8621
57	8·4039	2·2095	69	14·2174	57·0318	75·5	21·1027	56·4311
59	9·0620	4·5768	69·5	14·5969	60·5757	76	21·8890	51·4749
61	9·8457	8·9122	70	14·9945	63·8058	76·5	22·7323	46·1399
62	10·2642	12·1219	70·5	15·4116	66·6226	77	23·6392	40·5882
63	10·7110	16·1846	71	15·8498	68·9275	78	25·6757	29·5106
64	11·1896	21·1833	71·5	16·3109	70·6270	79	28·0766	19·5119
65	11·7037	27·1414	72	16·7966	71·6367	80	30·9513	11·5241
66	12·2578	33·9870	72·5	17·3091	71·8859	81	34·4576	5·9385
66·5	12·5515	37·6856	73	17·8508	71·3242	82	38·8324	2·5862
67	12·8572	41·5184	73·5	18·4243	69·9243	83	44·4481	0·9103
67·5	13·1757	45·4356	74	19·0327	67·6879	84	62·3800	0·0442

The Constants of the second of these theoretic distributions of σ^2 of samples of 10 being

$$\text{Mean} = 11{\cdot}465{,}313, \qquad {}_2B_1 = 1{\cdot}050{,}297,$$
$${}_2B_2 = 4{\cdot}541{,}956, \qquad \Sigma_2 = 5{\cdot}593{,}261,$$

its Skewness is ·522 and it is represented by a Type III curve with equation of the form

$$y = y_0 e^{-\gamma x}\left(1 + \frac{x}{a}\right)^p, \text{ where } p = \gamma a.$$

In the usual notation we have for this curve

$$\gamma = 0{\cdot}348{,}880, \qquad p = 2{\cdot}808{,}447,$$
$$a = 8{\cdot}049{,}894.$$

Also $$y_0 = \frac{N}{a}\frac{p^{p+1}}{e^p\,\Gamma(p+1)}, \text{ i.e. } y_0 = 80{\cdot}6342.$$

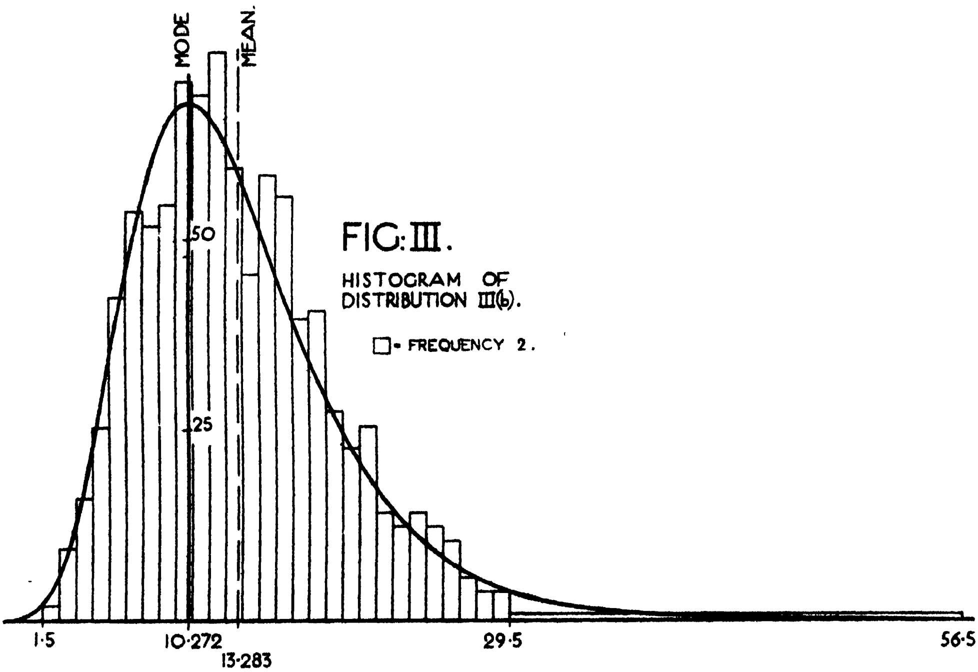
The equation of the curve is therefore

$$y = 80{\cdot}6342\, e^{-0{\cdot}348,880\,x}\left(1 + \frac{x}{8{\cdot}049,894}\right)^{2{\cdot}808,447}$$

The Mean of the distribution is 11·465,313.

The Origin (which is the Mode) = 8·598,998.

The actual Distribution IV (b) ranges from $\nu_1 = 0{\cdot}5$ to $\nu_2 = 40{\cdot}5$, so it is sufficient to plot from $x = -8{\cdot}0$ to $x = 32{\cdot}0$.



σ^2's of Samples in $\frac{1}{100}$ sq. inch of Barometric Height.

Distribution of σ^2 in 1000 Samples of 10 from the Skew Distribution of Barometric Heights (A). Fitted Curve from actual moments of (A).

The Plotted Ordinates of this Curve, which is shown in Fig. IV, are

x	y	x	y	x	y	x	y
−*8*	0·0008	−*0·5*	80·1800	*8*	34·3576	*24*	0·9022
−*7*	3·0386	*0·0*	80·6342	*10*	23·7808	*26*	0·5322
−*6*	14·0355	*0·5*	80·2157	*12*	15·8987	*28*	0·3109
−*5*	30·2227	*1·0*	79·0353	*14*	10·3348	*30*	0·1801
−*4*	47·2824	*2*	74·8409	*16*	6·5639	*32*	0·1035
−*3*	61·9925	*3*	68·9160	*18*	4·0885		
−*2*	72·6420	*4*	62·0113	*20*	2·5047		
−*1*	78·7497	*6*	47·5042	*22*	1·5126		

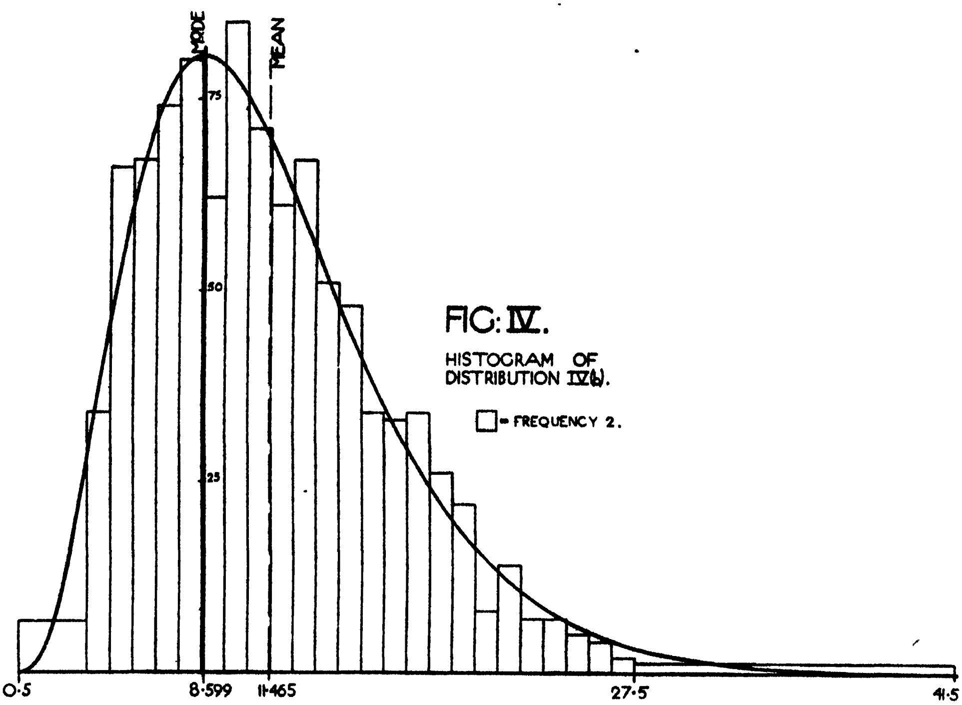
Tests for Goodness of Fit.

(1) For theoretical curve of σ^2 of samples of 10 from Population (A) against experimental Distribution I (*b*), i.e. ticket drawing:

29 Groups. $\chi^2 = 169{\cdot}95$. $P = 0$.

(2) For theoretical curve of σ^2 of samples of 10 from Population (A) against experimental Distribution III (*b*):

29 Groups. $\chi^2 = 23{\cdot}97$. $P = {\cdot}6831$.



σ^2's of Samples in squared units of original Distribution (B) with sub-range one unit.

Distribution of σ^2 in 1000 Samples of 10 from the Smooth Skew Distribution (B).
Fitted Curve from actual moments of (B).

(3) For theoretical curve of σ^2 of samples of 10 from Population (B) against experimental Distribution IV (*b*):

26 Groups. $\chi^2 = 22{\cdot}46$. $P = {\cdot}6090$.

From these it may be concluded that the theoretical Pearson Curve of σ^2 of samples of N from any "infinite" population, as obtained from its moments given by the Tchouproff formulae (1), (2), (3) and (4), does give an adequate representation of actual distributions for cases where N is small, provided the sampling leading to the actual distribution is of a reasonably random character. Moreover there is nothing to suggest that this representation will be any less accurate for larger values of N, always providing the assumption of an "infinite" sampled population be legitimate.

Population (A) is a sampled population with frequency groups of irregular size and with a very pronounced "tail," whereas Population (B) is one with graduated frequency groups and no pronounced "tail." From the fits obtained here it is evident that Tchouproff's formulae give as good results in one case as the other, that is: the presence of a "tail" has little significance for the purpose of sampling provided the sampling is random. It is very interesting, however, to see the divergences its presence causes between the actual and theoretical values of the higher moments of the distribution of the σ^2 of samples and still further the negligible ultimate effect that arises from these divergences as measured by χ^2.

(*c*) From the previous section Tchouproff's formulae may be deemed to give an efficient representation in practice of actual distributions of σ^2 of samples from any "infinite" population, and therefore it is of importance to ascertain whether their practical use can be extended by eliminating the laborious process of determining the first eight moments of the sampled population.

Now the sampled population being in general merely a sample from a larger population, the actual values of its higher moments are problematical as they are subject to large Probable Errors. Since Tchouproff's formulae can be expressed in terms of the betas and second moment of the sampled population, it is possible therefore that quite good approximate results for the moments of the distribution of σ^2 of samples may be obtained if, instead of the actual higher betas of the sampled population, any convenient good approximations to them are used. The convenience here will consist mainly in the ease and speed of the calculation necessary to obtain these approximate values.

Now assuming the applicability of Pearson's Frequency Curves, the values of $\bar{\beta}_3$, $\bar{\beta}_4$ and $\bar{\beta}_6$ can be obtained theoretically from the actual values of $\bar{\beta}_1$ and $\bar{\beta}_2$ by the use of the difference formulae

$$\beta_n \text{ (even)} = (n+1)\left\{\tfrac{1}{2}\beta_{n-1} \quad + (1+\tfrac{1}{2}\alpha)\beta_{n-2}\right\}/(1-\tfrac{1}{2}\overline{n-1}\alpha),$$

$$\beta_n \text{ (odd)} \ = (n+1)\left\{\tfrac{1}{2}\beta_1\beta_{n-1} + (1+\tfrac{1}{2}\alpha)\beta_{n-2}\right\}/(1-\tfrac{1}{2}\overline{n-1}\alpha),$$

where

$$\alpha = (2\beta_2 - 3\beta_1 - 6)/(\beta_2 + 3),$$

and the purpose of this section is to determine whether the representation of actual distributions of σ^2 of samples is as good, when in Tchouproff's formulae the actual $\bar{\beta}_3$, $\bar{\beta}_4$ and $\bar{\beta}_6$ of the sampled population are replaced by their approximate values obtained by the use of the above difference formulae, as it was when these higher betas were given their actual values as in the previous section.

Before the actual work performed with this object is given, it may be noted that the Tables XLII (*a*), (*b*), (*c*) and (*d*) of the higher betas calculated from β_1 and β_2 by these difference formulae, given in *Tables for Statisticians*, pp. 78—79, are in one or two cases in error. They have been re-calculated lately by Professor K. Yasukawa and are re-issued in the present number of *Biometrika*. Also, what is of more importance here, these difference formulae are based on the assumption that the sampled population is represented by a smooth curve, that is it is a continuous distribution. Therefore to apply them properly the $\bar{\nu}_2$, $\bar{\nu}_3$, $\bar{\nu}_4$ of the

sampled population should be first corrected by Sheppard's corrections and the $\bar{\beta}_1$ and $\bar{\beta}_2$ calculated from these values, then the difference formulae (or tables) should be applied to obtain the higher betas of the sampled population and the Tchouproff formulae applied to them to obtain the first four moments of the distribution of σ^2 of samples of N. Sheppard's corrections reversed should then be applied to these four moments and from the resulting uncorrected moments the two working values of the betas of this distribution obtained.

This procedure is such that, if carried out, the time and labour involved would be little less, if any, than that involved in the calculation of the first eight moments of the sampled population. Consequently, unless the straightforward application of these difference formulae as they stand gives a reliable representation of actual distributions of σ^2 of samples there is, for practical statistics, little point in their use.

With this consideration in view, in the following work, the difference formulae have been applied quite straightforwardly to the first two betas of the sampled populations to obtain the values of the higher betas to be substituted in the Tchouproff formulae.

The Constants of the original Populations (A) and (B), when the higher betas were calculated by the difference formulae, were

(1) For Population (A):

$$\bar{\nu}_2 = 14\cdot758{,}657, \qquad \bar{\beta}_1 = 0\cdot189{,}424,$$
$$\bar{\beta}_2 = 3\cdot160{,}978, \qquad \bar{\beta}_3 = 1\cdot865{,}497,$$
$$\bar{\beta}_4 = 19\cdot012{,}510, \qquad \bar{\beta}_6 = 182\cdot728{,}377.$$

(2) For Population (B):

$$\bar{\nu}_2 = 12\cdot739{,}237, \qquad \bar{\beta}_1 = 0\cdot219{,}333,$$
$$\bar{\beta}_2 = 3\cdot157{,}676, \qquad \bar{\beta}_3 = 2\cdot120{,}111,$$
$$\bar{\beta}_4 = 19\cdot058{,}578, \qquad \bar{\beta}_6 = 182\cdot723{,}685.$$

On application of Tchouproff's formulae it was found that the Constants of the Distribution of σ^2 of Samples of 10 from Population (A) were

$$\text{Mean} = 13\cdot282{,}791, \qquad {}_2M_2 = 42\cdot047{,}408,$$
$${}_2M_3 = 292\cdot350{,}383, \qquad {}_2M_4 = 8{,}615\cdot709{,}286,$$

which gave $\Sigma_2 = 6\cdot484{,}397, \qquad {}_2B_1 = 1\cdot149{,}714,$

and ${}_2B_2 = 4\cdot873{,}181;$

whilst the Constants of the Distribution of σ^2 of Samples of 10 from Population (B) were

$$\text{Mean} = 11\cdot465{,}313, \qquad {}_2M_2 = 31\cdot284{,}573,$$
$${}_2M_3 = 186\cdot066{,}245, \qquad {}_2M_4 = 4{,}709\cdot529{,}436,$$

which gave $\Sigma_2 = 5\cdot593{,}261, \qquad {}_2B_1 = 1\cdot130{,}522,$

and ${}_2B_2 = 4\cdot811{,}178.$

The fitting of Pearson Curves to these two distributions gave the following results:

For the first, $\kappa = 3{\cdot}736$; consequently a Type VI curve gives the best fit.

The equation is of the form $y = y_0 (x - a)^{q_2}/x^{q_1}$, and in this case

$$r = -54{\cdot}978{,}810, \qquad \epsilon = -276{\cdot}156{,}622;$$

the quadratic is $\quad z^2 + 54{\cdot}978{,}810z - 276{\cdot}156{,}622 = 0;$

its roots are $\quad -59{\cdot}611{,}422$ and $4{\cdot}632{,}612$.

Therefore $\quad q_1 = 60{\cdot}611{,}422, \qquad q_2 = 3{\cdot}632{,}612 \quad$ (${}_2M_3$ is positive);

also
$$ {}_2M_2 = \frac{a^2\epsilon}{r^2(r+1)}.$$

Accordingly $\quad a = 157{\cdot}615{,}481;$

also $\quad \log y_0 = 133{\cdot}033{,}978.$

Thus we have the curve:

$$y = 10^{133{\cdot}033{,}978}(x - 157{\cdot}615{,}481)^{3{\cdot}632{,}612}\, x^{-60{\cdot}611{,}422}.$$

The Origin $= -157{\cdot}613{,}650$.

The Mode $=$ $10{\cdot}050{,}406$.

The actual Distributions I (*b*) and III (*b*) range from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 55{\cdot}5$, so it is sufficient to plot the curve from $x = 158{\cdot}0$ to $x = 213{\cdot}0$.

The Plotted Ordinates of this Curve, which is shown in Fig. V, are

x	y	x	y	x	y	x	y
158	0·0183	*167*	69·8821	*175*	38·4848	*188*	3·8014
159	1·3106	*167·66*	70·4309	*176*	33·3829	*190*	2·5232
160	6·4578	*168*	70·2998	*177*	28·7042	*193*	1·3469
161	15·7972	*169*	68·5138	*178*	24·4908	*196*	0·7107
162	27·7962	*170*	65·0586	*179*	20·7534	*199*	0·3720
163	40·3752	*171*	60·4533	*180*	17·4800	*202*	0·1937
164	51·7445	*172*	55·1579	*182*	12·2091	*205*	0·1005
165	60·7301	*173*	49·5505	*184*	8·3823	*208*	0·0521
166	66·7905	*174*	43·9209	*186*	5·6762	*213*	0·0174

For the second, $\kappa = 4{\cdot}713$, so these data are likewise best fitted by a Type VI curve.

Here $\quad r = -69{\cdot}690{,}784, \qquad \epsilon = -326{\cdot}992{,}335;$

the quadratic is $\quad z^2 + 69{\cdot}690{,}784z - 326{\cdot}992{,}335 = 0;$

its roots are $\quad -74{\cdot}103{,}432$ and $4{\cdot}412{,}648$.

Therefore $\quad q_1 = 75{\cdot}103{,}432$ and $q_2 = 3{\cdot}412{,}648 \quad$ (${}_2M_3$ is positive);

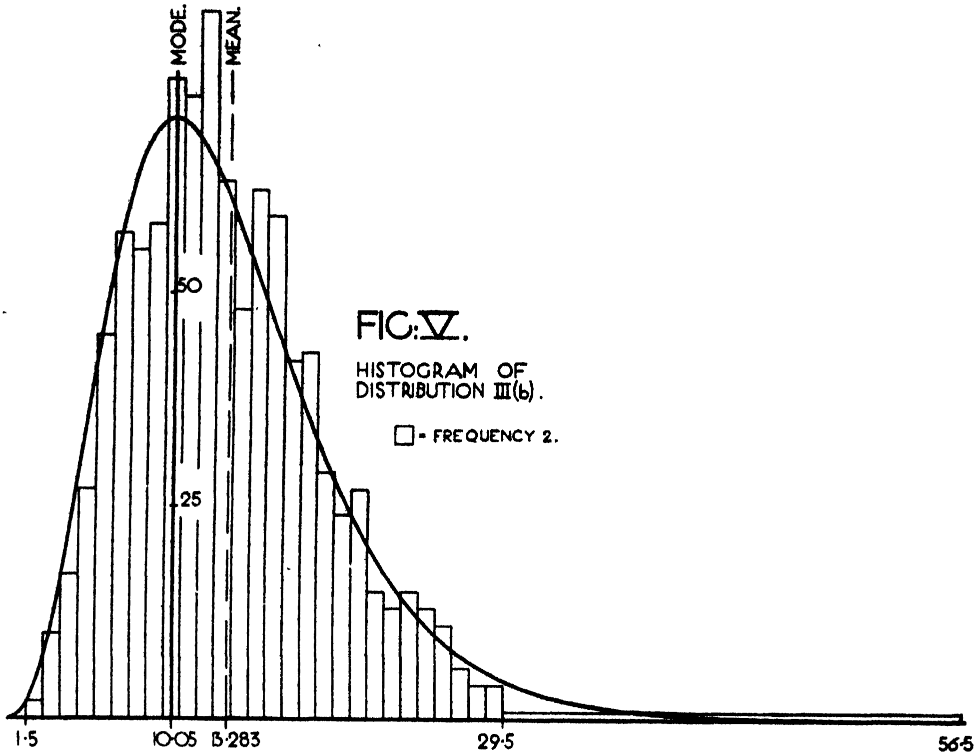
also $\quad a = 178{\cdot}657{,}362$ and $\log y_0 = 169{\cdot}391{,}180$,

and the curve is $\quad y = 10^{169{\cdot}391{,}180}(x - 178{\cdot}657{,}362)^{3{\cdot}412{,}648}\, x^{-75{\cdot}103{,}432}.$

The Origin $= -178{\cdot}504{,}191$.

The Mode $=$ $8{\cdot}657{,}465$.

The actual Distribution IV (b) ranges from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 40{\cdot}5$, so it is sufficient to plot the curve from $x = 179{\cdot}0$ to $x = 219{\cdot}0$.



σ^2's of Samples in $\frac{1}{100}$ sq. inch of Barometric Height.

Distribution of σ^2 in 1000 Samples of 10 from the Skew Distribution of Barometric Heights (A). Fitted Curve from β_1 and β_2 of Distribution (A), higher β's by the Difference Formulae.

The Plotted Ordinates of this Curve, which is shown in Fig. VI, are

x	y	x	y	x	y	x	y
179	0·0404	*187*	81·6889	*194*	41·3504	*206*	3·2752
180	2·8129	*187·16*	81·7382	*195*	34·8630	*208*	2·0171
181	12·4006	*188*	80·5368	*196*	29·0766	*210*	1·2312
182	27·5800	*189*	76·5008	*197*	24·0240	*212*	0·7462
183	44·6443	*190*	70·5232	*198*	19·6863	*214*	0·4497
184	60·1402	*191*	63·4358	*200*	12·9475	*216*	0·2698
185	71·8902	*192*	55·9085	*202*	8·3252	*219*	0·1247
186	79·0347	*193*	48·4341	*204*	5·2587		

The tests for Goodness of Fit gave the following results:

(1) For theoretical curve of σ^2 of samples of 10 from Population (A), when the higher betas of this population have been replaced by the values obtained by the difference formulae, against experimental Distribution I (b), i.e. ticket drawing:

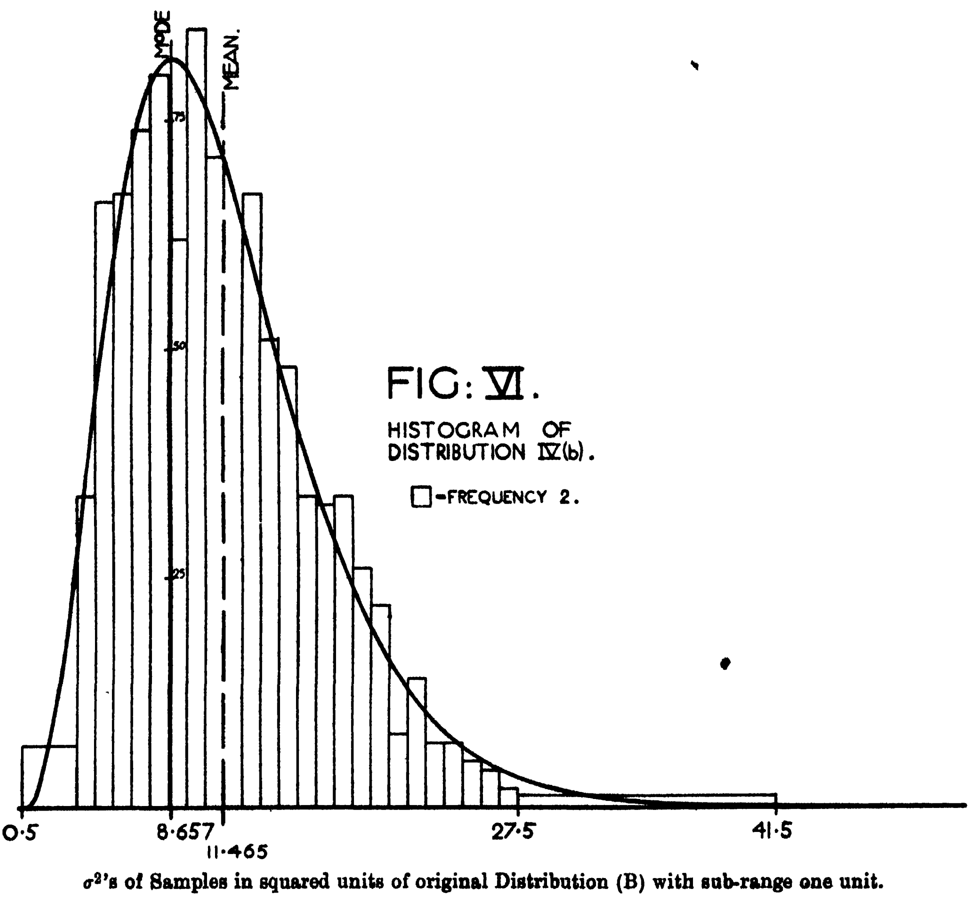
29 Groups. $\chi^2 = 166{\cdot}72$. $P = 0$.

(2) For theoretical curve of σ^2 of samples of 10 from Population (A), when the higher betas of this population have been replaced by the values obtained by the difference formulae, against experimental Distribution III (*b*):

29 Groups. $\chi^2 = 25{\cdot}78$. $P = {\cdot}5851$.

(3) For theoretical curve of σ^2 of samples of 10 from Population (B), when the higher betas of this population have been replaced by the values obtained by the difference formulae, against experimental Distribution IV (*b*):

26 Groups. $\chi^2 = 23{\cdot}99$. $P = {\cdot}5200$.



σ^2's of Samples in squared units of original Distribution (B) with sub-range one unit.

Distribution of σ^2 in 1000 Samples of 10 from the Smooth Skew Distribution (B).
Fitted Curve from β_1 and β_2 of Distribution (B), higher β's by the Difference Formulae.

The fits here are, for all intents and purposes, identical with those obtained in the previous section, consequently the chief difficulty in the application of Tchouproff's formulae is abolished.

We thus conclude that the moments of the distribution of σ^2 of samples of N from any "infinite" population may be deduced with reasonable speed and ease by the application of Tchouproff's formulae for the actual second moment, and

by using the first two betas of the sampled population and the approximate values of its higher betas as obtained by the assumption of Pearson's Frequency Types, together with the straightforward application of the difference relations which follows that assumption. Moreover the values of the moments of the distribution of σ^2 of samples so obtained will lead to a representation of this distribution by a Pearson Frequency Curve, which will describe adequately any actual distributions of σ^2 of samples obtained from an actual set of samples that are the result of truly random sampling.

PART IV. THE DISTRIBUTION OF MEANS OF SAMPLES OF N FROM A FINITE POPULATION OF M.

If, in drawing a sample of N, the individuals are drawn separately and replaced each time before the next is drawn, the sampled population, as we have seen before, can be treated as infinite in all its categories. If however the whole sample be drawn before replacement, the sampled population, in theory, must be treated as of finite total frequency M. However, as we shall see later, even when the samples are actually so drawn in practice, the resulting distributions can be represented approximately under certain circumstances by theoretical curves obtained by treating the sampled population as infinite, the loss in accuracy being small whilst the saving of time and labour is considerable. The determination of reliable conditions under which this can be done is a matter of some difficulty and will be discussed more fully later; but, in order that this consideration should not affect the discussion, in Parts IV and V a sampled population has been chosen which clearly cannot be treated as infinite when the complete sample is drawn before replacement.

This population, which will be called Population (C), is the binomial distribution $(\frac{1}{6}+\frac{5}{6})^2$, having a total frequency of 36.

The distribution, its moments and betas are as follows:

Designation of the Frequency Group	A	B	C
Central Abscissa	0	1	2
Frequency	1	10	25

$$\text{Mean} = 1{\cdot}666{,}667, \qquad \nu_2 = 0{\cdot}277{,}778,$$
$$\nu_3 = -0{\cdot}185{,}185, \qquad \nu_4 = 0{\cdot}277{,}778,$$
$$\nu_5 = -0{\cdot}390{,}950, \qquad \nu_6 = 0{\cdot}620{,}722,$$
$$\nu_8 = 1{\cdot}664{,}817; \qquad \sigma = 0{\cdot}527{,}046,$$
$$\beta_1 = 1{\cdot}600{,}000, \qquad \beta_2 = 3{\cdot}600{,}000,$$
$$\beta_3 = 12{\cdot}160{,}074, \qquad \beta_4 = 28{\cdot}959{,}690$$

and

$$\beta_6 = 279{\cdot}624{,}976.$$

Now the various types of samples of 10 from this population of 36, with their frequencies and means, are:

Type of Sample			Frequency	Mean
0 A	0 B	10 C	3,268,760	2·0
0 A	1 B	9 C	20,429,750	1·9
0 A	2 B	8 C	48,670,875	1·8
0 A	3 B	7 C	57,684,000	1·7
0 A	4 B	6 C	37,191,000	1·6
0 A	5 B	5 C	13,388,760	1·5
0 A	6 B	4 C	2,656,500	1·4
0 A	7 B	3 C	276,000	1·3
0 A	8 B	2 C	13,500	1·2
0 A	9 B	1 C	250	1·1
0 A	10 B	0 C	1	1·0
1 A	0 B	9 C	2,042,975	1·8
1 A	1 B	8 C	10,815,750	1·7
1 A	2 B	7 C	21,631,500	1·6
1 A	3 B	6 C	21,252,000	1·5
1 A	4 B	5 C	11,157,300	1·4
1 A	5 B	4 C	3,187,800	1·3
1 A	6 B	3 C	483,000	1·2
1 A	7 B	2 C	36,000	1·1
1 A	8 B	1 C	1,125	1·0
1 A	9 B	0 C	10	0·9

The total frequency of the samples is, of course, $36!/(10!\,26!) = 254{,}186{,}856$.

Taking the means of samples, we have the following discrete frequency distribution, which is the true or complete distribution of means of samples in this case.

The Complete Distribution of Means of Samples of 10 *from* (C).

Value of the Mean	Frequency
0·9	10
1·0	1,126
1·1	36,250
1·2	496,500
1·3	3,463,800
1·4	13,813,800
1·5	34,640,760
1·6	58,822,500
1·7	68,499,750
1·8	50,713,850
1·9	20,429,750
2·0	3,268,760
Total	254,186,856

The constants are:

$$\text{Mean} = 1{\cdot}666{,}667, \qquad {}_1M_2 = 0{\cdot}020{,}635,$$
$${}_1M_3 = -\,0{\cdot}000{,}647, \qquad {}_1M_4 = 0{\cdot}001{,}199,$$
$$\Sigma_1 = 0{\cdot}143{,}649, \qquad {}_1B_1 = 0{\cdot}047{,}697,$$
$${}_1B_2 = 2{\cdot}816{,}989.$$

Now the formulae for the Mean, Second Moment-Coefficient and first two Betas of the Distribution of Means of Samples from a finite population are as given below, but their ultimate origin is doubtful, for I have been unable to ascertain the person who first deduced them.

The formulae are:

Mean of Distribution of Means of Samples = Mean of Sampled Population...(1),

$$ {}_1M_2 = \frac{M-N}{N(M-1)}\bar{\nu}_2 \qquad \text{(2)}, $$

$$ {}_1B_1 = \frac{(M-1)(M-2N)^2}{N(M-N)(M-2)^2}\bar{\beta}_1 \qquad \text{(3)}, $$

$$ {}_1B_2 = \frac{(M-1)\left[(M^2-6MN+M+6N^2)\bar{\beta}_2 + 3M(M-N-1)(N-1)\right]}{N(M-2)(M-3)(M-N)} \qquad \text{(4)}, $$

where $\bar{\nu}_2$, $\bar{\beta}_1$ and $\bar{\beta}_2$ are the corresponding constants of the sampled population.

On application of these formulae to the constants of Population (C) the results were:

$$ \begin{aligned} \text{Mean of Distribution of Means} &= 1{\cdot}666{,}667, \\ {}_1M_2 &= 0{\cdot}020{,}635, \\ {}_1B_1 &= 0{\cdot}047{,}697, \\ {}_1B_2 &= 2{\cdot}816{,}989, \end{aligned} $$

which agree with the values given above, thus supplying a numerical check to the formulae.

In the consideration of the Distribution of Means from any population when the complete sample is drawn before replacement, as the above formulae give the betas of the distribution, three considerations arise:

(*a*) The adequate representation of this complete distribution by a Pearson Curve, the constants of which are those given by formulae (1), (2), (3) and (4) of this part.

(*b*) The possible adequate representation of this distribution by a Pearson Curve, the constants of which are the values of ${}_1M_2$, ${}_1B_1$ and ${}_1B_2$ given by the simpler formulae of Part I, which involve the assumption of an infinite sampled population.

(*c*) A Distribution of Means will not be obtained actually in practice by theoretical methods as employed here to obtain the complete distribution from Population (C), but by the actual drawing of a limited number of samples. The question thus arises as to whether the Pearson Curve, as obtained in the first of these considerations, will represent adequately such a distribution of Means obtained by actual sampling.

(*a*) Now it does not follow *a priori* that, because the constants obtained by the formulae of this part are the actual constants of the complete distribution of Means of Samples obtained from the population, the resulting Pearson Curve will represent the distribution, for it is possible for different distributions to have the same second moment and first two betas.

As we shall see in the case of the Distribution of σ^2 it is possible, in extreme cases, for the representation by a Pearson Curve to fail completely. Now, undoubtedly, Population (C) is extreme both in the paucity of its categories and in its smallness, whilst its skewness and lack of a true mode are abnormal. Therefore, if the true distribution of Means of Samples be adequately represented by a Pearson Curve in this case, we are practically justified in assuming that these formulae and a consequent Pearson Curve will represent, for practical purposes, any Distribution of Means of Samples from a finite population.

A Pearson Curve is thus required whose constants are

$$\text{Mean} = 1{\cdot}666{,}667, \qquad \Sigma_1 = 0{\cdot}143{,}640,$$
$$_1B_1 = 0{\cdot}047{,}697, \qquad _1B_2 = 2{\cdot}816{,}989,$$

the unit being that of Population (C).

The curve is clearly of Type I and therefore, in the usual notation, we have

$$r = 20{\cdot}850{,}646, \qquad \epsilon = 101{\cdot}459{,}700,$$
$$m'^2 - 20{\cdot}850{,}646m' + 101{\cdot}459{,}700 = 0,$$

with roots 13·113,664 and 7·736,982; as $_1M_3$ is negative, therefore

$$m_1' = 13{\cdot}113{,}664 \text{ and } m_2' = 7{\cdot}736{,}982;$$

therefore $$m_1 = 12{\cdot}113{,}664 \text{ and } m_2 = 6{\cdot}736{,}982.$$

Changing, for convenience, the unit to $\frac{1}{100}$ of that of (C), we have

$$b = 13{\cdot}899{,}77, \quad a_1 = 8{\cdot}932{,}13 \text{ and } a_2 = 4{\cdot}967{,}58.$$

Now as the curve is to be used again later and, moreover, the total frequency 254,186,856 of the complete Distribution of Means of Samples of 10 from (C) is so large, the total frequency of the curve has been taken as 1000.

Accordingly: $$y_0 = 269{\cdot}791.$$

The curve is thus

$$y = 269{\cdot}791\left(1 + \frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}664}\left(1 - \frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}982}.$$

The origin, which is the mode, is at $16{\cdot}666{,}67 + 0{\cdot}190{,}15 = 16{\cdot}856{,}82$, whilst the range is roughly from $x = -9{\cdot}0$ to $x = 5{\cdot}0$.

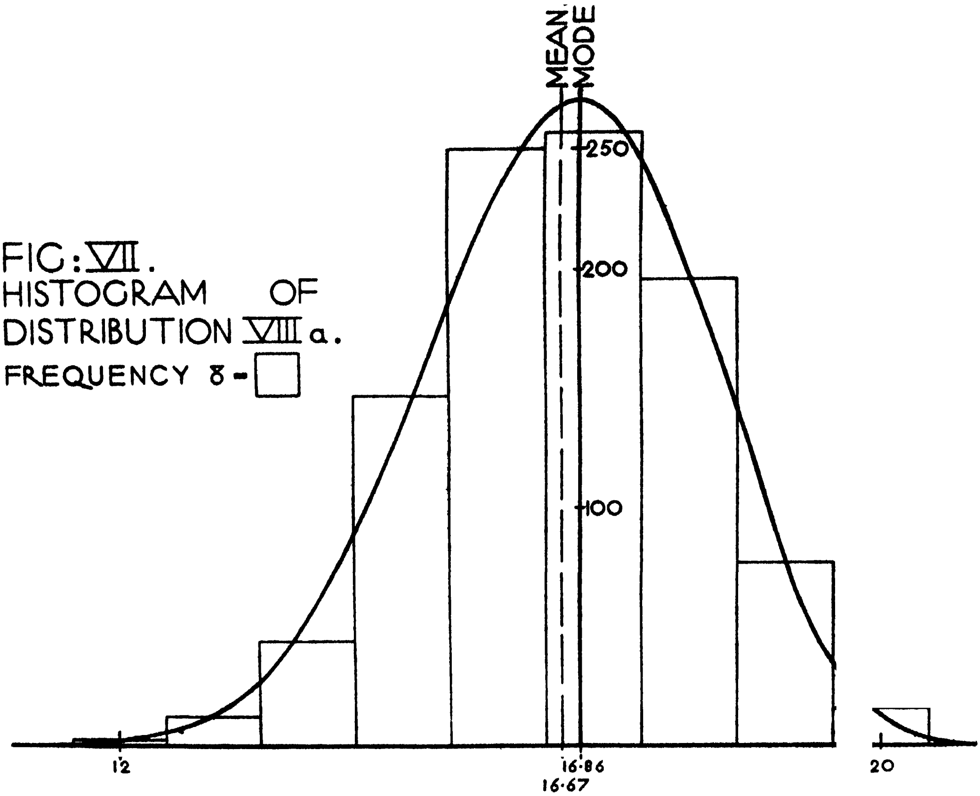
The curve is shown in Fig. VII, and its ordinates are

Abscissa	Ordinate	Abscissa	Ordinate
−7·0	0·001	−0·5	256·229
−6·0	0·077	0·0	269·791
−5·0	1·420	0·5	255·379
−4·0	10·837	1·0	214·612
−3·0	45·603	2·0	96·793
−2·0	122·873	3·0	17·570
−1·0	220·297	4·0	0·256

For the representation of the complete Distribution of Means of Samples of 10 from (C) by this curve, we have

Value of Mean	Frequency in Curve	Actual Frequency
1·3 and under	17·40	15·73
1·4	55·56	54·34
1·5	135·48	136·28
1·6	227·76	231·42
1·7	261·48	269·49
1·8	206·40	199·52
1·9	80·88	80·37
2·0	15·04	12·86

FIG: VII.
HISTOGRAM OF DISTRIBUTION VIII a.
FREQUENCY 8 - □



The small frequencies at the tails have been clubbed together so that the Goodness of Fit test may be applied. The actual frequencies are the true frequencies of the complete Distribution of Means of Samples of 10 from (C) multiplied by $\frac{1000}{254,186,856}$ to bring the total frequency down to 1000, as is necessary for comparison. This complete Distribution of Means is, as we have seen, a set of discrete frequencies, but as these discrete frequencies occur at equal intervals, we can represent it by a histogram by drawing rectangles, whose

areas are the separate discrete frequencies and whose mid-ordinates correspond to the values of the mean for which the discrete frequencies occur. Hence we can obtain a comparison between the curve and the distribution which is quite reasonable.

The application of the Goodness of Fit test gave $P = \cdot 993$ with $\chi^2 = 1\cdot 04$ for 8 Groups. Thus, even in this extreme case, a Pearson Curve represents perfectly adequately the true Distribution of Means of Samples.

Still further, an inspection of the betas of this distribution shows that, even in this exceptional type of sampled population, the Distribution of Means approximates very closely to normality. This is so clear, especially in the light of the fits to the Normal Curve of previous distributions of means in this paper, that the determination of the actual fit is not worth the labour.

Thus it may be concluded that the Distribution of Means of Samples from practically any "finite" population can be treated for practical purposes as a normal distribution, or, if a better representation is desired, can be represented by a Pearson Curve obtained from its first four moments.

(*b*) From the previous consideration, formulae (1), (2), (3) and (4) of this part, together with a Pearson Curve, will represent adequately an actual distribution of means from a "finite" population, i.e. a distribution of means of samples where each sample was drawn completely before replacement. Now, it has been shown that the formulae

$$ {}_1M_2 = \frac{\bar{\nu}_2}{N} \qquad \text{...(5)}, $$

$$ {}_1B_1 = \frac{\bar{\beta}_1}{3} \qquad \text{...(6)}, $$

$$ {}_1B_2 = \frac{\bar{\beta}_2 - 3}{N} + 3 \qquad \text{.................................(7)}, $$

together with a Pearson Curve, will likewise represent an actual distribution of means from an "infinite" population, i.e. a distribution of means where the individuals of the sample were drawn singly and replaced.

It will be useful if a practical indication can be obtained as to when the results of these two sets of formulae are not significantly different, that is, an indication as to when the Pearson Curve obtained from either set will represent reasonably well in practice, not only distributions of means obtained by sampling conducted in accordance with the assumption made in obtaining the curve, but also distributions obtained by sampling conducted in accordance with the opposite assumption. When this is so, it follows that a Pearson Curve obtained from the constants given by the simpler formulae (5), (6) and (7) will represent adequately in practice any actual distribution of means of samples obtained from sampling conducted by either method, that is, the assumption of an infinite sampled population will be legitimate, in practice, for the representation of distributions of means arising from sets of samples obtained either by drawing the complete sample, or individuals singly, before replacement.

The following tables give approximate values of ${}_1B_1$ and ${}_1B_2$ for values of M, N, $\bar{\beta}_1$ and $\bar{\beta}_2$ covering a fairly extensive range. For convenience, the value M of the total frequency of the sampled population has been expressed as a multiple of the magnitude N of the sample. In each table the row opposite $\frac{M}{N} = \infty$ gives the values of the constants for samples drawn from an infinite population, that is, the values calculated by formulae (5), (6) and (7).

Tables giving the values of ${}_1B_1$ for various values of M, N and $\bar{\beta}_1$.

$\bar{\beta}_1 = 0{\cdot}1.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·050000	0·020000	0·010000	0·002000	0·001000
100	0·049242	0·019519	0·009730	0·001941	0·000970
50	0·048470	0·019036	0·009461	0·001883	0·000941
10	0·041701	0·015123	0·007330	0·001431	0·000713
5	0·031641	0·010208	0·004785	0·000911	0·000453

$\bar{\beta}_1 = 0{\cdot}5.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·250000	0·100000	0·050000	0·010000	0·005000
100	0·246212	0·097595	0·048651	0·009707	0·004852
50	0·242348	0·095181	0·047304	0·009415	0·004705
10	0·208505	0·075617	0·036651	0·007154	0·003566
5	0·158203	0·051040	0·023926	0·004555	0·002264

$\bar{\beta}_1 = 1{\cdot}0.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·500000	0·200000	0·100000	0·020000	0·010000
100	0·492425	0·195191	0·097302	0·019414	0·009704
50	0·484696	0·190363	0·094608	0·018831	0·009410
10	0·417010	0·151235	0·073303	0·014308	0·007133
5	0·316406	0·102079	0·047852	0·009109	0·004527

$\bar{\beta}_1 = 1{\cdot}5.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·750000	0·300000	0·150000	0·030000	0·015000
100	0·738637	0·292786	0·145953	0·029121	0·014556
50	0·727044	0·285544	0·141912	0·028246	0·014115
10	0·625514	0·226852	0·109954	0·021462	0·010699
5	0·474609	0·153119	0·071777	0·013664	0·006791

$\bar{\beta}_1 = 2{\cdot}0.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	1·000000	0·400000	0·200000	0·040000	0·020000
100	0·984849	0·390381	0·194604	0·038827	0·019408
50	0·969392	0·380726	0·189216	0·037662	0·018819
10	0·834019	0·302469	0·146606	0·028616	0·014265
5	0·632812	0·204159	0·095703	0·018218	0·009054

Tables giving the values of $_1B_2$ *for various values of* M, N *and* $\bar{\beta}_2$.

$\beta_2 = 1.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	2·000000	2·600000	2·800000	2·960000	2·980000
100	2·010101	2·606406	2·803593	2·960780	2·980394
50	2·020408	2·612823	2·807170	2·961554	2·980784
10	2·111111	2·664303	2·835052	2·967449	2·983751
5	2·250001	2·727273	2·867021	2·973887	2·986972

$\bar{\beta}_2 = 2.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	2·500000	2·800000	2·900000	2·980000	2·990000
100	2·497398	2·798365	2·899085	2·979802	2·989900
50	2·494586	2·796657	2·898142	2·979599	2·989798
10	2·462963	2·780142	2·889438	2·977799	2·988894
5	2·390626	2·750988	2·875166	2·975004	2·987501

$\bar{\beta}_2 = 3.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	3·000000	3·000000	3·000000	3·000000	3·000000
100	2·984695	2·990323	2·994578	2·998823	2·999406
50	2·968763	2·980490	2·989113	2·997644	2·998811
10	2·814815	2·895981	2·943825	2·988148	2·994037
5	2·531264	2·774704	2·883311	2·976122	2·988030

$\bar{\beta}_2 = 4.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	3·500000	3·200000	3·100000	3·020000	3·010000
100	3·471993	3·182281	3·090071	3·017844	3·008912
50	3·442940	3·164323	3·080085	3·015689	3·007824
10	3·166667	3·011820	2·998212	2·998497	2·990180
5	2·671876	2·798419	2·891456	2·977240	2·988560

$\bar{\beta}_2 = 6.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	4·500000	3·600000	3·300000	3·060000	3·030000
100	4·444659	3·566198	3·281057	3·055887	3·027923
50	4·391295	3·531990	3·260293	3·051780	3·025851
10	3·870370	3·243499	3·106985	3·019196	3·009465
5	2·953127	2·845850	2·907746	2·979476	2·989618

To determine approximately whether a value in any column of one of these tables is significantly different or not from the value at the head of the column, we are practically compelled to use the ordinary criterion of the Probable Error. The difference between these values must be compared with the P.E. of the value considered and the difference counted significant or not according as it is of the same order as, or considerably less than, three times this P.E.

The value of the P.E. will depend, of course, upon the total frequency desired for the distribution of means to be represented, but, apart from this, the value of the P.E. is still very doubtful as it has been obtained on the assumption that the higher betas of this distribution of means are given by the well-known difference formulae referred to in previous sections of this paper. This is, in general, only very approximately true, consequently the value of the P.E. can only be taken as a rough guide. However, this very roughness makes it possible to interpolate approximately in and between the tables, concise as they are, and so to obtain rough values of ${}_1B_1$ and ${}_1B_2$ for values of M, N, $\bar{\beta}_1$ and $\bar{\beta}_2$ not explicitly occurring. These, used in conjunction with this somewhat doubtful value of the P.E., will give at any rate an indication as to whether the difference in the two values of ${}_1B_1$ or ${}_1B_2$, as the case may be, is significant or not.

It is thus probable that, by interpolation in and between these tables together with the evaluation of the P.E.* of the value of the constant so approximately obtained, a rough but fairly reliable indication can be obtained quite quickly as to whether in any given case the population sampled may be treated as infinite or not, that is, as to whether the method (complete samples or individuals drawn before replacement) of sampling is immaterial or not.

In order to obtain some confirmation of this, two particular cases were investigated:

(1) Distributions of Means of Samples from Population (A).

(2) The Distribution of Means of Samples from Population (C).

(1) *Distributions of Means of* 1000 *Samples of* 10 *from Population* (A).

Now we have the following four series of Distributions of Means for 1000 Samples of 10 from this population:

I (*a*) The Means of 1000 Samples of 10, where the actual sampling was conducted by drawing marked "tickets" singly and replacing each time.

* See *Tables for Statisticians and Biometricians*, pp. 68—71.

II (a) The Means of 1000 Samples of 10, where the actual sampling was conducted by drawing marked "tickets," the complete sample being drawn each time before replacement.

III (a) The Means of 1000 Samples of 10, where the sampling was conducted by Tippett's Random Sample.

V (a) The Means of 1000 Samples of 10, where the sampling was conducted by drawing small commercial coloured beads, the complete sample being drawn each time before replacement.

Now for Population (A) $\bar{\beta}_1 = 0{\cdot}189{,}424$, $\bar{\beta}_2 = 3{\cdot}160{,}978$, and the total frequency is 4011. Thus it is easily seen by a rough application of the tables and the use of the P.E. obtained on the basis of 1000 samples, that the differences between the values of ${}_1B_1$ and of ${}_1B_2$ are in each case non-significant.

Confirming this by actual evaluation from the formulae, we have

(a) By formulae (2), (3) and (4) of this Part (Finite Sampled Population),

$${}_1B_1 = 0{\cdot}018{,}815 \pm {\cdot}007{,}018, \quad {}_1B_2 = 3{\cdot}014{,}569 \pm {\cdot}111{,}921.$$

(b) By formulae (5), (6) and (7) of this Part (Infinite Sampled Population),

$${}_1B_1 = 0{\cdot}018{,}942 \pm {\cdot}007{,}146, \quad {}_1B_2 = 3{\cdot}016{,}098 \pm {\cdot}111{,}224.$$

Clearly the differences are in each case non-significant. We should thus expect that the Pearson Curve with betas $0{\cdot}018{,}942$ and $3{\cdot}016{,}098$ would represent not only Distributions I (a) and III (a), but also Distributions II (a) and V (a) quite adequately. This curve has been given in Part II, Section (c), its equation being

$$y = 328{\cdot}5432\left(1 + \frac{x}{54{\cdot}216{,}202}\right)^{389{\cdot}606{,}521}\left(1 - \frac{x}{13{\cdot}207{,}422}\right)^{94{\cdot}911{,}385}.$$

Now applying the Goodness of Fit test to the representation of these distributions by this curve, we have

I (a)	15 Groups,	$\chi^2 = 84{\cdot}07$,	$P = {\cdot}0000$.
II (a)	19 Groups,	$\chi^2 = 24{\cdot}28$,	$P = {\cdot}1466$.
III (a)	16 Groups,	$\chi^2 = 7{\cdot}01$,	$P = {\cdot}9573$.
V (a)	16 Groups,	$\chi^2 = 13{\cdot}74$,	$P = {\cdot}5455$.

The fit to II (a) is not very good, but it is much better than that to I (a), where the sampling was conducted in accordance with theory, and thus the poorness of fit can be ascribed principally to the difficulty of shuffling properly a large mass of tickets. The fit to V (a) is quite good, consequently two conclusions are suggested:

Firstly that, when the tables of this section indicate the differences between the values of ${}_1B_1$ and ${}_1B_2$ are clearly non-significant, the population can be treated theoretically as infinite and the method of actual sampling (complete samples or individuals drawn between replacements) is immaterial.

Secondly, when Tippett's Random Sample is inapplicable or not obtainable, the inaccuracies involved in the use of "tickets" for actual sampling arising from the great difficulty of reliable shuffling can be very considerably reduced, for large populations, by using some more symmetrical and homogeneous material like small coloured commercial beads.

(2) *The Distribution of Means of Samples of* 10 *from Population* (C).

Now the betas of (C) are $\beta_1 = 1\cdot6$ and $\beta_2 = 3\cdot6$, whilst its total frequency is 36; thus it is easily seen, by rough application of the tables and calculation of the corresponding P.E.'s on the basis of say 1000 samples, that the differences between the two values of ${}_1B_1$ and the two values of ${}_1B_2$ are distinctly significant. Therefore the complete Distribution of Means of Samples of 10 from this population obtained at the commencement of Part IV cannot be represented by a curve or distribution obtained on the assumption that the population is to be considered infinite.

For this population, as it is a binomial, not only have we the complete discrete Distribution of Means mentioned above, but we have also the discrete Distribution of Means obtained by considering the population infinite. That is, in this case, although we have seen the representation by a Pearson Curve would be adequate, it is not necessary.

Assuming (C) to be infinite, i.e. in any sampling individuals only are drawn before replacement, the Distribution of Means of 1000 Samples of 10 is the binomial*

$$1000\left(\tfrac{1}{6}+\tfrac{5}{6}\right)^{20}, \text{ the unit being } \tfrac{1}{10} \text{ of that of (C)}.$$

Expanding this, and changing the unit back to that of (C), this distribution is

Value of Mean	*0·6*	*0·7*	*0·8*	*0·9*	*1·0*	*1·1*	*1·2*	*1·3*
Frequency	·0002	·0017	·0135	·0897	·4935	2·2431	8·4116	25·8820
Value of Mean	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*	
Frequency	64·7050	129·4100	202·2030	237·8860	198·2380	104·3360	26·0840	

The two Distributions of Means are thus

Value of the Mean	Frequency on the assumption (C) is infinite	Frequency on the assumption (C) is finite
0·6	0·0002	0·0000
0·7	0·0017	0·0000
0·8	0·0135	0·0000
0·9	0·0897	0·0000
1·0	0·4935	0·0044
1·1	2·2431	0·1426
1·2	8·4116	1·9533
1·3	25·8820	13·6272
1·4	64·7050	54·3461
1·5	129·4100	136·2833
1·6	202·2030	231·4189
1·7	237·8860	269·4910
1·8	198·2380	199·5179
1·9	104·3360	80·3745
2·0	26·0840	12·8599

* It may not be generally familiar, and it is indeed doubtful as to whether it has been published or not, although known in the Biometric Laboratory for some considerable time, that the Distribution of Means of Samples of N from an infinite population represented by the binomial $(p+q)^n$ is the binomial $(p+q)^{nN}$, plotted to $1/N$ the unit of the original binomial.

In the second case the frequencies have been multiplied by $\frac{1000}{254,186,856}$ for comparison.

The fact that the first distribution in no way represents the second is obvious, but applying the Goodness of Fit test we have, for 8 Groups,

$$\chi^2 = 35{\cdot}07, \quad P = {\cdot}0000.$$

We thus see that the tables of this section, together with the use of the Probable Errors of the constants, are quite likely to give a rough but useful indication as to when a given population can be treated as infinite and the sampling conducted by either method, that is, as far as the consideration of the Distribution of Means of Samples is concerned.

(*c*) The representation of Distributions of Means of Samples, obtained as the results of actual sampling, by the Pearson Curve, whose constants are given by formulae (1), (2), (3) and (4) of this part.

Now it was seen in (*b*) that a closer agreement between theory and practice was obtained when the actual sampling was carried out by small commercial coloured beads than when "tickets" were used. It is perfectly clear, *a priori*, that for perfect shuffling perfectly homogeneous and symmetrical material is required, but, when the sampled population is large, questions of cost and total weight of material are important. Tickets, which are cheap and light, have been frequently used but the results obtained are in general quite unsatisfactory, as has been shown in previous parts of this paper. Coloured beads, when small and having merely the rough homogeneity of size and shape that obtains in cheap commercial products, have been shown to be quite satisfactory when sampling from large populations, and moreover the variety of shades of colour enables them to be used when the sampled population, like Population (A), has a large number of categories.

Coming now to actual sampling from small populations, it seemed clear that "tickets" would be unsuitable, so an attempt was made to use the same beads as had been used with success when sampling from the large Population (A).

Population (C) has three categories A, B and C, of frequencies 1, 10 and 25, and these were represented by the appropriate number of red, orange and yellow beads taken from the 4011 used for Population (A). These 36 beads, it may be remarked, were by no means uniform in shape; they were none of them really spherical, and since their weights varied from ·512 g. to ·864 g., so there was also little uniformity of size and weight.

Before giving the results of the actual sets of samples made, it may be well to re-state the values of the constants of the theoretic Distribution of Means as given by the formulae. They are

$$\text{Mean} = 1{\cdot}666{,}667, \qquad \Sigma_1 = 0{\cdot}143{,}649,$$

$$_1B_1 = 0{\cdot}047{,}697, \qquad _1B_2 = 2{\cdot}816{,}989,$$

and the Pearson Curve with these constants is

$$y_0 = 269{\cdot}791\left(1+\frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}664}\left(1-\frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}962},$$

whose ordinates etc. have been given before.

Distribution VI (a). The Means of 1000 Samples of 10 from Population (C) obtained by drawing beads from a bowl, the complete sample being drawn before replacement.

Value of Mean	*1·2*	*1·3*	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*
Frequency	5	13	74	197	241	226	163	72	9

The constants are as follows, the origin being 1·6 and the working unit 10 times that of the distribution, i.e. 10 times that of Population (C):

$$\begin{array}{lll} \nu_1' = 0{\cdot}400{,}000, & \nu_2 = 2{\cdot}200{,}000, & \text{Mean} = 1{\cdot}640{,}000, \\ \nu_2' = 2{\cdot}360{,}000, & \nu_3 = -0{\cdot}114{,}000, & \Sigma_1 = 0{\cdot}148{,}324, \\ \nu_3' = 2{\cdot}590{,}000, & \nu_4 = 12{\cdot}728{,}800, & {}_1B_1 = 0{\cdot}001{,}221, \\ \nu_4' = 14{\cdot}684{,}000, & & {}_1B_2 = 2{\cdot}629{,}917. \end{array}$$

Distribution VII (a). The Means of 1000 Samples of 10 from Population (C) drawn as in VI (a), but a *different* set of 36 beads being used. These beads were, however, merely another 36 from the 4011 used in V (a).

Value of Mean	*1·2*	*1·3*	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*
Frequency	4	7	43	108	244	276	204	99	15

The constants are (working unit as before):

$$\begin{array}{lll} \nu_1' = 0{\cdot}810{,}000, & \nu_2 = 1{\cdot}973{,}900, & \text{Mean} = 1{\cdot}681{,}000, \\ \nu_2' = 2{\cdot}630{,}000, & \nu_3 = -0{\cdot}684{,}018, & \Sigma_1 = 0{\cdot}140{,}496, \\ \nu_3' = 4{\cdot}644{,}000, & \nu_4 = 11{\cdot}801{,}297, & {}_1B_1 = 0{\cdot}060{,}836, \\ \nu_4' = 17{\cdot}786{,}000, & & {}_1B_2 = 3{\cdot}028{,}862. \end{array}$$

These are likewise discrete distributions, but as the intervals are equal, they were represented, as in the previous case, by histograms and tested for Goodness of Fit with the curve $y = 269{\cdot}791\left(1+\frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}664}\left(1-\frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}962}$ given above.

The results were

VI (a) $\chi^2 = 52{\cdot}18$, $P = {\cdot}000{,}000$ with 8 Groups.

VII (a) $\chi^2 = 16{\cdot}82$, $P = {\cdot}018{,}786$ with 8 Groups.

The fits here are very bad.

A glance at the values of the constants of VI (a) and VII (a) will show that these poor fits are largely due to displacements of the means of these distributions from the theoretical value 1·666,667. Now each of these two distributions was obtained by taking four sets of 250 samples, each 250 samples being completed at one sitting. When the fits were discovered to be poor owing to displacements of the means, the separate means of these 8 sets of 250 samples were evaluated with the following results:

VI (a)	1st set of 250 Samples	Mean = 1·6448.
	2nd " " " "	Mean = 1·6440.
	3rd " " " "	Mean = 1·6260.
	4th " " " "	Mean = 1·6452.
VII (a) (Different set of beads)	1st " " " "	Mean = 1·6740.
	2nd " " " "	Mean = 1·6916.
	3rd " " " "	Mean = 1·6724.
	4th " " " "	Mean = 1·6860.

Now the P.E. of the Mean 1·666,667 on the basis of 1000 samples is 0·003,064, consequently it is clear that with the set of beads used in VI (a) the mean was consistently significantly below its true value, whilst with the set used in VII (a) the mean was consistently significantly above its true value. It is clear therefore that the lack of homogeneity of shape and size of the beads tends to produce a definite bias in the sampling, and this bias will depend on the particular beads chosen.

We may thus conclude that, although the beads may be sufficiently homogeneous to produce effective sampling from a large population, yet when samples are taken from a small population, material which is very close to the ideal of complete homogeneity of size and weight together with complete symmetry of shape is absolutely requisite.

To this end 1000 fresh samples were made with clay marbles as the medium instead of beads; steel spheres would have been ideal as regards the conditions of uniformity of size and shape, but they would have been difficult to mark so that the categories of the population were readily distinguishable. These marbles were not quite of uniform size and weight (their weights ranged from 3·1535 g. to 3·8955 g.), but they were much more spherical and therefore much more uniform in shape than the beads.

Distribution VIII (a). The Means of 1000 Samples of 10 from Population (C) drawn as in VI (a), but by the use of coloured spherical clay marbles.

Value of Mean	*1·2*	*1·3*	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*
Frequency	2	12	44	147	250	257	196	77	15

The constants are (working unit as before):

$\nu_1' = 0{\cdot}661{,}000$,	$\nu_2 = 2{\cdot}000{,}079$,	Mean $= 1{\cdot}666{,}100$,
$\nu_2' = 2{\cdot}437{,}000$,	$\nu_3 = -0{\cdot}341{,}961$,	$\Sigma_1 = 0{\cdot}141{,}424$,
$\nu_3' = 3{\cdot}913{,}000$,	$\nu_4 = 11{\cdot}274{,}984$,	${}_1B_1 = 0{\cdot}014{,}615$,
$\nu_4' = 15{\cdot}805{,}000$,		${}_1B_2 = 2{\cdot}818{,}523$.

Representing this discrete distribution by a histogram (the intervals are equal) and testing for the Goodness of Fit of the representation by the curve

$$y = 269{\cdot}791\left(1 + \frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}684}\left(1 - \frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}796{,}962},$$

the result was

$$\text{VIII}\,(a)\ \ \chi^2 = 7{\cdot}01, \quad P = {\cdot}4288 \text{ with 8 Groups.}$$

The fit here (shown in Fig. VII, see p. 359) is quite reasonable, but doubtless could be improved by performing the sampling with even more homogeneous and symmetrical material.

Distribution VIII (a), being an experimental distribution of Means of Samples obtained by actual sampling from (C), can be fitted to the true Distribution of Means given at the commencement of this part, as a verification of the efficacy of this material.

The fit was $\chi^2 = 5{\cdot}63$, $P = {\cdot}5842$ with 8 Groups, providing a satisfactory comparison, but in general practice the true Distribution of Means would be unknown and the comparison could not be carried out, at least not until our knowledge is extended to the direct establishment of the Distribution of Means from any finite population.

We see thus that for sampling from a small population, if there is to be any good agreement between theory and its representation as regards the distribution of means obtained from actual sets of samples, the material employed must approximate closely to the ideal of complete homogeneity of shape, size and weight. This material must be chosen therefore with more care than perhaps has been deemed necessary in the past; for instance, I am convinced, cardboard tickets are quite useless.

PART V. THE DISTRIBUTION OF σ^2 OF SAMPLES OF N FROM A POPULATION OF M.

The Mean and Second Moment of this Distribution have been given by Dr Neyman in the paper to which previous reference has been made, but he did not attempt the much more laborious task of the determination of the Third and Fourth Moment-Coefficients, which are necessary if the distribution is to be represented by a Pearson Curve.

Dr Neyman's results are

$$ {}_2M_1' = \frac{(N-1)\,M}{N\,(M-1)}\,\bar{\nu}_2 \quad\text{......(1)}, $$

$$ {}_2M_2 = \frac{M\,(M-N)\,(N-1)\,\bar{\nu}_2^2}{N^3\,(M-1)^2\,(M-2)\,(M-3)}\,[(MN-M-N-1)\,(M-1)\,\bar{\beta}_2 - (M^2N-3M^2+6M-3N-3)] \quad\text{......(2)}. $$

To obtain the Third and Fourth Moments ${}_2M_3$ and ${}_2M_4$ the following expansions are required in addition to those given in *Biometrika*, Vol. XVII. p. 80.

S being the summation over any finite distribution of X's we have, for the evaluation of ${}_2M_3$:

$$
\begin{aligned}
S(X_1^5)\,S(X_1) &= S(X_1^6) + S(X_1^5X_2),\\
S(X_1^4)\,S(X_1^2) &= S(X_1^6) + S(X_1^4X_2^2),\\
S(X_1^4)\,\{S(X_1)\}^2 &= S(X_1^6) + 2S(X_1^5X_2) + S(X_1^4X_2^2) + 2S(X_1^4X_2X_3),\\
S(X_1^3)\,\{S(X_1)\}^3 &= S(X_1^6) + 3S(X_1^5X_2) + 3S(X_1^4X_2^2) + 6S(X_1^4X_2X_3)\\
&\quad + 2S(X_1^3X_2^3) + 3S(X_1^3X_2^2X_3) + 6S(X_1^3X_2X_3X_4),\\
\{S(X_1^3)\}^2 &= S(X_1^6) + 2S(X_1^3X_2^3),\\
\{S(X_1^2)\}^2\,\{S(X_1)\}^2 &= S(X_1^6) + 2S(X_1^5X_2) + 3S(X_1^4X_2^2) + 2S(X_1^4X_2X_3)\\
&\quad + 4S(X_1^3X_2^3) + 4S(X_1^3X_2^2X_3) + 6S(X_1^2X_2^2X_3^2)\\
&\quad + 4S(X_1^2X_2^2X_3X_4),\\
S(X_1^2)\,\{S(X_1)\}^4 &= S(X_1^6) + 4S(X_1^5X_2) + 7S(X_1^4X_2^2) + 12S(X_1^4X_2X_3)\\
&\quad + 8S(X_1^3X_2^3) + 16S(X_1^3X_2^2X_3) + 24S(X_1^3X_2X_3X_4)\\
&\quad + 18S(X_1^2X_2^2X_3^2) + 24S(X_1^2X_2^2X_3X_4)\\
&\quad + 24S(X_1^2X_2X_3X_4X_5),\\
S(X_1)\,S(X_1^2)\,S(X_1^3) &= S(X_1^6) + S(X_1^5X_2) + S(X_1^4X_2^2) + 2S(X_1^3X_2^3)\\
&\quad + S(X_1^3X_2^2X_3),\\
S(X_1^7)\,S(X_1) &= S(X_1^8) + S(X_1^7X_2),\\
S(X_1^6)\,S(X_1^2) &= S(X_1^8) + S(X_1^6X_2^2),\\
S(X_1^6)\,\{S(X_1)\}^2 &= S(X_1^8) + 2S(X_1^7X_2) + S(X_1^6X_2^2) + 2S(X_1^6X_2X_3),\\
S(X_1^5)\,S(X_1^3) &= S(X_1^8) + S(X_1^5X_2^3),\\
S(X_1^5)\,S(X_1^2)\,S(X_1) &= S(X_1^8) + S(X_1^7X_2) + S(X_1^6X_2^2) + S(X_1^5X_2^3)\\
&\quad + S(X_1^5X_2^2X_3),\\
S(X_1^5)\,\{S(X_1)\}^3 &= S(X_1^8) + 3S(X_1^7X_2) + 3S(X_1^6X_2^2) + 6S(X_1^6X_2X_3)\\
&\quad + S(X_1^5X_2^3) + 3S(X_1^5X_2^2X_3) + 6S(X_1^5X_2X_3X_4),\\
\{S(X_1^4)\}^2 &= S(X_1^8) + 2S(X_1^4X_2^4),\\
S(X_1^4)\,S(X_1^3)\,S(X_1) &= S(X_1^8) + S(X_1^7X_2) + S(X_1^5X_2^3) + 2S(X_1^4X_2^4)\\
&\quad + S(X_1^4X_2^3X_3),\\
S(X_1^4)\,\{S(X_1^2)\}^2 &= S(X_1^8) + 2S(X_1^6X_2^2) + 2S(X_1^4X_2^4) + 2S(X_1^4X_2^2X_3^2),
\end{aligned}
$$

$$\begin{aligned}
S(X_1^4)\,S(X_1^2)\,\{S(X_1)\}^2 = {} & S(X_1^8) + 2S(X_1^7X_2) + 2S(X_1^6X_2^2) + 2S(X_1^6X_2X_3) \\
& + 2S(X_1^5X_2^3) + 2S(X_1^5X_2^2X_3) + 2S(X_1^4X_2^4) \\
& + 2S(X_1^4X_2^3X_3) + 2S(X_1^4X_2^2X_3^2) + 2S(X_1^4X_2^2X_3X_4), \\
S(X_1^4)\,\{S(X_1)\}^4 = {} & S(X_1^8) + 4S(X_1^7X_2) + 6S(X_1^6X_2^2) + 12S(X_1^6X_2X_3) \\
& + 4S(X_1^5X_2^3) + 12S(X_1^5X_2^2X_3) + 24S(X_1^5X_2X_3X_4) \\
& + 2S(X_1^4X_2^4) + 4S(X_1^4X_2^3X_3) + 6S(X_1^4X_2^2X_3^2) \\
& + 12S(X_1^4X_2^2X_3X_4) + 24S(X_1^4X_2X_3X_4X_5), \\
\{S(X_1^3)\}^2\,S(X_1^2) = {} & S(X_1^8) + S(X_1^6X_2^2) + 2S(X_1^5X_2^3) + 2S(X_1^3X_2^3X_3^2), \\
\{S(X_1^3)\}^2\,\{S(X_1)\}^2 = {} & S(X_1^8) + 2S(X_1^7X_2) + S(X_1^6X_2^2) + 2S(X_1^6X_2X_3) \\
& + 2S(X_1^5X_2^3) + 4S(X_1^4X_2^4) + 4S(X_1^4X_2^3X_3) \\
& + 2S(X_1^3X_2^3X_3^2) + 4S(X_1^3X_2^3X_3X_4), \\
S(X_1^3)\,\{S(X_1^2)\}^2\,S(X_1) = {} & S(X_1^8) + S(X_1^7X_2) + 2S(X_1^6X_2^2) + 3S(X_1^5X_2^3) \\
& + 2S(X_1^5X_2^2X_3) + 2S(X_1^4X_2^4) + S(X_1^4X_2^3X_3) \\
& + 2S(X_1^4X_2^2X_3^2) + 4S(X_1^3X_2^3X_3^2) + 2S(X_1^3X_2^2X_3^2X_4), \\
S(X_1^3)\,S(X_1^2)\,\{S(X_1)\}^3 = {} & S(X_1^8) + 3S(X_1^7X_2) + 4S(X_1^6X_2^2) + 6S(X_1^6X_2X_3) \\
& + 5S(X_1^5X_2^3) + 6S(X_1^5X_2^2X_3) + 6S(X_1^5X_2X_3X_4) \\
& + 6S(X_1^4X_2^4) + 9S(X_1^4X_2^3X_3) + 6S(X_1^4X_2^2X_3^2) \\
& + 6S(X_1^4X_2^2X_3X_4) + 8S(X_1^3X_2^3X_3^2) \\
& + 12S(X_1^3X_2^3X_3X_4) + 6S(X_1^3X_2^2X_3^2X_4) \\
& + 6S(X_1^3X_2^2X_3X_4X_5), \\
S(X_1^3)\,\{S(X_1)\}^5 = {} & S(X_1^8) + 5S(X_1^7X_2) + 10S(X_1^6X_2^2) + 20S(X_1^6X_2X_3) \\
& + 11S(X_1^5X_2^3) + 30S(X_1^5X_2^2X_3) + 60S(X_1^5X_2X_3X_4) \\
& + 10S(X_1^4X_2^4) + 25S(X_1^4X_2^3X_3) + 30S(X_1^4X_2^2X_3^2) \\
& + 60S(X_1^4X_2^2X_3X_4) + 120S(X_1^4X_2X_3X_4X_5) \\
& + 20S(X_1^3X_2^3X_3^2) + 40S(X_1^3X_2^3X_3X_4) + 30S(X_1^3X_2^2X_3^2X_4) \\
& + 60S(X_1^3X_2^2X_3X_4X_5) + 120S(X_1^3X_2X_3X_4X_5X_6).
\end{aligned}$$

The Third and Fourth Moment-Coefficients of the Distribution of σ^2 of Samples of N from a population of M.

For this we have

$$\frac{M!}{N!\,(M-N)!}\,{}_2M_3' = \Sigma\left[\frac{S(X_1^2)}{N} - \left\{\frac{S(X_1)}{N}\right\}^2\right]^3,$$

where S represents the summation over the N values of the variate forming the sample and Σ represents the summation for the $\frac{M!}{N!\,(M-N)!}$ samples, the sampled population $X_1X_2 \ldots X_M$ being referred to its Mean as origin.

Therefore
$$\frac{M!}{N!\,(M-N)!}\,{}_2M_3' = \Sigma\left[\frac{\{S(X_1^2)\}^3}{N^3} - 3\,\frac{\{S(X_1^2)\}^2\{S(X_1)\}^2}{N^4} + \frac{3S(X_1^2)\{S(X_1)\}^4}{N^5} - \frac{\{S(X_1)\}^6}{N^6}\right].$$

Using the appropriate expansions from the above and from *Biometrika*, Vol. XVII. p. 80, we have

$$\begin{aligned}
N^6 \frac{M!}{N!\,(M-N)!}\,{}_2M_3' = {} & (N-1)^3\, \Sigma S\,(X_1^6) - 6\,(N-1)^2\, \Sigma S\,(X_1^5 X_2) \\
& + 3\,(N-1)\,[(N-1)^2+4]\, \Sigma S\,(X_1^4 X_2^2) \\
& - 6\,(N-1)\,(N-5)\, \Sigma S\,(X_1^4 X_2 X_3) - 4\,[3\,(N-1)^2+2]\, \Sigma S\,(X_1^3 X_2^3) \\
& - 12\,[(N-2)^2+1]\, \Sigma S\,(X_1^3 X_2^2 X_3) + 24\,[3\,(N-2)+1]\, \Sigma S\,(X_1^3 X_2 X_3 X_4) \\
& + 6\,[(N-1)^3+6\,(N-3)+4]\, \Sigma S\,(X_1^2 X_2^2 X_3^2) - 12\,[(N-3)^2+6]\, \Sigma S\,(X_1^2 X_2^2 X_3 X_4) \\
& + 72\,(N-5)\, \Sigma S\,(X_1^2 X_2 X_3 X_4 X_5) - 720\, \Sigma S\,(X_1 X_2 X_3 X_4 X_5 X_6).
\end{aligned}$$

Now taking any product $X_1^\alpha X_2^\beta \dots X_r^\lambda$, the number of samples in which it will occur is clearly ${}^{N-r}C_{M-r} = \dfrac{(M-r)!}{(N-r)!\,(M-N)!}$; therefore

$$\Sigma S\,(X_1^\alpha X_2^\beta \dots X_r^\lambda) = \frac{(M-r)!}{(N-r)!\,(M-N)!}\, S\,(X_1^\alpha X_2^\beta \dots X_r^\lambda),$$

where S is now the summation over the sampled population $X_1 X_2 \dots X_M$.

Therefore we have

$$\begin{aligned}
{}_2M_3' = \frac{1}{N^5 \underline{|M}} \Big\{ & (N-1)^3\, \underline{|M-1}\, S\,(X_1^6) - 6\,(N-1)^3\, \underline{|M-2}\, S\,(X_1^5 X_2) \\
& + 3\,(N-1)^2\,[(N-1)^2+4]\, \underline{|M-2}\, S\,(X_1^4 X_2^2) \\
& - 6\,(N-1)^2\,(N-2)\,(N-5)\, \underline{|M-3}\, S\,(X_1^4 X_2 X_3) \\
& - 4\,(N-1)\,[3\,(N-1)^2+2]\, \underline{|M-2}\, S\,(X_1^3 X_2^3) \\
& - 12\,(N-1)\,(N-2)\,[(N-2)^2+1]\, \underline{|M-3}\, S\,(X_1^3 X_2^2 X_3) \\
& + 24\,(N-1)\,(N-2)\,(N-3)\,[3\,(N-2)^2+1]\, \underline{|M-4}\, S\,(X_1^3 X_2 X_3 X_4) \\
& + 6\,(N-1)\,(N-2)\,[(N-1)^3+6\,(N-3)+4]\, \underline{|M-3}\, S\,(X_1^2 X_2^2 X_3^2) \\
& - 12\,(N-1)\,(N-2)\,(N-3)\,[(N-3)^2+6]\, \underline{|M-4}\, S\,(X_1^2 X_2^2 X_3 X_4) \\
& + 72\,(N-1)\,(N-2)\,(N-3)\,(N-4)\,(N-5)\, \underline{|M-5}\, S\,(X_1^2 X_2 X_3 X_4 X_5) \\
& - 720\,(N-1)\,(N-2)\,(N-3)\,(N-4)\,(N-5)\, \underline{|M-6}\, S\,(X_1 X_2 X_3 X_4 X_5 X_6) \Big\},
\end{aligned}$$

where S is the summation over the sampled population $X_1 X_2 \dots X_M$.

Now since the origin is the mean of this sampled population we have $S\,(X_1)=0$, and thus the following equations:

$$\begin{aligned}
M\bar{\nu}_6 &= S\,(X_1^6), \\
0 &= S\,(X_1^5)\, S\,(X_1), \\
M^2 \bar{\nu}_4 \bar{\nu}_2 &= S\,(X_1^4)\, S\,(X_1^2), \\
0 &= S\,(X_1^4)\, \{S\,(X_1)\}^2, \\
M^2 \bar{\nu}_3^2 &= \{S\,(X_1^3)\}^2, \\
M^3 \bar{\nu}_2^3 &= \{S\,(X_1^2)\}^3, \\
0 &= S\,(X_1^3)\, S\,(X_1^2)\, S\,(X_1), \\
0 &= S\,(X_1^3)\, \{S\,(X_1)\}^3, \\
0 &= \{S\,(X_1^2)\}^2\, \{S\,(X_1)\}^2, \\
0 &= S\,(X_1^2)\, \{S\,(X_1)\}^4, \\
0 &= \{S\,(X_1)\}^6,
\end{aligned}$$

where $\bar{\nu}_s$ is the sth uncorrected moment about its own mean of the sampled population.

Expanding the right-hand sides of these equations by use of the expansions above and in *Biometrika*, Vol. XVII. p. 80, we obtain 11 linear equations to express the 11 sums

$$S(X_1^6),\ S(X_1^5X_2),\ S(X_1^4X_2^2),\ S(X_1^4X_2X_3),\ S(X_1^3X_2^3),\ S(X_1^3X_2^2X_3),$$
$$S(X_1^3X_2X_3X_4),\ S(X_1^2X_2^2X_3^2),\ S(X_1^2X_2^2X_3X_4),\ S(X_1^2X_2X_3X_4X_5),$$

and
$$S(X_1X_2X_3X_4X_5X_6)$$

in terms of $\bar{\nu}_6$, $\bar{\nu}_4\bar{\nu}_2$, $\bar{\nu}_3^2$ and $\bar{\nu}_2^3$.

Solving these we have, after considerable manipulation:

$$\begin{aligned}
S(X_1^6) &= M\bar{\nu}_6,\\
S(X_1^5X_2) &= -M\bar{\nu}_6,\\
S(X_1^4X_2^2) &= M^2\bar{\nu}_4\bar{\nu}_2 - M\bar{\nu}_6,\\
2S(X_1^4X_2X_3) &= 2M\bar{\nu}_6 - M^2\bar{\nu}_4\bar{\nu}_2,\\
2S(X_1^3X_2^3) &= M^2\bar{\nu}_3^2 - M\bar{\nu}_6,\\
S(X_1^3X_2^2X_3) &= 2M\bar{\nu}_6 - M^2\bar{\nu}_4\bar{\nu}_2 - M^2\bar{\nu}_3^2,\\
6S(X_1^3X_2X_3X_4) &= 3M^2\bar{\nu}_4\bar{\nu}_2 + 2M^2\bar{\nu}_3^2 - 6M\bar{\nu}_6,\\
6S(X_1^2X_2^2X_3^2) &= 2M\bar{\nu}_6 + M^3\bar{\nu}_2^3 - 3M^2\bar{\nu}_4\bar{\nu}_2,\\
4S(X_1^2X_2^2X_3X_4) &= 5M^2\bar{\nu}_4\bar{\nu}_2 + 2M^2\bar{\nu}_3^2 - M^3\bar{\nu}_2^3 - 6M\bar{\nu}_6,\\
24S(X_1^2X_2X_3X_4X_5) &= 24M\bar{\nu}_6 - 18M^2\bar{\nu}_2\bar{\nu}_4 - 8M^2\bar{\nu}_3^2 + 3M^3\bar{\nu}_2^3,\\
720S(X_1X_2X_3X_4X_5X_6) &= 90M^2\bar{\nu}_4\bar{\nu}_2 + 40M^2\bar{\nu}_3^2 - 15M^3\bar{\nu}_2^3 - 120M\bar{\nu}_6.
\end{aligned}$$

Substituting the values for these sums in the last expression for ${}_2M_3'$ we obtain:

$$\begin{aligned}
{}_2M_3' = \frac{N-1}{N^5}\Bigg\{&\bar{\nu}_6\Bigg[(N-1)^2 - \frac{(3N^3-21N^2+45N-31)}{M-1}\\
&+\frac{2(N-2)(N-3)(N^2-15N+30)}{(M-1)(M-2)} + \frac{6(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)}\\
&+\frac{72(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} + \frac{120(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}\Bigg]\\
&+\frac{M\bar{\nu}_4\bar{\nu}_2}{M-1}\Bigg[3(N-1)(N^2-2N+5) - \frac{3(N-2)(N^3-8N^2+31N-40)}{M-2}\\
&-\frac{3(N-2)(N-3)(5N^2-42N+95)}{(M-2)(M-3)} - \frac{54(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}\\
&-\frac{90(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\Bigg]\\
&+\frac{M\bar{\nu}_3^2}{M-1}\Bigg[2(3N^2-6N+5) - \frac{12(N-2)(N^2-4N+5)}{M-2}\\
&+\frac{2(N-2)(N-3)(3N^2-30N+65)}{(M-2)(M-3)} + \frac{24(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}\\
&+\frac{40(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\Bigg]
\end{aligned}$$

$$+\frac{M^2\bar{\nu}_2^3}{(M-1)(M-2)}\left[(N-2)(N^3-3N^2+9N-15)+\frac{3(N-2)(N-3)(N^2-6N+15)}{M-3}\right.$$
$$\left.+\frac{9(N-2)(N-3)(N-4)(N-5)}{(M-3)(M-4)}+\frac{15(N-2)(N-3)(N-4)(N-5)}{(M-3)(M-4)(M-5)}\right]\Bigg\}.$$

Transferring the origin to the mean of this distribution of σ^2 of samples by ${}_2M_3 = {}_2M_3' - 3\,.\,{}_2M_2\,.\,{}_2M_1' - ({}_2M_1')^3$, we have, using Neyman's results (1) and (2):

$${}_2M_3 = \frac{N-1}{N^5}\Bigg\{\bar{\nu}_6\left[(N-1)^2 - \frac{(3N^3-21N^2+45N-31)}{M-1}\right.$$
$$+\frac{2(N-2)(N-3)(N^2-15N+30)}{(M-1)(M-2)}+\frac{6(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)}$$
$$\left.+\frac{72(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)}+\frac{120(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}\right]$$
$$-\frac{3M\bar{\nu}_4\bar{\nu}_2}{M-1}\left[(N-1)(N-5)-\frac{N(N-1)(N^2-6N+7)}{M-1}\right.$$
$$+\frac{(N-2)(N^3-8N^2+31N-40)}{M-2}-\frac{4N(N-1)(N-2)(N-3)}{(M-1)(M-2)}$$
$$+\frac{(N-2)(N-3)(5N^2-42N+95)}{(M-2)(M-3)}-\frac{6N(N-1)(N-2)(N-3)}{(M-1)(M-2)(M-3)}$$
$$\left.+\frac{18(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}+\frac{30(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\right]$$
$$-\frac{2M\bar{\nu}_3^2}{M-1}\left[(3N^2-6N+5)-\frac{6(N-2)(N^2-4N+5)}{M-2}\right.$$
$$+\frac{(N-2)(N-3)(3N^2-30N+65)}{(M-2)(M-3)}+\frac{12(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}$$
$$\left.+\frac{20(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\right]$$
$$\frac{M^2\bar{\nu}_2^3}{M-1}\left[\frac{N(N-1)(N^2-4N+9)}{M-1}-\frac{(N-2)(N^3-3N^2+9N-15)}{M-2}-\frac{2N^2(N-1)^2}{(M-1)^2}\right.$$
$$+\frac{6N(N-1)(N-2)(N-3)}{(M-1)(M-2)}-\frac{3(N-2)(N-3)(N^2-6N+15)}{(M-2)(M-3)}$$
$$+\frac{9N(N-1)(N-2)(N-3)}{(M-1)(M-2)(M-3)}-\frac{9(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}$$
$$\left.-\frac{15(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\right]\Bigg\}\quad\ldots\ldots\ldots\ldots(3).$$

Again

$$\frac{M!}{N!(M-N)!}\,{}_2M_4' = \Sigma\left[\frac{S(X_1^2)}{N}-\frac{\{S(X_1)\}^2}{N^2}\right]^4$$
$$=\Sigma\left[\frac{\{S(X_1^2)\}^4}{N^4}-\frac{4\{S(X_1^2)\}^3\{S(X_1)\}^2}{N^5}+\frac{6\{S(X_1^2)\}^2\{S(X_1)\}^4}{N^6}\right.$$
$$\left.-\frac{4\{S(X_1^2)\}\{S(X_1)\}^6}{N^7}+\frac{\{S(X_1)\}^8}{N^8}\right].$$

Using again the appropriate expansions from those at the commencement of this portion of the work and from those in *Biometrika*, Vol. XVII. p. 80, we have, after simplification:

$$\begin{aligned}
N^8 \frac{M!}{N!\,(M-N)!}\cdot {}_2M_4' = \Sigma\,\{&S(X_1^8)[N^4-4N^3+6N^2-4N+1]\\
&+S(X_1^7X_2)[-8N^3+24N^2-24N+8]\\
&+S(X_1^6X_2^2)[4N^4-16N^3+48N^2-64N+28]\\
&+S(X_1^6X_2X_3)[-8N^3+72N^2-120N+56]\\
&+S(X_1^5X_2^3)[-24N^3+72N^2-104N+56]\\
&+S(X_1^5X_2^2X_3)[-24N^3+120N^2-264N+168]\\
&+S(X_1^5X_2X_3X_4)[144N^2-480N+336]\\
&+S(X_1^4X_2^4)[6N^4-24N^3+84N^2-120N+70]\\
&+S(X_1^4X_2^2X_3^2)[12N^4-48N^3+192N^2-480N+420]\\
&+S(X_1^4X_2^3X_3)[-24N^3+168N^2-360N+280]\\
&+S(X_1^4X_2^2X_3X_4)[-24N^3+216N^2-840N+840]\\
&+S(X_1^4X_2X_3X_4X_5)[144N^2-1440N+1680]\\
&+S(X_1^3X_2^3X_3^2)[-48N^3+240N^2-560N+560]\\
&+S(X_1^3X_2^2X_3^2X_4)[-48N^3+336N^2-1200N+1680]\\
&+S(X_1^3X_2^3X_3X_4)[288N^2-960N+1120]\\
&+(X_1^3X_2^2X_3X_4X_5)[288N^2-1920N+3360]\\
&+S(X_1^3X_2X_3X_4X_5X_6)[-2880N+6720]\\
&+S(X_1^2X_2^2X_3^2X_4^2)[24N^4-96N^3+432N^2-1440N+2520]\\
&+S(X_1^2X_2^2X_3^2X_4X_5)[-48N^3+432N^2-2160N+5040]\\
&+S(X_1^2X_2^2X_3X_4X_5X_6)[288N^2-2880N+10080]\\
&+S(X_1^2X_2X_3X_4X_5X_6X_7)[-2880N+20160]\\
&+S(X_1X_2X_3X_4X_5X_6X_7X_8)\,.\,40320\}.
\end{aligned}$$

Again, since $\Sigma S(X_1^{\alpha}X_2^{\beta}\ldots X_r^{\lambda}) = \dfrac{(M-r)!}{(N-r)!\,(M-N)!}\,S(X_1^{\alpha}X_2^{\beta}\ldots X_r^{\lambda})$, where S is now the summation over the sampled population $X_1X_2\ldots X_M$, we have

$$\begin{aligned}
{}_2M_4' = \frac{N-1}{N^7\,\underline{|M}}\,\{&S(X_1^8)(N-1)^3\,\underline{|M-1}\\
&+S(X_1^7X_2)(-8N^3+24N^2-24N+8)\,\underline{|M-2}\\
&+S(X_1^6X_2^2)(4N^4-16N^3+48N^2-64N+28)\,\underline{|M-2}\\
&+S(X_1^6X_2X_3)(N-2)(-8N^3+72N^2-120N+56)\,\underline{|M-3}\\
&+S(X_1^5X_2^3)(-24N^3+72N^2-104N+56)\,\underline{|M-2}\\
&+S(X_1^5X_2^2X_3)(N-2)(-24N^3+120N^2-264N+168)\,\underline{|M-3}
\end{aligned}$$

$$+ S(X_1^5 X_2 X_3 X_4)(N-2)(N-3)(144N^2 - 480N + 336)\,|\underline{M-4}$$
$$+ S(X_1^4 X_2^4)(6N^4 - 24N^3 + 84N^2 - 120N + 70)\,|\underline{M-2}$$
$$+ S(X_1^4 X_2^2 X_3^2)(N-2)(12N^4 - 48N^3 + 192N^2 - 480N + 420)\,|\underline{M-3}$$
$$+ S(X_1^4 X_2^3 X_3)(N-2)(-24N^3 + 168N^2 - 360N + 280)\,|\underline{M-3}$$
$$+ S(X_1^4 X_2^2 X_3 X_4)(N-2)(N-3)(-24N^3 + 216N^2 - 840N + 840)\,|\underline{M-4}$$
$$+ S(X_1^4 X_2 X_3 X_4 X_5)(N-2)(N-3)(N-4)(144N^2 - 1440N + 1680)\,|\underline{M-5}$$
$$+ S(X_1^3 X_2^3 X_3^2)(N-2)(-48N^3 + 240N^2 - 560N + 560)\,|\underline{M-3}$$
$$+ S(X_1^3 X_2^2 X_3^2 X_4)(N-2)(N-3)(-48N^3 + 336N^2 - 1200N + 1680)\,|\underline{M-4}$$
$$+ S(X_1^3 X_2^3 X_3 X_4)(N-2)(N-3)(288N^2 - 960N + 1120)\,|\underline{M-4}$$
$$+ S(X_1^3 X_2^2 X_3 X_4 X_5)(N-2)(N-3)(N-4)(288N^2 - 1920N + 3360)\,|\underline{M-5}$$
$$+ S(X_1^3 X_2 X_3 X_4 X_5 X_6)(N-2)(N-3)(N-4)(N-5)(-2880N + 6720)\,|\underline{M-6}$$
$$+ S(X_1^2 X_2^2 X_3^2 X_4^2)(N-2)(N-3)(24N^4 - 96N^3 + 432N^2 - 1440N + 2520)\,|\underline{M-4}$$
$$+ S(X_1^2 X_2^2 X_3^2 X_4 X_5)(N-2)(N-3)(N-4)(-48N^3 + 432N^2 - 2160N + 5040)\,|\underline{M-5}$$
$$+ S(X_1^2 X_2^2 X_3 X_4 X_5 X_6)(N-2)(N-3)(N-4)(N-5)(288N^2 - 2880N + 10080)\,|\underline{M-6}$$
$$+ S(X_1^2 X_2 X_3 X_4 X_5 X_6 X_7)(N-2)(N-3)(N-4)(N-5)(N-6)(-2880N + 20160)\,|\underline{M-7}$$
$$+ S(X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)\,.\,40320\,|\underline{M-8}\}.$$

Now the origin being the mean of the sampled population, we have as before $S(X_1) = 0$, and thus we have the equations:

$$M\bar{\nu}_8 = S(X_1^8),$$
$$0 = S(X_1^7)\,S(X_1),$$
$$M^2\bar{\nu}_6\bar{\nu}_2 = S(X_1^6)\,S(X_1^2),$$
$$0 = S(X_1^6)\,\{S(X_1)\}^2,$$
$$M^2\bar{\nu}_5\bar{\nu}_3 = S(X_1^5)\,S(X_1^3),$$
$$0 = S(X_1^5)\,S(X_1^2)\,S(X_1),$$
$$0 = S(X_1^5)\,\{S(X_1)\}^3,$$
$$M^2\bar{\nu}_4^2 = \{S(X_1^4)\}^2,$$
$$0 = S(X_1^4)\,S(X_1^3)\,S(X_1),$$
$$M^3\bar{\nu}_4\bar{\nu}_2^2 = S(X_1^4)\,\{S(X_1^2)\}^2,$$
$$0 = S(X_1^4)\,\{S(X_1)\}^4,$$
$$0 = S(X_1^4)\,S(X_1^2)\,\{S(X_1)\}^2,$$

$$
\begin{aligned}
M^2\bar{\nu}_3^2\bar{\nu}_2 &= \{S(X_1^3)\}^2 S(X_1^2),\\
0 &= \{S(X_1^3)\}^2 \{S(X_1)\}^2,\\
0 &= S(X_1^3)\,\{S(X_1^2)\}^2 S(X_1),\\
0 &= S(X_1^3)\, S(X_1^2)\,\{S(X_1)\}^3,\\
0 &= S(X_1^3)\,\{S(X_1)\}^5,\\
M^4\bar{\nu}_2^4 &= \{S(X_1^2)\}^4,\\
0 &= \{S(X_1^2)\}^3 \{S(X_1)\}^2,\\
0 &= \{S(X_1^2)\}^2 \{S(X_1)\}^4,\\
0 &= S(X_1^2)\,\{S(X_1)\}^6,\\
0 &= \{S(X_1)\}^8.
\end{aligned}
$$

Expanding the right-hand sides of these equations by the appropriate expansions, we obtain 22 linear equations to express the 22 sums $S(X_1^8)$, $S(X_1^7X_2)$, $S(X_1^6X_2^2)$, $S(X_1^6X_2X_3)$, $S(X_1^5X_2^3)$, $S(X_1^5X_2^2X_3)$, $S(X_1^5X_2X_3X_4)$, $S(X_1^4X_2^4)$, $S(X_1^4X_2^2X_3^2)$, $S(X_1^4X_2^3X_3)$, $S(X_1^4X_2^2X_3X_4)$, $S(X_1^4X_2X_3X_4X_5)$, $S(X_1^3X_2^3X_3^2)$, $S(X_1^3X_2^2X_3^2X_4)$, $S(X_1^3X_2^3X_3X_4)$, $S(X_1^3X_2^2X_3X_4X_5)$, $S(X_1^3X_2X_3X_4X_5X_6)$, $S(X_1^2X_2^2X_3^2X_4^2)$, $S(X_1^2X_2^2X_3^2X_4X_5)$, $S(X_1^2X_2^2X_3X_4X_5X_6)$, $S(X_1^2X_2X_3X_4X_5X_6X_7)$, and $S(X_1X_2X_3X_4X_5X_6X_7X_8)$ in terms of the moments $\bar{\nu}_8$, $\bar{\nu}_6\bar{\nu}_2$, $\bar{\nu}_5\bar{\nu}_3$, $\bar{\nu}_4^2$, $\bar{\nu}_4\bar{\nu}_2^2$, $\bar{\nu}_3^2\bar{\nu}_2$ and $\bar{\nu}_2^4$ of the sampled population.

Solving these equations we obtain, after considerable manipulation:

$$
\begin{aligned}
S(X_1^8) &= M\bar{\nu}_8,\\
S(X_1^7X_2) &= -M\bar{\nu}_8,\\
S(X_1^6X_2^2) &= M^2\bar{\nu}_6\bar{\nu}_2 - M\bar{\nu}_8,\\
2S(X_1^6X_2X_3) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_6\bar{\nu}_2,\\
S(X_1^5X_2^3) &= M^2\bar{\nu}_5\bar{\nu}_3 - M\bar{\nu}_8,\\
S(X_1^5X_2^2X_3) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_6\bar{\nu}_2 - M^2\bar{\nu}_5\bar{\nu}_3,\\
6S(X_1^5X_2X_3X_4) &= 3M^2\bar{\nu}_6\bar{\nu}_2 + 2M^2\bar{\nu}_5\bar{\nu}_3 - 6M\bar{\nu}_8,\\
2S(X_1^4X_2^4) &= M^2\bar{\nu}_4^2 - M\bar{\nu}_8,\\
2S(X_1^4X_2^2X_3^2) &= 2M\bar{\nu}_8 - 2M^2\bar{\nu}_6\bar{\nu}_2 - M^2\bar{\nu}_4^2 + M^3\bar{\nu}_4\bar{\nu}_2^2,\\
S(X_1^4X_2^3X_3) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_4^2 - M^2\bar{\nu}_5\bar{\nu}_3,\\
2S(X_1^4X_2^2X_3X_4) &= 3M^2\bar{\nu}_6\bar{\nu}_2 + 2M^2\bar{\nu}_5\bar{\nu}_3 + 2M^2\bar{\nu}_4^2 - M^3\bar{\nu}_4\bar{\nu}_2^2 - 6M\bar{\nu}_8,\\
24S(X_1^4X_2X_3X_4X_5) &= 24M\bar{\nu}_8 - 12M^2\bar{\nu}_6\bar{\nu}_2 - 8M^2\bar{\nu}_5\bar{\nu}_3 - 6M^2\bar{\nu}_4^2 + 3M^3\bar{\nu}_4\bar{\nu}_2^2,\\
2S(X_1^3X_2^3X_3^2) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_6\bar{\nu}_2 - 2M^2\bar{\nu}_5\bar{\nu}_3 + M^3\bar{\nu}_3^2\bar{\nu}_2,\\
2S(X_1^3X_2^2X_3^2X_4) &= 4M^2\bar{\nu}_6\bar{\nu}_2 + 4M^2\bar{\nu}_5\bar{\nu}_3 + M^2\bar{\nu}_4^2 - M^3\bar{\nu}_4\bar{\nu}_2^2\\
&\qquad - 2M^3\bar{\nu}_3^2\bar{\nu}_2 - 6M\bar{\nu}_8,\\
4S(X_1^3X_2^3X_3X_4) &= M^2\bar{\nu}_6\bar{\nu}_2 + 4M^2\bar{\nu}_5\bar{\nu}_3 + 2M^2\bar{\nu}_4^2 - M^3\bar{\nu}_3^2\bar{\nu}_2 - 6M\bar{\nu}_8,\\
6S(X_1^3X_2^2X_3X_4X_5) &= 24M\bar{\nu}_8 - 12M^2\bar{\nu}_6\bar{\nu}_2 - 14M^2\bar{\nu}_5\bar{\nu}_3 - 6M^2\bar{\nu}_4^2\\
&\qquad + 3M^3\bar{\nu}_4\bar{\nu}_2^2 + 5M^3\bar{\nu}_3^2\bar{\nu}_2,
\end{aligned}
$$

$$120S(X_1^3X_2X_3X_4X_5X_6) = 120M\bar{v}_8 + 60M^2\bar{v}_6\bar{v}_2 + 64M^2\bar{v}_5\bar{v}_3 + 30M^2\bar{v}_4^2 - 15M^3\bar{v}_4\bar{v}_2^2 - 20M^3\bar{v}_3^2\bar{v}_2$$

$$24S(X_1^2X_2^2X_3^2X_4^2) = 8M^2\bar{v}_6\bar{v}_2 + 3M^2\bar{v}_4^2 - 6M^3\bar{v}_4\bar{v}_2^2 + M^4\bar{v}_2^4 - 6M\bar{v}_8,$$

$$12S(X_1^2X_2^2X_3^2X_4X_5) = 24M\bar{v}_8 - 20M^2\bar{v}_6\bar{v}_2 - 12M^2\bar{v}_5\bar{v}_3 - 6M^2\bar{v}_4^2 + 9M^3\bar{v}_4\bar{v}_2^2 + 6M^3\bar{v}_3^2\bar{v}_2 - M^4\bar{v}_2^4,$$

$$48S(X_1^2X_2^2X_3X_4X_5X_6) = 84M^2\bar{v}_6\bar{v}_2 + 64M^2\bar{v}_5\bar{v}_3 + 30M^2\bar{v}_4^2 - 33M^3\bar{v}_4\bar{v}_2^2 - 28M^3\bar{v}_3^2\bar{v}_2 + 3M^4\bar{v}_2^4 - 120M\bar{v}_8,$$

$$720S(X_1^2X_2X_3X_4X_5X_6X_7) = 180M^3\bar{v}_4\bar{v}_2^2 + 160M^3\bar{v}_3^2\bar{v}_2 - 15M^4\bar{v}_2^4 - 180M^2\bar{v}_4^2 - 384M^2\bar{v}_5\bar{v}_3 - 480M^2\bar{v}_6\bar{v}_2 - 720M\bar{v}_8,$$

$$40320S(X_1X_2X_3X_4X_5X_6X_7X_8) = 21840M\bar{v}_8 + 3360M^2\bar{v}_6\bar{v}_2 + 2688M^2\bar{v}_5\bar{v}_3 + 1260M^2\bar{v}_4^2 - 1260M^3\bar{v}_4\bar{v}_2^2 - 1120M^3\bar{v}_3^2\bar{v}_2 + 105M^4\bar{v}_2^4.$$

Substituting in the last expression for ${}_2M_4'$, we have

$$
\begin{aligned}
{}_2M_4' = \frac{N-1}{N^7}\Bigg\{&(N-1)^3\bar{v}_8 + \frac{(8N^3-24N^2+24N-8)}{M-1}\bar{v}_8 \\
&+ \frac{(4N^4-16N^3+48N^2-64N+28)}{M-1}(M\bar{v}_6\bar{v}_2 - \bar{v}_8) \\
&+ \frac{(N-2)(-4N^3+36N^2-60N+28)}{(M-1)(M-2)}(2\bar{v}_8 - M\bar{v}_6\bar{v}_2) \\
&+ \frac{(-24N^3+72N^2-104N+56)}{M-1}(M\bar{v}_5\bar{v}_3 - \bar{v}_8) \\
&+ \frac{(N-2)(-24N^3+120N^2-264N+168)}{(M-1)(M-2)}(2\bar{v}_8 - M\bar{v}_6\bar{v}_2 - M\bar{v}_5\bar{v}_3) \\
&+ \frac{(N-2)(N-3)(24N^2-80N+56)}{(M-1)(M-2)(M-3)}(3M\bar{v}_6\bar{v}_2 + 2M\bar{v}_5\bar{v}_3 - 6\bar{v}_8) \\
&+ \frac{(3N^4-12N^3+42N^2-60N+35)}{M-1}(M\bar{v}_4^2 - \bar{v}_8) \\
&+ \frac{(N-2)(6N^4-24N^3+96N^2-240N+210)}{(M-1)(M-2)}(2\bar{v}_8 - 2M\bar{v}_6\bar{v}_2 - M\bar{v}_4^2 + M^2\bar{v}_4\bar{v}_2^2) \\
&+ \frac{(N-2)(-24N^3+168N^2-360N+280)}{(M-1)(M-2)}(2\bar{v}_8 - M\bar{v}_4^2 - M\bar{v}_5\bar{v}_3) \\
&+ \frac{(N-2)(N-3)(-12N^3+108N^2-420N+420)}{(M-1)(M-2)(M-3)}(3M\bar{v}_6\bar{v}_2 + 2M\bar{v}_5\bar{v}_3 \\
&\qquad + 2M\bar{v}_4^2 - M^2\bar{v}_4\bar{v}_2^2 - 6\bar{v}_8) \\
&+ \frac{(N-2)(N-3)(N-4)(6N^2-60N+70)}{(M-1)(M-2)(M-3)(M-4)}(24\bar{v}_8 - 12M\bar{v}_6\bar{v}_2 - 8M\bar{v}_5\bar{v}_3 \\
&\qquad - 6M\bar{v}_4^2 + 3M^2\bar{v}_4\bar{v}_2^2) \\
&+ \frac{(N-2)(-24N^3+120N^2-280N+280)}{(M-1)(M-2)}(2\bar{v}_8 + M^2\bar{v}_3^2\bar{v}_2 - M\bar{v}_6\bar{v}_2 - 2M\bar{v}_5\bar{v}_3) \\
&+ \frac{(N-2)(N-3)(-24N^3+168N^2-600N+840)}{(M-1)(M-2)(M-3)}(4M\bar{v}_6\bar{v}_2 + 4M\bar{v}_5\bar{v}_3 \\
&\qquad + M\bar{v}_4^2 - M^2\bar{v}_4\bar{v}_2^2 - 2M^2\bar{v}_3^2\bar{v}_2 - 6\bar{v}_8)
\end{aligned}
$$

$$+\frac{(N-2)(N-3)(72N^2-240N+280)}{(M-1)(M-2)(M-3)}(M\bar{\nu}_6\bar{\nu}_2+4M\bar{\nu}_5\bar{\nu}_3+2M\bar{\nu}_4^2 - M^2\bar{\nu}_3^2\bar{\nu}_2-6\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(48N^2-320N+560)}{(M-1)(M-2)(M-3)(M-4)}(24\bar{\nu}_8-12M\bar{\nu}_6\bar{\nu}_2 - 14M\bar{\nu}_5\bar{\nu}_3-6M\bar{\nu}_4^2+3M^2\bar{\nu}_4\bar{\nu}_2^2+5M^2\bar{\nu}_3^2\bar{\nu}_2)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(-24N+56)}{(M-1)(M-2)(M-3)(M-4)(M-5)}(120\bar{\nu}_8+60M\bar{\nu}_6\bar{\nu}_2 + 64M\bar{\nu}_5\bar{\nu}_3+30M\bar{\nu}_4^2-15M^2\bar{\nu}_4\bar{\nu}_2^2-20M^2\bar{\nu}_3^2\bar{\nu}_2)$$

$$+\frac{(N-2)(N-3)(N^4-4N^3+18N^2-60N+105)}{(M-1)(M-2)(M-3)}(8M\bar{\nu}_6\bar{\nu}_2+3M\bar{\nu}_4^2 - 6M^2\bar{\nu}_4\bar{\nu}_2^2+M^3\bar{\nu}_2^4-6\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(-4N^3+36N^2-180N+420)}{(M-1)(M-2)(M-3)(M-4)}(24\bar{\nu}_8-20M\bar{\nu}_6\bar{\nu}_2 - 12M\bar{\nu}_5\bar{\nu}_3-6M\bar{\nu}_4^2+9M^2\bar{\nu}_4\bar{\nu}_2^2+6M^2\bar{\nu}_3^2\bar{\nu}_2-M^3\bar{\nu}_2^4)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(6N^2-60N+210)}{(M-1)(M-2)(M-3)(M-4)(M-5)}(84M\bar{\nu}_6\bar{\nu}_2 + 64M\bar{\nu}_5\bar{\nu}_3+30M\bar{\nu}_4^2-33M^2\bar{\nu}_4\bar{\nu}_2^2-28M^2\bar{\nu}_3^2\bar{\nu}_2+3M^3\bar{\nu}_2^4-120\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(N-6)(-4N+28)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)}(180M^2\bar{\nu}_4\bar{\nu}_2^2 + 160M^2\bar{\nu}_3^2\bar{\nu}_2-15M^3\bar{\nu}_2^4-180M\bar{\nu}_4^2-384M\bar{\nu}_5\bar{\nu}_3-480M\bar{\nu}_6\bar{\nu}_2-720\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)} (21840\bar{\nu}_8+3360M\bar{\nu}_6\bar{\nu}_2+2688M\bar{\nu}_5\bar{\nu}_3+1260M\bar{\nu}_4^2-1260M^2\bar{\nu}_4\bar{\nu}_2^2 - 1120M^2\bar{\nu}_3^2\bar{\nu}_2+105M^3\bar{\nu}_2^4)\Bigg\},$$

and hence

$$ {}_2M_4' = \frac{N-1}{N^7}\Bigg\{\bar{\nu}_8\Bigg[(N-1)^3-\frac{7N^4-60N^3+186N^2-252N+127}{M-1}$$
$$+\frac{4(N-2)(3N^4-50N^3+270N^2-602N+483)}{(M-1)(M-2)}$$
$$-\frac{6(N-2)(N-3)(N^4-40N^3+390N^2-1400N+1701)}{(M-1)(M-2)(M-3)}$$
$$-\frac{48(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-1)(M-2)(M-3)(M-4)}$$
$$-\frac{240(N-2)(N-3)(N-4)(N-5)(3N^2-18N+77)}{(M-1)(M-2)(M-3)(M-4)(M-5)}$$
$$+\frac{2880(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$+\frac{21840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$

$$
\begin{aligned}
&+\frac{4M\bar{\nu}_6\bar{\nu}_2}{M-1}\Bigg[(N-1)^2(\overline{N-1}^2+6)-\frac{(N-2)(3N^4-25N^3+117N^2-271N+224)}{M-2}\\
&\qquad+\frac{(N-2)(N-3)(2N^4-41N^3+321N^2-1155N+1477)}{(M-2)(M-3)}\\
&\qquad+\frac{2(N-2)(N-3)(N-4)(10N^3-171N^2+1020N-1995)}{(M-2)(M-3)(M-4)}\\
&\qquad+\frac{6(N-2)(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-2)(M-3)(M-4)(M-5)}\\
&\qquad+\frac{480(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&\qquad+\frac{840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]\\
&-\frac{8M\bar{\nu}_5\bar{\nu}_3}{M-1}\Bigg[(N-1)(3\overline{N-1}^2+4)-\frac{2(N-2)(6N^3-33N^2+74N-63)}{M-2}\\
&\qquad+\frac{(N-2)(N-3)(15N^3-153N^2+545N-679)}{(M-2)(M-3)}\\
&\qquad-\frac{(N-2)(N-3)(N-4)(6N^3-144N^2+890N-1680)}{(M-2)(M-3)(M-4)}\\
&\qquad-\frac{16(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}\\
&\qquad-\frac{192(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&\qquad-\frac{336(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]\\
&+\frac{M\bar{\nu}_4^2}{M-1}\Bigg[3(N-1)^4+24(N-1)^2+8\\
&\qquad-\frac{2(N-2)(3N^4-24N^3+132N^2-300N+245)}{M-2}\\
&\qquad+\frac{(N-2)(N-3)(3N^4-60N^3+582N^2-2100N+2555)}{(M-2)(M-3)}\\
&\qquad+\frac{12(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-2)(M-3)(M-4)}\\
&\qquad+\frac{60(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}\\
&\qquad+\frac{720(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&\qquad+\frac{1260(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg].
\end{aligned}
$$

$$\frac{6M^2(N-2)\bar{\nu}_4\bar{\nu}_2^2}{(M-1)(M-2)}\Bigg[(N-1)^4+10(N-1)^2-16(N-1)+8$$

$$-\frac{(N-3)(N^4-10N^3+64N^2-230N+315)}{M-3}$$

$$-\frac{(N-3)(N-4)(6N^3-81N^2+460N-945)}{(M-3)(M-4)}$$

$$-\frac{(N-3)(N-4)(N-5)(33N^2-390N+1295)}{(M-3)(M-4)(M-5)}$$

$$-\frac{120(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)}$$

$$-\frac{210(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$

$$\frac{8M^2(N-2)\bar{\nu}_3^2\bar{\nu}_2}{(M-1)(M-2)}\Bigg[3(N-1)^3-6(N-1)^2+14(N-1)-12$$

$$-\frac{(N-3)(6N^3-51N^2+180N-245)}{M-3}$$

$$+\frac{(N-3)(N-4)(3N^3-57N^2+335N-665)}{(M-3)(M-4)}$$

$$+\frac{(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-3)(M-4)(M-5)}$$

$$+\frac{80(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)}$$

$$+\frac{140(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$

$$+\frac{M^3(N-2)(N-3)\bar{\nu}_2^4}{(M-1)(M-2)(M-3)}\Bigg[(N-1)^4+12(N-1)^2-32(N-1)+60$$

$$+\frac{4(N-4)(N^3-9N^2+45N-105)}{M-4}$$

$$+\frac{9(N-4)(N-5)(2N^2-20N+70)}{(M-4)(M-5)}$$

$$+\frac{60(N-4)(N-5)(N-6)(N-7)}{(M-4)(M-5)(M-6)}$$

$$+\frac{105(N-4)(N-5)(N-6)(N-7)}{(M-4)(M-5)(M-6)(M-7)}\Bigg]\Bigg\}.$$

Transferring to the mean of this distribution of σ^2 of samples as origin, by

$${}_2M_4 = {}_2M_4' - 4\,.\,{}_2M_3\,.\,{}_2M_1' - 6\,.\,{}_2M_2\,.\,({}_2M_1')^2 - ({}_2M_1')^4,$$

we have

$$
\begin{aligned}
{}_2M_4,\quad \frac{N-1}{N^7}\Bigg\{\bar{\nu}_8\Bigg[&(N-1)^3-\frac{7N^4-60N^3+186N^2-252N+127}{M-1}\\
&+\frac{4(N-2)(3N^4-50N^3+270N^2-602N+483)}{(M-1)(M-2)}\\
&-\frac{6(N-2)(N-3)(N^4-40N^3+390N^2-1400N+1701)}{(M-1)(M-2)(M-3)}\\
&-\frac{48(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-1)(M-2)(M-3)(M-4)}\\
&-\frac{240(N-2)(N-3)(N-4)(N-5)(3N^2-18N+77)}{(M-1)(M-2)(M-3)(M-4)(M-5)}\\
&+\frac{2880(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&+\frac{21840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]\\
-\frac{4M\bar{\nu}_6\bar{\nu}_2}{M-1}\Bigg[&(N-1)^2(N-7)-\frac{N(N-1)(3N^3-21N^2+45N-31)}{M-1}\\
&+\frac{(N-2)(3N^4-25N^3+117N^2-217N+224)}{M-2}\\
&+\frac{2N(N-1)(N-2)(N-3)(N^2-15N+30)}{(M-1)(M-2)}\\
&-\frac{(N-2)(N-3)(2N^4-41N^3+321N^2-1155N+1477)}{(M-2)(M-3)}\\
&+\frac{6N(N-1)(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)}\\
&-\frac{2(N-2)(N-3)(N-4)(10N^3-171N^2+1020N-1995)}{(M-2)(M-3)(M-4)}\\
&+\frac{72N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)}\\
&-\frac{6(N-2)(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-2)(M-3)(M-4)(M-5)}\\
&+\frac{120N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}\\
&-\frac{480(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&-\frac{840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{8M\bar{v}_5\bar{v}_3}{M-1}\Bigg[(N-1)(3N-1^2+4)-\frac{2(N-2)(6N^3-33N^2+74N-63)}{M-2}\\
&\qquad+\frac{(N-2)(N-3)(15N^3-153N^2+545N-679)}{(M-2)(M-3)}\\
&\qquad-\frac{(N-2)(N-3)(N-4)(6N^3-144N^2+890N-1680)}{(M-2)(M-3)(M-4)}\\
&\qquad-\frac{16(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}\\
&\qquad-\frac{192(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&\qquad-\frac{336(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]\\
&+\frac{M\bar{v}_4^2}{M-1}\Bigg[3(N-1)^4+24(N-1)^2+8\\
&\qquad-\frac{2(N-2)(3N^4-24N^3+132N^2-300N+245)}{M-2}\\
&\qquad+\frac{(N-2)(N-3)(3N^4-60N^3+582N^2-2100N+2555)}{(M-2)(M-3)}\\
&\qquad+\frac{12(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-2)(M-3)(M-4)}\\
&\qquad+\frac{60(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}\\
&\qquad+\frac{720(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}\\
&\qquad+\frac{1260(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]\\
&-\frac{6M^2\bar{v}_4\bar{v}_2^2}{M-1}\Bigg[\frac{N(N-1)^2(N^2-3N+10)}{M-1}\\
&\qquad-\frac{(N-2)(N^4-4N^3+16N^2-40N+35)}{M-2}\\
&\qquad+\frac{N^2(N-1)^2(N^2-6N+7)}{(M-1)^2}\\
&\qquad-\frac{2N(N-1)(N-2)(N^3-8N^2+31N-40)}{(M-1)(M-2)}\\
&\qquad+\frac{(N-2)(N-3)(N^4-10N^3+64N^2-230N+315)}{(M-2)(M-3)}\\
&\qquad+\frac{4N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2N(N-1)(N-2)(N-3)(5N^2-42N+95)}{(M-1)(M-2)(M-3)} \\
&+ \frac{(N-2)(N-3)(N-4)(6N^3-81N^2+460N-945)}{(M-2)(M-3)(M-4)} \\
&+ \frac{6N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)(M-3)} \\
&- \frac{36N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} \\
&+ \frac{(N-2)(N-3)(N-4)(N-5)(33N^2-390N+1295)}{(M-2)(M-3)(M-4)(M-5)} \\
&- \frac{60N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)} \\
&+ \frac{120(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)} \\
&+ \left. \frac{210(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)} \right]
\end{aligned}
$$

$$
\begin{aligned}
+ \frac{8M^3\bar{v}_3^2\bar{v}_2}{M-1} \Bigg[& \frac{N(N-1)(3N^2-6N+5)}{M-1} \\
&- \frac{(N-2)(3N^3-15N^2+35N-35)}{M-2} \\
&- \frac{6N(N-1)(N-2)(N^2-4N+5)}{(M-1)(M-2)} \\
&+ \frac{(N-2)(N-3)(6N^3-51N^2+180N-245)}{(M-2)(M-3)} \\
&+ \frac{N(N-1)(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)} \\
&- \frac{(N-2)(N-3)(N-4)(3N^3-57N^2+335N-665)}{(M-2)(M-3)(M-4)} \\
&+ \frac{12N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} \\
&- \frac{(N-2)(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-2)(M-3)(M-4)(M-5)} \\
&+ \frac{20N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)} \\
&- \frac{80(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)} \\
&- \frac{140(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)} \Bigg]
\end{aligned}
$$

$$+\frac{M^2\bar{\nu}_2^4}{M-1}\left[\frac{3N^2(N-1)^2(N^2-3N+6)}{(M-1)^2}\right.$$
$$-\frac{4N(N-1)(N-2)(N^3-3N^2+9N-15)}{(M-1)(M-2)}$$
$$+\frac{(N-2)(N-3)(N^4-4N^3+18N^2-60N+105)}{(M-2)(M-3)}$$
$$-\frac{3N^3(N-1)^3}{(M-1)^3}+\frac{12N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)}$$
$$-\frac{12N(N-1)(N-2)(N-3)(N^2-6N+15)}{(M-1)(M-2)(M-3)}$$
$$+\frac{4(N-2)(N-3)(N-4)(N^3-9N^2+45N-105)}{(M-2)(M-3)(M-4)}$$
$$+\frac{18N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)(M-3)}$$
$$-\frac{36N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)}$$
$$+\frac{9(N-2)(N-3)(N-4)(N-5)(2N^2-20N+70)}{(M-2)(M-3)(M-4)(M-5)}$$
$$-\frac{60N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}$$
$$+\frac{60(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$\left.\left.+\frac{105(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\right]\right\} \quad \text{......................(4).}$$

The Tchouproff formulae (1), (2), (3) and (4) of Part III are particular cases of formulae (1), (2), (3) and (4) of this part and are deduced by putting M infinite.

The great length of these formulae (3) and (4) is of interest from several points of view. The method employed to obtain them can be employed to obtain the higher moments ${}_2M_5$, ${}_2M_6$, etc. of the Distribution of σ^2 of Samples, and, indeed, any moment of a distribution of a higher moment of the sample, for instance the moments of the distribution of the fourth moments of samples. It is however immediately apparent when the problem is considered, that the algebra involved would be extremely laborious and the results obtained of too great length to be of any value. It is thus evident that the expression of the moments of various distributions, of the nature considered here, in terms of the moments (or betas) of the sampled population, although quite possible, leads to unduly complex results and some alternative method of expression is required.

To return to ${}_2M_1$, ${}_2M_2$, ${}_2M_3$ and ${}_2M_4$, the size of the last two formulae, together with the type of term involved, means that ${}_2M_4$ is in practice impossible of calculation, even with a machine, when M is over 40 or 50; consequently any sampled population of fair size must be treated as infinite and Tchouproff's formulae used even though the assumption be not really legitimate and thus the results

obtained only approximate. The labour involved, even when this assumption is made, is moreover still very considerable.

Practical considerations thus compel us, when a representation of the Distribution of σ^2 of Samples by a Pearson Curve is required, to use Tchouproff's formulae when the population is anything except very small. When it is very small however the formulae given above may be used and a Pearson Curve obtained to represent the distribution. The possibility of such a representation will now be considered.

The Distribution of σ^2 of Samples of 10 *from Population* (C) *of total frequency* 36, *together with its representation by a Pearson Curve.*

Now the types of samples to be obtained from (C), together with the σ^2 of each type of sample, are:

Type of Sample			Frequency	σ^2
0 A	0 B	10 C	3,268,760	·00
0 A	1 B	9 C	20,429,750	·09
0 A	2 B	8 C	48,670,875	·16
0 A	3 B	7 C	57,684,000	·21
0 A	4 B	6 C	37,191,000	·24
0 A	5 B	5 C	13,388,760	·25
0 A	6 B	4 C	2,656,500	·24
0 A	7 B	3 C	276,000	·21
0 A	8 B	2 C	13,500	·16
0 A	9 B	1 C	250	·09
0 A	10 B	0 C	1	·00
1 A	0 B	9 C	2,042,975	·36
1 A	1 B	8 C	10,815,750	·41
1 A	2 B	7 C	21,631,500	·44
1 A	3 B	6 C	21,252,000	·45
1 A	4 B	5 C	11,157,300	·44
1 A	5 B	4 C	3,187,800	·41
1 A	6 B	3 C	483,000	·36
1 A	7 B	2 C	36,000	·29
1 A	8 B	1 C	1,125	·20
1 A	9 B	0 C	10	·09

Thus dividing the samples to be obtained into two groups according as they do or do not contain the element of the first category of (C), we have the two distributions of σ^2 of samples as follows:

Group I		Group II	
Value of σ^2	Frequency	Value of σ^2	Frequency
·09	10	*·00*	3,268,761
·20	1,125	*·09*	20,430,000
·29	36,000	*·16*	48,684,375
·36	2,525,975	*·21*	57,960,000
·41	14,003,550	*·24*	39,847,500
·44	32,788,800	*·25*	13,388,760
·45	21,252,000		

Adding these we have for the complete Distribution of σ^2 of Samples of 10 from Population (C):

Value of σ^2	Frequency
·00	3,268,761
·09	20,430,010
·16	48,684,375
·20	1,125
·21	57,960,000
·24	39,847,500
·25	13,388,760
·29	36,000
·36	2,525,975
·41	14,003,550
·44	32,788,800
·45	21,252,000
Total Frequency = 254,186,856	

The constants of this distribution are:

Working origin = ·24. Working unit, the squared unit of (C).

$$\nu_1' = \cdot 017{,}142{,}86, \qquad {}_2M_2 = \cdot 014{,}274{,}0,$$
$$\nu_2' = \cdot 014{,}567{,}91, \qquad {}_2M_3 = \cdot 000{,}801{,}8,$$
$$\nu_3' = \cdot 001{,}540{,}90, \qquad {}_2M_4 = \cdot 000{,}428{,}2,$$
$$\nu_4' = \cdot 000{,}508{,}45,$$
$$\text{Mean} = 0\cdot 257{,}143, \qquad \Sigma_2 = 0\cdot 119{,}474,$$
$${}_2B_1 = 0\cdot 221{,}053, \qquad {}_2B_2 = 2\cdot 101{,}626.$$

A consideration of the two groups of σ^2 of samples above and a comparison of them with the corresponding groups of the Distribution of Means of Samples from (C) given in Part IV illustrate clearly a number of important points that arise in sampling from a finite population.

If the samples be divided into groups by the consideration of the numbers of individuals from certain categories of the sampled population which are to enter into the sample, these groups will lead to a number of sub-frequency distributions of means, or σ^2 of samples as the case may be, which in turn combine into the complete distributions of those constants.

The very nature or definition of a mean ensures that in the case of the distribution of means of samples, these sub-frequency distributions will be moderately skew and uni-modal, whilst the modes of consecutive sub-frequencies will occur at comparatively small intervals of the range of the mean and the ranges of consecutive sub-frequencies will overlap except for small intervals at their ends. Thus the complete Distribution of Means of Samples of N will approximate quite closely to normality even when the number of categories in the sampled population is small and therefore the number of sub-frequencies of means small likewise. In fact for all sampled populations small or large the Distribution of Means of Samples of N, where N is not large, will be strikingly normal in character. This

has been shown in the previous parts of this paper and has been well known before, at least for samples from large populations.

When we turn to the Distribution of σ^2 of Samples of N, where N is not large, matters are quite different. The value of σ^2 measures the dispersion of the individuals of a sample from its mean, thus in any group of samples as defined above there is a tendency for many values of σ^2 to occur twice causing the corresponding sub-frequency of σ^2 of samples to be much more skew than in the case of the means of samples, although it is still uni-modal. However, this also causes the modes and ranges of consecutive sub-frequencies to differ comparatively widely; and thus even when the sampled population is large with many categories the Distribution of σ^2 of Samples is still far from normal, being in general quite distinctly skew. When however the sampled population is very small with but a few categories, these wide differences between the modes and ranges of the sub-frequencies of σ^2 of samples cause the Distribution of σ^2 of Samples itself to be bi- or tri-modal and moreover to be a discrete frequency distribution with unequal intervals between frequencies. As a Pearson Curve is based on the assumption of a uni-modal frequency, it is thus not to be expected that, in this case, a representation by such a curve will be reasonably accurate; accordingly when these formulae do give the moments of the Distribution of σ^2 of Samples with moderate ease and rapidity their practical use is nullified by the failure of any adequate representation.

This failure is clearly shown in the distribution of σ^2 of samples of 10 from (C). Its moment-coefficients are

$$ {}_2M_1' = 0{\cdot}257{,}143, \qquad {}_2M_2 = 0{\cdot}014{,}274, $$
$$ {}_2M_3 = 0{\cdot}000{,}802, \qquad {}_2M_4 = 0{\cdot}000{,}428, $$

and, when the values of the first eight moment-coefficients of (C), together with $M = 36$ and $N = 10$, are substituted in formulae (1), (2), (3) and (4) of this part, the values obtained for ${}_2M_1'$, etc. are precisely the same. Consequently the formulae are quite efficacious, but as the Pearson Curve derived from these moment-coefficient values above proves to be a J-curve, the completeness of the failure is at once indicated.

We can thus sum up the considerations which arise when dealing with the Distribution of σ^2 of Samples from a finite population. Owing to the unwieldy character of the expressions for ${}_2M_3$ and ${}_2M_4$, it is a practical impossibility, even with a calculating machine, to evaluate them both when M is greater than 40 or 50. Thus, when sampling is conducted on the basis of M finite (i.e. complete samples are drawn) and M is of moderate size, we are compelled by sheer necessities of time and labour to represent the resulting Distribution of σ^2 of Samples by a Pearson Curve based on the doubtful assumption that M is infinite, because Tchouproff's formulae, as used in Part III, provide the only reasonable method of calculating the values of ${}_2M_2$, ${}_2M_3$ and ${}_2M_4$. When M is very large, e.g. several thousands, this probably matters little as is indicated by the Distribution II (b) of this paper, but when M is only of moderate size, the loss of accuracy in representation is not clearly known and may be considerable; however the assumption appears at present a necessity.

When M is quite small, these four formulae will give the moment-coefficients of the Distribution of σ^2 of Samples moderately easily, but their practical use is liable to cease at this stage. As illustrated here the Distribution of σ^2 of Samples, when M is quite small, may be multi-modal, and moreover is almost certain to be a distribution with discrete frequency groups at unequal intervals.

Taking the first of these difficulties, the plurality of modes, this may at first sight be overcome by developing a set of frequency curves of the Pearson type with more modes than one or by using curves based upon some method such as Thiele's semi-invariants. Now if methods like the latter are to produce multi-modal curves, it appears to me, although I have no practical experience in this field, that more constants than the apparently usual number of four would be required, for these constants are after all equivalent to moments in their essentials. The formula for ${}_2M_4$ is severe enough and it is easily seen that formulae for any other higher moment-coefficients or combinations of such moment-coefficients would be impossibly severe; thus this method would not obviate the difficulty of the evaluation of the necessary constants and would fail in practice at any rate on this account. It might be possible to develop a quasi-Pearson Curve with two or more modes whose constants only involve the first four moment-coefficients of the distribution, but, even if developed, it is a very open question as to whether it would be efficacious in the representation of actual distributions of this type, and until the formulae can be simplified, the use of higher moment-coefficients is not practicable.

The second of the difficulties, that is, the frequency groups becoming discrete and at unequal intervals, presents a problem when the representation of the distribution is required, which is also very difficult and for which, at present, no one appears to have obtained any satisfactory solution.

Enough has been said to emphasise the fact that the problem of the representation of Distributions of σ^2 of Samples from a quite small finite population is one which will call for a large amount of research before any practical advance towards its solution is likely to be made.

In conclusion a comparison is given between the Distribution of σ^2 of Samples of 10 from Population (C) and the corresponding Pearson Curve obtained from the values of ${}_2M_2$, ${}_2M_3$ and ${}_2M_4$ given by the formulae of this part, not because it can be expected that a Pearson Curve will represent the distribution successfully, for a Pearson Curve is designedly uni-modal, but merely to emphasise the unusual character of the distribution when M is very small and also the need for some other method of representation.

For this attempted representation of the Distribution of σ^2 of Samples of 10 from (C) we have seen that the constants of the curve are to be:

$$\text{Mean} = 0{\cdot}257{,}143, \qquad {}_2M_2 = 0{\cdot}014{,}274,$$
$$ {}_2B_1 = 0{\cdot}221{,}053, \qquad {}_2B_2 = 2{\cdot}101{,}626.$$

The curve is a J-curve of Type I, hence with the usual notation

$$r = 2{\cdot}147{,}820, \qquad \epsilon = 1{\cdot}072{,}312,$$
$$m'^2 - 2{\cdot}147{,}820\, m' + 1{\cdot}072{,}312 = 0.$$

The last equation has roots 1·358,463 and 0·789,356, and as ${}_2M_3$ is positive

$$m_1' = \quad 0{\cdot}789{,}356, \qquad m_2' = 1{\cdot}358{,}463;$$

therefore

$$m_1 = -0{\cdot}210{,}644, \qquad m_2 = 0{\cdot}358{,}463.$$

Taking as unit $\frac{1}{100}$ the squared unit of (C)

$$b = 43{\cdot}9659, \qquad a_1 = -62{\cdot}6514, \qquad a_2 = 106{\cdot}6173.$$

Taking the total frequency to be 1000, $y_0 = 29{\cdot}6752$, and the curve is

$$y = 29{\cdot}6752\left(\frac{x}{62{\cdot}6514} - 1\right)^{-0{\cdot}210{,}644}\left(1 - \frac{x}{106{\cdot}6173}\right)^{0{\cdot}358{,}463}.$$

The Origin = Mode = 25·7143 − 78·8095 = − 53·0952. Hence, in units of Population (C), the range is from $\sigma^2 = {\cdot}096$ to $\sigma^2 = {\cdot}535$.

Keeping to the unit of the curve, the ordinates of the curve, which is shown in Fig. VIII, are

Abscissa	Ordinate	Abscissa	Ordinate
63	64·2940	90	18·1488
66	38·9144	95	15·4086
70	31·7737	100	12·2180
75	27·2467	103	9·6807
80	23·8151	105	7·1806
85	20·8089	106·62	0·0000

Now Distribution VIII (a) indicated that, when marbles were used, the Distribution of Means of 1000 Samples of 10 obtained from (C) by actual sampling agreed well with the true Distribution of Means of Samples. Consequently the corresponding Distribution of σ^2 of these 1000 Samples was obtained.

VIII (b). The σ^2 of 1000 Samples of 10 from Population (C), the sampling being conducted by the use of coloured marbles.

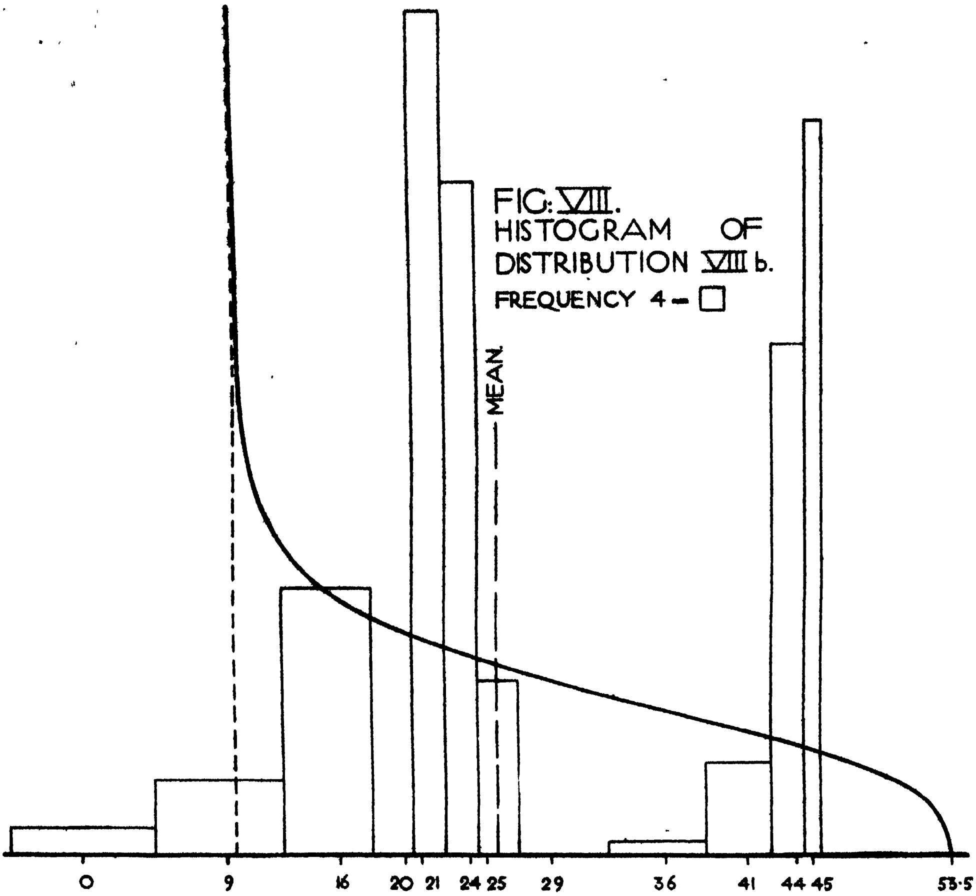
Value of σ^2	*·00*	*·09*	*·16*	*·20*	*·21*	*·24*	*·25*	*·29*	*·36*	*·41*	*·44*	*·45*
Frequency	15	76	188	0	214	171	56	0	10	47	130	93

The constants are:

Working origin = ·24. Working unit, the squared unit of (C).

$\nu_1' = {\cdot}018{,}820{,}0$, Mean = 0·258,820,0, $\Sigma_2 = 0{\cdot}120{,}080$,
$\nu_2' = {\cdot}014{,}773{,}4$, ${}_2M_2 = 0{\cdot}014{,}419{,}2$, ${}_2B_1 = 0{\cdot}194{,}083$,
$\nu_3' = {\cdot}001{,}583{,}6$, ${}_2M_3 = 0{\cdot}000{,}762{,}8$, ${}_2B_2 = 2{\cdot}108{,}627$,
$\nu_4' = {\cdot}000{,}526{,}3$, ${}_2M_4 = 0{\cdot}000{,}438{,}2$.

Note. In the representation of this distribution in Fig. VIII the unit of σ^2 is $\frac{1}{100}$ the squared unit of (C).



This is a discrete distribution like the true Distribution of σ^2 of Samples of 10 from (C) given before in this part, of which distribution it is itself a sample of course. If we reduce the total frequency of this true distribution of σ^2 from 254,186,856 to 1000, we can compare this sample with it. Carrying out this reduction, we have

Value of σ^2	Frequency in VIII (b)	True Frequency reduced by $\frac{1000}{254,186,856}$
·00	15	12·86
·09	76	80·37
·16	188	191·53
·20	0	0·00
·21	214	228·02
·24	171	156·76
·25	56	52·67
·29	0	0·14
·36	10	9·94
·41	47	55·09
·44	130	128·99
·45	93	83·61

Applying the Goodness of Fit test, we have

$$\chi^2 = 5{\cdot}41,\ P = {\cdot}9075 \text{ with 12 Groups.}$$

This shows that VIII (b) is an excellent sample of the true Distribution of σ^2 of Samples of 10 from (C), and consequently an attempt has been made to compare VIII (b) with the J-curve.

Now VIII (b) is a discrete distribution with unequal intervals, so the frequency of any value of σ^2 was represented by a rectangle, whose base was the distance between the two points of bisection of the intervals on either side of this value of σ^2. The height of the rectangle being properly chosen the frequency of this value of σ^2 is thus represented by an area, which is the first step in the comparison because it is areas under the J-curve that represent frequencies. However, as the intervals between successive values of σ^2 in VIII (b) are unequal, this method causes the value of σ^2 considered *not* to be on the mid-ordinate of the rectangle whose area represents the frequency of the value. Thus while the representation is purely approximate and pictorial, it suffices to show that the J-curve in no way represents the Distribution of σ^2 of Samples of 10 from (C).

Summary of Conclusions reached.

(a) The Distribution of Means of Small Samples from an infinite Population.

(1) Even when the sampled population is quite skew, this distribution exhibits a surprising tendency to normality. Hence its approximate representation by a normal curve will be quite good in general and the assumption that, for small samples as well as large, the distribution of means from a quite skew infinite population is normal will be quite useful in practical statistics.

(2) By the aid of moment-coefficients a good representation of this distribution by a Pearson Curve can be obtained showing quite well the skewness, small as this is in general.

(3) The true curve of means of samples is known when the sampling is from a population represented by a Pearson Type III Curve.

(b) The Distribution of σ^2 of Small Samples from an infinite Population.

(1) Tchouproff's corrected formulae for the moment-coefficients of this distribution will lead, with the aid of a Pearson Curve, to a good representation of the distribution in practical cases.

(2) The labour involved in the calculation of the higher moment-coefficients of the sampled population, necessitated by the exact use of Tchouproff's formulae, can be obviated by the assumption that a Pearson Curve will represent the population, which enables approximate values for these higher moment-coefficients to be calculated rapidly by the well-known difference formulae. This approximation does not appear to affect the accuracy of the representation of this distribution of σ^2 of samples to any appreciable extent.

(c) The Distribution of Means of Small Samples from a finite Population.

(1) Here again, even when the sampled population is very skew and as small as 36, the Distribution of Means of Samples still shows a very strong tendency to normality. In fact, it appears that the assumption of the normality of this distribution is legitimate for practical purposes, no matter what the size of the sample or the sampled population, and no matter upon which basis the sampling is conducted. It is only likely to fail as a practical hypothesis if the sampled population follows a really extreme form of skewness.

(2) This distribution can be represented satisfactorily by a Pearson Curve based on the formulae for its moment-coefficients given in Part IV.

(3) Subject to conditions that have been stated broadly, it has been possible to show that this distribution can be represented approximately by a Pearson Curve based upon the values of its moment-coefficients which result from the assumption that the sampled population is infinite in all its categories. That is to say, it has been possible to indicate the conditions under which, in dealing with the Distribution of Means of Samples, it is immaterial whether the sampling be conducted by drawing each time complete samples or individuals only before replacements.

(d) The Distribution of σ^2 of Small Samples from a finite Population.

(1) The two formulae for the 3rd and 4th moment-coefficients of this distribution have been obtained, and thus with Dr Neyman's results the first four moments of the distribution are known. The great length of these two formulae however contributes materially to the difficulty of accurate representation of this distribution.

(2) When the sampled population is large or of moderate size, the length of the formulae for ${}_2M_3$ and ${}_2M_4$, together with their general unwieldiness, makes the evaluation of these constants by their aid almost an arithmetical impossibility at any rate in practical statistics. We are thus compelled to resort to an approximate representation of the distribution by assuming the sampled population infinite and calculating approximate values for the moment-coefficients of the distribution by means of Tchouproff's formulae as in Part III. This approximation is probably quite good when the total frequency of the sampled population is of the order of several thousands, but for moderate-sized sampled populations with total frequency several hundreds its usefulness and accuracy are certainly questionable.

(3) When the sampled population is very small in its total frequency, the formulae for the moment-coefficients of the Distribution of σ^2 of Samples give the values of these constants quite accurately and with moderate ease, but the representation by a Pearson Curve is liable to fail. Since the sampled population when it is small, has only a few categories, the distribution of σ^2 may become multi-modal and hence, as the Pearson Curves involve the assumption of a uni-modal frequency, they are naturally incapable of representing the distribution; whilst further, not only does it tend to become one with discrete frequencies, but these

frequencies occur at unequal intervals, rendering its representation by any curve a matter of doubtful validity and accuracy.

(*e*) Considerations upon the actual practice of sampling.

When the actual sampling is conducted on the basis of an infinite sampled population, that is, individuals only are drawn before replacement, the use of one sample whose randomness is established, such as Mr L. C. Tippett's Random Numbers, provides a really reliable method. Further, it is one which takes precedence over any system of drawing objects from a bag, bowl or urn because of its much greater speed and ease in operation.

When the sampling is conducted on the basis of a finite sampled population, that is, complete samples are drawn each time before replacement, or when a typical random sample is not available, the use of marked tickets appears to be of little or no value because of the almost insuperable difficulty of obtaining really effective shuffling between consecutive draws. If the sampled population be very large, roughly homogeneous and symmetrical material such as commercial coloured glass beads, which provide easy means of distinguishing between the various categories of the sampled population, appears to be quite satisfactory in practice. If however the sampled population be quite small, the lack of true homogeneity even in such material as this will cause bias in actual sampling and, if good results are to be obtained, some material the individuals of which really approximate quite closely to the ideal of complete homogeneity must be employed.

TABLE OF THE RATIO: AREA TO BOUNDING ORDINATE, FOR ANY PORTION OF NORMAL CURVE.

BY JOHN P. MILLS.

*Introduction**. It has been recognized for some time that a table of the ratio, area to ordinate, for the normal curve would be useful. It was pointed out, for example, in *Biometrika*, Vol. XVI. p. 164, that the area A of the tail of a frequency curve could be found approximately by the formula $A = \phi(0)\,\mathcal{R}_x$, where $\phi(0)$ is the ordinate of the curve at the stump and x and $\mathcal{R}_x$ depend on the first and second derivatives at that point. Particular examples of application to the point binomial were given in succeeding pages of that volume, and to the hypergeometric series in Vol. XVII. pp. 61 *et seq.* In all these cases the need for a tabulation of $\mathcal{R}_x$ was apparent because for certain values of x its value could not be obtained easily and with sufficient accuracy from existing tables of the area and the ordinate. The ratio $\mathcal{R}_x$ is also important because its reciprocal equals the mean value of the area of the tail, from x onwards, of the normal curve $y = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}$. The methods by which this table were computed were different in different parts. For the interval, $0 \leq x \leq 1{\cdot}84$, Sheppard's well-known tables gave correct results, save in six cases, which were examined separately by the methods used later. For $1{\cdot}84 \leq x \leq 4{\cdot}20$ it was necessary to use the tables of Dr Jas. Burgess, *Trans. R. S. Edinburgh*, Vol. XXXIX. He tabulated

$$H_t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt, \quad \log \zeta_t = \frac{2}{\sqrt{\pi}}\,e^{-t^2}.$$

If
$$\rho_t = \frac{1 - H_t}{\zeta_t}, \text{ then } \mathcal{R}_x = \rho_t\sqrt{2} \text{ at } t = x/\sqrt{2}.$$

Accordingly ρ was first computed, the values of ζ being taken from an eight place logarithm table, and the divisions made on a machine. For $1{\cdot}300 \leq t \leq 1{\cdot}500$, H and ζ were extracted at intervals of $\Delta t = 0{\cdot}005$, and ρ_t and $\mathcal{R}_x$ computed. This gave a provisional table of $\mathcal{R}_x$ at the inconvenient points, $1{\cdot}300\sqrt{2}$, $1{\cdot}305\sqrt{2}$, etc. What was desired was a set of values at $x = 1{\cdot}80$, $1{\cdot}81$, etc. These could be found by interpolation from the provisional table already obtained, but to produce the desired accuracy easily, it was desirable first to obtain twice as many values of $\mathcal{R}_x$ at intervals half as great. These were found quickly by Pearson's 4-point mid-panel formula. For the interval, $1{\cdot}50 \leq t \leq 2{\cdot}15$, sufficient accuracy could be obtained by first extracting t at intervals of $0{\cdot}01$, and then halving them as before. For $2{\cdot}15 \leq t \leq 3{\cdot}00$, a somewhat greater simplification was possible. By computing several values of ρ_t near the point $t = 2{\cdot}20\sqrt{2}$ the value of $\mathcal{R}_x$ was found by interpolation at $2{\cdot}20$; and then, in similar manner, at $2{\cdot}30$, $2{\cdot}40$, etc. Then the intervening tenths of the x interval were filled in by using Pearson's 5-point formula. From $x = 4{\cdot}25$ ($t = 3$) to the end, the values were supplied by the Editor

* By B. H. Camp, under whose direction this table was prepared.

from material already available at the Galton Laboratory. Some of this part was also computed by the author from the formula

$$\mathcal{R}_x = \frac{1}{x}\left(1 - \frac{1}{x^2} + \frac{1.3}{x^4} - \frac{1.3.5}{x^6} + \ldots\right).$$

The entire computation was checked by differencing, and by overlapping the several methods. The errors introduced were investigated. Eight places of decimals were retained throughout, and the original computation therefore gave a table of $\mathcal{R}_x$ to eight places. In no case did the possible error in the eighth place jeopardize the accuracy of the fifth place retained in the final table; save only in the six instances already mentioned near the beginning of the table, and these were eventually calculated by the later methods to nine places.

$$\left(\textit{The Ratio } \mathcal{R}_x = e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}x^2}\,dx.\right)$$

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
·00	1·25331	993	·40	·93567	623
·01	1·24338	982	·41	·92944	615
·02	1·23356	969	·42	·92329	609
·03	1·22387	957	·43	·91720	602
·04	1·21430	946	·44	·91118	596
·05	1·20484	934	·45	·90522	590
·06	1·19550	923	·46	·89932	583
·07	1·18627	911	·47	·89349	577
·08	1·17716	900	·48	·88772	571
·09	1·16816	890	·49	·88201	565
·10	1·15926	878	·50	·87636	558
·11	1·15048	869	·51	·87078	553
·12	1·14179	858	·52	·86525	548
·13	1·13321	847	·53	·85977	541
·14	1·12474	838	·54	·85436	536
·15	1·11636	827	·55	·84900	530
·16	1·10809	818	·56	·84370	525
·17	1·09991	808	·57	·83845	519
·18	1·09183	799	·58	·83326	514
·19	1·08384	790	·59	·82812	509
·20	1·07594	780	·60	·82303	504
·21	1·06814	771	·61	·81799	498
·22	1·06043	762	·62	·81301	494
·23	1·05281	754	·63	·80807	488
·24	1·04527	745	·64	·80319	484
·25	1·03782	736	·65	·79835	478
·26	1·03046	728	·66	·79357	474
·27	1·02318	719	·67	·78883	469
·28	1·01599	712	·68	·78414	465
·29	1·00887	703	·69	·77949	460
·30	1·00184	696	·70	·77489	455
·31	0·99488	687	·71	·77034	451
·32	·98801	680	·72	·76583	446
·33	·98121	673	·73	·76137	442
·34	·97448	665	·74	·75695	438
·35	·96783	657	·75	·75257	433
·36	·96126	651	·76	·74824	430
·37	·95475	643	·77	·74394	425
·38	·94832	636	·78	·73969	421
·39	·94196	629	·79	·73548	417

TABLE (*continued*).

x	R_x	$-\Delta$	x	R_x	$-\Delta$
·80	·73131	413	1·30	·56487	265
·81	·72718	409	1·31	·56222	262
·82	·72309	405	1·32	·55960	261
·83	·71904	401	1·33	·55699	258
·84	·71503	397	1·34	·55441	256
·85	·71106	394	1·35	·55185	254
·86	·70712	390	1·36	·54931	252
·87	·70322	387	1·37	·54679	249
·88	69935	382	1·38	·54430	248
·89	·69553	380	1·39	·54182	246
·90	·69173	375	1·40	·53936	244
·91	·68798	373	1·41	·53692	242
·92	·68425	368	1·42	·53450	240
·93	·68057	366	1·43	·53210	238
·94	·67691	362	1·44	·52972	237
·95	·67329	358	1·45	·52735	234
·96	·66971	356	1·46	·52501	233
·97	66615	352	1·47	·52268	230
·98	·66263	349	1·48	·52038	229
·99	·65914	346	1·49	·51809	227
1·00	·65568	343	1·50	·51582	226
1·01	·65225	340	1·51	·51356	223
1·02	·64885	336	1·52	·51133	222
1·03	·64549	334	1·53	·50911	221
1·04	·64215	330	1·54	·50690	218
1·05	·63885	328	1·55	·50472	217
1·06	·63557	325	1·56	·50255	215
1·07	·63232	322	1·57	·50040	214
1·08	·62910	319	1·58	·49826	212
1·09	·62591	317	1·59	·49614	210
1·10	·62274	313	1·60	·49404	209
1·11	·61961	311	1·61	·49195	207
1·12	·61650	308	1·62	·48988	206
1·13	·61342	306	1·63	·48782	204
1·14	·61036	303	1·64	·48578	202
1·15	·60733	300	1·65	·48376	201
1·16	·60433	298	1·66	·48175	200
1·17	·60135	295	1·67	·47975	198
1·18	·59840	292	1·68	·47777	197
1 19	·59548	291	1·69	·47580	195
1·20	·59257	287	1·70	·47385	193
1·21	·58970	286	1·71	·47192	193
1·22	·58684	282	1·72	·46999	191
1·23	·58402	281	1·73	·46808	189
1·24	·58121	278	1·74	·46619	188
1·25	·57843	276	1·75	·46431	187
1·26	·57567	273	1·76	·46244	186
1·27	·57294	272	1·77	·46058	184
1·28	·57022	268	1·78	·45874	182
1·29	·56754	267	1·79	·45692	182

TABLE (*continued*).

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
1·80	·45510	180	2·30	·37858	129
1·81	·45330	179	2·31	·37729	128
1·82	·45151	178	2·32	·37601	127
1·83	·44973	176	2·33	·37474	126
1·84	·44797	175	2·34	·37348	126
1·85	·44622	174	2·35	·37222	125
1·86	·44448	173	2·36	·37097	124
1·87	·44275	171	2·37	·36973	123
1·88	·44104	170	2·38	·36850	123
1·89	·43934	169	2·39	·36727	122
1·90	·43765	168	2·40	·36605	121
1·91	·43597	167	2·41	·36484	120
1·92	·43430	165	2·42	·36364	120
1·93	·43265	165	2·43	·36244	119
1·94	·43100	163	2·44	·36125	118
1·95	42937	162	2·45	·36007	118
1·96	·42775	161	2·46	·35889	116
1·97	·42614	160	2·47	·35773	116
1·98	·42454	159	2·48	·35657	116
1·99	·42295	158	2·49	·35541	114
2·00	·42137	157	2·50	·35427	114
2·01	·41980	155	2·51	·35313	114
2·02	·41825	155	2·52	·35199	112
2·03	·41670	154	2·53	·35087	112
2·04	·41516	152	2·54	·34975	112
2·05	·41364	152	2·55	·34863	110
2·06	·41212	150	2·56	·34753	110
2·07	·41062	150	2·57	·34643	110
2·08	·40912	148	2·58	·34533	108
2·09	·40764	148	2·59	·34425	109
2·10	·40616	146	2·60	·34316	107
2·11	·40470	146	2·61	·34209	107
2·12	·40324	145	2·62	·34102	106
2·13	·40179	143	2·63	·33996	106
2·14	·40036	143	2·64	·33890	105
2·15	·39893	142	2·65	·33785	104
2·16	·39751	141	2·66	·33681	104
2·17	·39610	140	2·67	·33577	103
2·18	·39470	139	2·68	·33474	103
2·19	·39331	138	2·69	·33371	102
2·20	·39193	138	2·70	·33269	101
2·21	·39055	136	2·71	·33168	101
2·22	·38919	136	2·72	·33067	100
2·23	·38783	134	2·73	·32967	100
2·24	·38649	134	2·74	·32867	98
2·25	·38515	133	2·75	·32768	98
2·26	·38382	132	2·76	·32669	98
2·27	·38250	132	2·77	·32571	97
2·28	·38118	130	2·78	·32474	97
2·29	·37988	130	2·79	·32377	97

TABLE (*continued*).

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
2·80	·32280	96	3·30	·28064	74
2·81	·32184	95	3·31	·27990	73
2·82	·32089	95	3·32	·27917	73
2·83	·31994	94	3·33	·27844	72
2·84	·31900	94	3·34	·27772	73
2·85	·31806	93	3·35	·27699	72
2·86	·31713	93	3·36	·27627	71
2·87	·31620	92	3·37	·27556	71
2·88	·31528	92	3·38	·27485	71
2·89	·31436	91	3·39	·27414	71
2·90	·31345	91	3·40	·27343	70
2·91	·31254	90	3·41	·27273	70
2·92	·31164	90	3·42	·27203	69
2·93	·31074	89	3·43	·27134	69
2·94	·30985	89	3·44	·27065	69
2·95	·30896	88	3·45	·26996	69
2·96	·30808	88	3·46	·26927	68
2·97	·30720	88	3·47	·26859	68
2·98	·30632	87	3·48	·26791	67
2·99	·30545	86	3·49	·26724	67
3·00	·30459	86	3·50	·26657	67
3·01	·30373	86	3·51	·26590	67
3·02	·30287	85	3·52	·26523	66
3·03	·30202	84	3·53	·26457	66
3·04	·30118	84	3·54	·26391	65
3·05	·30034	84	3·55	·26326	66
3·06	·29950	84	3·56	·26260	65
3·07	·29866	82	3·57	·26195	64
3·08	·29784	83	3·58	·26131	65
3·09	·29701	82	3·59	·26066	64
3·10	·29619	81	3·60	·26002	63
3·11	·29538	82	3·61	·25939	64
3·12	·29456	80	3·62	·25875	63
3·13	·29376	81	3·63	·25812	63
3·14	·29295	80	3·64	·25749	63
3·15	·29215	79	3·65	·25686	62
3·16	·29136	79	3·66	·25624	62
3·17	·29057	79	3·67	·25562	62
3·18	·28978	78	3·68	·25500	61
3·19	·28900	78	3·69	·25439	61
3·20	·28822	78	3·70	·25378	61
3·21	·28744	77	3·71	·25317	61
3·22	·28667	77	3·72	·25256	60
3·23	·28590	76	3·73	·25196	60
3·24	·28514	76	3·74	·25136	60
3·25	·28438	75	3·75	·25076	59
3·26	·28363	76	3·76	·25017	60
3·27	·28287	74	3·77	·24957	59
3·28	·28213	75	3·78	·24898	58
3·29	·28138	74	3·79	·24840	59

TABLE (*continued*).

x	$\mathfrak{R}_x$	$-\Delta$	x	$\mathfrak{R}_x$	$-\Delta$
3·80	·24781	58	6·00	·16238	253
3·81	·24723	58	6·10	·15984	246
3·82	·24665	58	6·20	·15739	239
3·83	·24607	57	6·30	·15500	232
3·84	·24550	57	6·40	·15269	225
3·85	·24493	57	6·50	·15044	218
3·86	·24436	57	6·60	·14825	212
3·87	·24379	56	6·70	·14613	206
3·88	·24323	56	6·80	·14407	201
3·89	·24267	56	6·90	·14206	195
3·90	·24211	56	7·00	·14010	190
3·91	·24155	55	7·10	·13820	185
3·92	·24100	55	7·20	·13635	180
3·93	·24045	55	7·30	·13455	176
3·94	·23990	55	7·40	·13279	171
3·95	·23935	54	7·50	·13108	167
3·96	·23881	55	7·60	·12941	163
3·97	·23826	54	7·70	·12778	159
3·98	·23772	53	7·80	·12619	155
3·99	·23719	54	7·90	·12464	151
4·00*	·23665	264	8·00	·12313	148
4·05	·23401	258	8·10	·12166	144
4·10	·23143	253	8·20	·12021	141
4·15	·22890	248	8·30	·11880	138
4·20	·22642	243	8·40	·11743	135
4·25	·22399	238	8·50	·11608	132
4·30	·22161	233	8·60	·11477	129
4·35	·21928	228	8·70	·11348	126
4·40	·21700	224	8·80	·11222	123
4·45	·21476	219	8·90	·11099	120
4·50	·21257	215	9·00	·10979	118
4·55	·21042	211	9·10	·10861	115
4·60	·20831	207	9·20	·10745	113
4·65	·20624	203	9·30	·10632	111
4·70	·20421	199	9·40	·10522	108
4·75	·20222	195	9·50	·10413	106
4·80	·20027	192	9·60	·10307	104
4·85	·19835	188	9·70	·10203	102
4·90	·19647	185	9·80	·10101	100
4·95	·19462	181	9·90	·10001	98
5·00*	·19281	353	10·00	·09903	
5·10	·18928	341			
5·20	·18587	329			
5·30	·18258	318			
5·40	·17940	307			
5·50	·17632	297			
5·60	·17335	288			
5·70	·17047	278			
5·80	·16769	270			
5·90	·16499	261			

* Note the change in the x interval.

ON THE CORRELATION OF THE MEAN AND THE VARIANCE IN SAMPLES DRAWN FROM AN "INFINITE" POPULATION

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1. The relation between the two characters in a sample, mean $\bar{x}$ and standard deviation σ (or variance σ^2), is a very interesting one as several methods of judging whether a certain sample has been taken from a given population, or if two different samples have been taken from the same population, are based on the comparison of certain functions of $\bar{x}$ with certain functions of σ.

For the case where the original frequency distribution is normal, the question has been largely discussed, in particular in the paper of "Student"* in which the probability is found of having different values of $\frac{\bar{x}}{\sigma}$, where $\bar{x}$ is measured from the mean of the sampled population. We know that in the case of a normal population $\bar{x}$ and σ are independent, but we know further that the populations having a limited range which can be met with in practice are, speaking rigorously, never normal. In many cases, also, we are dealing with populations which differ from the normal ones in other characters besides the range. It is interesting therefore to consider the relation of $\bar{x}$ and σ for such populations in order to get some idea about the possible differences in these properties.

In dealing with this matter we may use the results of an earlier paper by the present author about skew regression published in this volume of Biometrika, and also the formula of K. Pearson giving the second order parabola of regression†. After deducing a second order parabola of regression of σ^2 on $\bar{x}$ in a sample taken from an indefinitely large population with any given distribution, we shall show that the calculation of higher parabolae is of little value, as their expressions for small values of n—the number of individuals in a sample—are too complicated for us to draw practical conclusions from. Further the difference between them and the expression of the second order parabola vanishes as $n \to \infty$.

Afterwards we shall draw attention to a method of judging whether a sample is likely to have been taken from a population whose distribution is supposed to be known.

2. We shall use a notation like that in the previous paper, namely: μ_k will denote the kth moment of the sampled population about its mean; λ_k—the mean value of $(\sigma^2 - \bar{\sigma}^2)\bar{x}^k$ and n—the number of individuals in the sample.

* *Biometrika*, Vol. VI. pp. 1 *et seq.*

† *Biometrika*, Vol. XIII. pp. 296—300.

It is convenient to measure the deviations of the two variates σ^2 and $\bar{x}$ from their respective means in terms of their standard deviations, namely

$$\sigma_{\bar{x}} = \sqrt{\frac{\mu_2}{n}}, \quad \sigma_{\sigma^2} = \frac{\mu_2}{n}\sqrt{\frac{n-1}{n}[(n-1)\beta_2 - n + 3]} \qquad (1).$$

Adopting these units we shall put in our formulae for the higher parabolae of regression, instead of λ_k, the expression

$$q_k = \frac{\lambda_k}{\sigma_{\sigma^2}\,\sigma_{\bar{x}}^k} \qquad (2).$$

The deviation of the mean in any sample from its mean value in samples we shall denote by $\bar{x}'$ and the ordinates of the parabola of regression measured from the mean of σ^2 by u'; when $\bar{x}'$ and u' are measured in terms of their standard deviations as given in (1) we shall term them $\bar{x}$ and u. We shall also need the usual frequency constants β for the sampled population and for the population of means of samples taken from the former. The first we shall denote by $\beta_1, \beta_2, \ldots$ and the second by $B_1, B_2, \ldots$.

The formula for the second order parabola of regression can be taken directly from the paper of K. Pearson (*loc. cit.*), and applying it to the variates under consideration we have

$$u = q_1\bar{x} + \frac{q_2 - q_1\sqrt{B_1}}{B_2 - B_1 - 1}(\bar{x}^2 - \sqrt{B_1}\,\bar{x} - 1) \qquad (3).$$

It remains to express all the constants q_k and the B's in terms of the β's. All necessary formulae are contained in my paper in *Biometrika*, Vol. XVII. pp. 472—479. We have

$$H_1 = q_1 = \rho' = \frac{\sqrt{n-1}\,\sqrt{\beta_1}}{\sqrt{(n-1)\beta_2 - n + 3}} \qquad (4),$$

$$q_2 = \frac{R'\sigma_1}{\sigma_{\bar{x}}^2} = \sqrt{\frac{n-1}{n}}\,\frac{\beta_2 - 3}{\sqrt{(n-1)\beta_2 - n + 3}} \qquad (5),$$

where ρ', R' and σ are symbols previously used.

Further
$$B_1 = \frac{\beta_1}{n}, \quad B_2 = \frac{3(n-1) + \beta_2}{n} \qquad (6).$$

Using these formulae, we have

$$u = \frac{\sqrt{n-1}\,\sqrt{\beta_1}}{\sqrt{(n-1)\beta_2 - n+3}}\bar{x} + \frac{\sqrt{n(n-1)}}{2(n-1)+\beta_2-\beta_1-1}\,\frac{\beta_2-\beta_1-3}{\sqrt{(n-1)\beta_2 - n+3}}\left(\bar{x}^2 - \sqrt{\frac{\beta_1}{n}}\,\bar{x} - 1\right) \qquad (7),$$

and the square of the corresponding second order correlation ratio is

$$H_2^2 = \frac{(n-1)\beta_1}{(n-1)\beta_2 - n + 3} + \frac{n-1}{2(n-1)+\beta_2-\beta_1-1}\,\frac{(\beta_2-\beta_1-3)^2}{(n-1)\beta_2 - n + 3}$$

$$= \frac{(n-1)\beta_1}{(n-1)\beta_2 - n + 3}\left\{1 + \frac{(\beta_2-\beta_1-3)^2}{\beta_1(2n+\beta_2-\beta_1-3)}\right\} \qquad (8a),$$

$$= 1 - \frac{2n^2(\beta_2-\beta_1-1)}{\{(n-1)\beta_2 - n + 3\}\{2n+\beta_2-\beta_1-3\}} \qquad (8b).$$

From (8a) we see that for large values of n the second order term has no importance provided β_1 is not zero. If $\beta_1 = 0$ the first order term disappears and the correlation depends only on the remaining term, unless $\beta_2 = 3$, at the Normal Point, when this term also vanishes. When $n \to \infty$, then $H_2^2 \to \frac{\beta_1}{\beta_2 - 1}$, that is to say the contours in the β_1, β_2 field of equal correlation between $\bar{x}$ and σ^2 are approximately straight lines radiating from the point $\beta_2 = 1$, $\beta_1 = 0$. Along the boundary line $\beta_1 = 0$ we have seen that $H_1^2 = 0$, while the form (8b) shows that along the other boundary of the field, namely, $\beta_2 - \beta_1 - 1 = 0$, there is perfect relationship, or $H_2^2 = 1$.

This latter equality arises from the well-known fact that the populations corresponding to β_1, β_2 points along this line contain only two classes of individuals; the same condition must hold in the sample, where let us say there are p individuals with character value a and $n-p$ with character value b. Then the mean and the variance of the sample will be definite functions of p, namely

$$\bar{x} = \frac{pa + (n-p)\,b}{n}, \quad \sigma^2 = \frac{(n-p)\,p}{n^2}(a+b)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

whence

$$r^2 = \left(\frac{a+b}{b-a}\right)^2 (b-\bar{x})(\bar{x}-a) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

If now we express a and b in terms of β_1 and $\beta_2 = \beta_1 + 1$, we shall be able to transform (10) into the regression parabola (7) corresponding to these values of the β's.

Turning to (7), it will be noticed that the sign of the coefficient of $\bar{x}^2$ depends upon the sign of $\kappa = \beta_2 - \beta_1 - 3$. That is to say for samples from populations whose distributions correspond to β's lying above* the line $\beta_2 - \beta_1 - 3 = 0$, the two infinite branches of the regression parabola are directed in the negative sense of the u-axis, while for samples from populations with β's lying on this line the second order term vanishes and (7) becomes linear; and for samples from populations with β's below this line, the branches of the parabola are directed in the positive sense. $\beta_2 - \beta_1 - 3 = 0$ is the line along which lie the β's of distributions representing Poisson's limit to the binomial, and it also divides the areas of positive and negative binomials.

Let us now consider the form of (7) a little more in detail. In general (that is to say, when the original frequency distribution is not a U- or J-shaped one) the most interesting points of the x-axis are those near the mean and the mode, and we shall consider the behaviour of the second order parabola of regression near these points. The abscissa of the mode of means in samples is not known, but we can take, as an approximation to it, the abscissa $M(\bar{x})$ of the mode of a Pearson curve fitted to the distribution of means. It is given by

$$M(\bar{x}) = -\frac{\sqrt{B_1}(B_2+3)}{2(5B_2 - 6B_1 - 9)} = -\sqrt{\frac{\beta_1}{n}}\,\frac{6(n-1) + \beta_2 + 3}{2[6(n-1) + 5\beta_2 - 6\beta_1 - 9]} \ldots\ldots(11).$$

* Above, in the sense of Fig. 2.

The sign of $M(\bar{x})$ depends upon the sign of $\sqrt{\beta_1}\,[6(n-1)+5\beta_2-6\beta_1-9]$. The expression in brackets is generally positive. In fact, if it be negative, then $5\beta_2-6\beta_1-9$ must be negative, which happens when the original frequency distribution is a J-shaped or a U-shaped one and is unusual. Further, as $\beta_2 \geqslant \beta_1+1$,

$$6(n-1)+5\beta_2-6\beta_1-9 \geqslant 6n-\beta_1-10$$

and if the left-hand side be negative then the right-hand one must be so and therefore $\beta_1 \geqslant 6n-10$, which for any value of $n \geqslant 2$ gives values of β_1 very unlikely to be met with in practice. Therefore we shall consider only the case, when the denominator in (11) is positive and so that the sign of $M(\bar{x})$ is that of $-\sqrt{\beta_1}$.

Let ξ be the abscissa of the vertex of the parabola (7). We find easily:

$$\xi = -\frac{\sqrt{n\beta_1}}{\beta_2-\beta_1-3}.$$

If $\beta_2-\beta_1-3<0$ then the signs of $M(\bar{x})$ and ξ are different and thus the mean of the population is between the mode $M(\bar{x})$ and the vertex of the parabola. We remember that in this case the infinite branches of the parabola are directed in the negative sense of the u-axis.

Suppose now that $\beta_2-\beta_1-3>0$. Then the signs of ξ and $M(\bar{x})$ are the same. Let us consider

$$\frac{\xi}{M(\bar{x})} = \frac{2n[6(n-1)+5\beta_2-6\beta_1-9]}{(\beta_2-\beta_1-3)[6(n-1)+\beta_2+3]} \quad\text{.................(12).}$$

and find the condition that it be less than unity.

This condition can be written

$$12n^2+2n(5\beta_2-6\beta_1-15) < 6n(\beta_2-\beta_1-3)+(\beta_2-3)(\beta_2-\beta_1-3),$$

and we see that it is satisfied if n lies between the two roots of the equation

$$12n^2+2n(2\beta_2-3\beta_1-6)-(\beta_2-3)(\beta_2-\beta_1-3)=0,$$

that is, (as n is positive), when

$$n < \frac{\sqrt{(2\beta_2-3\beta_1-6)^2+12(\beta_2-3)(\beta_2-\beta_1-3)}-(2\beta_2-3\beta_1-6)}{12}.$$

As $0 \leqslant \beta_1 \leqslant \beta_2-3$, it follows that for $\xi < M(\bar{x})$ it is necessary that

$$n < \frac{5(\beta_2-3)}{12} \quad\text{or}\quad \beta_2 > \frac{12n+15}{5} \geqslant 7{\cdot}8.$$

In practice we meet very rarely a β_2 with so high a value as 7·8 (which corresponds to the case $n=2$!) and therefore see that with rare exceptions the mode $M(\bar{x})$ lies between the vertex and the mean of the population. Further, in the case we are considering $(\beta_2-\beta_1-3>0)$ the infinite branches of the parabola are directed in the positive sense of the u-axis.

We conclude therefore that for samples from populations likely to be met with in practice, that is to say when

$$n > \frac{\beta_1+10}{6}, \quad\text{and}\quad n > \frac{5(\beta_2-3)}{12},$$

the ordinate, u, of the regression parabola of the variance decreases as we pass from the mean value of $\bar{x}$ to its modal value in samples. Further if the size of the sample n is considerable, since the ratio $\xi/M(\bar{x})$ is of order n, the variation of u in the neighbourhood of the modal and mean $\bar{x}$ will be approximately linear. The position can be illustrated diagrammatically as in Fig. 1, which corresponds to the case of a sample drawn from a population with positive skewness (μ_3 or $\sqrt{\beta_1}$ positive) and having $\beta_2 - \beta_1 - 3 < 0$.

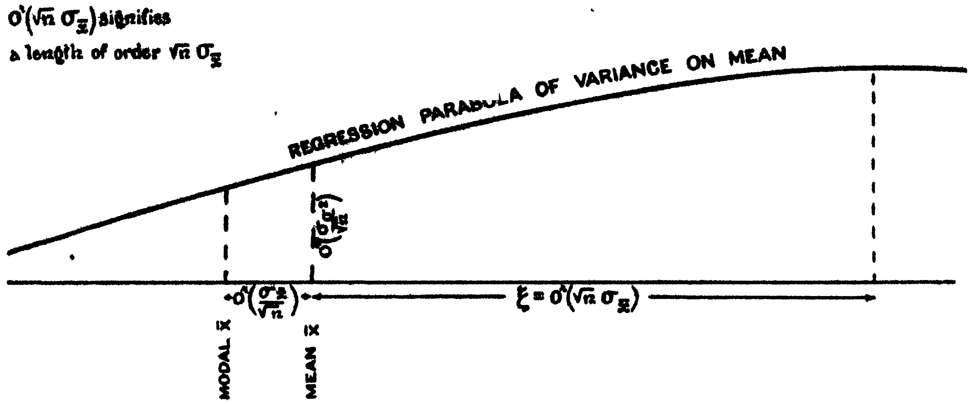

Fig. 1. Regression of Variance on Mean in Sample of n.

It is in fact only in small samples that the regression curve diverges far from a straight line within the range of significant frequency, i.e. within $\bar{x} = \pm 3\sigma_{\bar{x}}$, and $u = \pm 3\sigma_{\sigma^2}$, say. This is brought out clearly in the Tables on pp. 411—413, which give the values of $H_1 = |r|$, H_2, $100 \times \dfrac{H_2^2 - H_1^2}{H_2^2}$ for samples of 10, 50, 100 and 1000, and for population values of β_1 and β_2 covering the range commonly met with. Table III shows how the second order term becomes of slight importance for samples over 100, except for very small values of β_1 and for extreme values of β_2. Tables I and II make it clear that we may be very far wrong when considering a sample from a population of unknown form, if we assume that the normal rule of no correlation between the mean and the standard deviation in the sample is applicable.

Let us consider the locus of points on the $\beta_1\beta_2$ field corresponding to a given value of H_2. We have from (8b)

$$2n^2(\beta_2 - \beta_1 - 1) = (1 - H_2^2)[(n-1)\beta_2 - n + 3][2n + \beta_2 - \beta_1 - 3] \quad \text{...(13)},$$

which is the equation of a hyperbola passing through the point

$$\beta_1 = -\frac{2}{n-1}, \quad \beta_2 = \frac{n-3}{n-1}.$$

This hyperbola is independent of H_2, and has the asymptotes

$$(1 - H_2^2)[(n-1)\beta_2 - n + 3] = 2n^2 \quad \text{......................(14)},$$

$$\beta_2 - \beta_1 + 2n - 3 = 0 \quad \text{..............................(15)}.$$

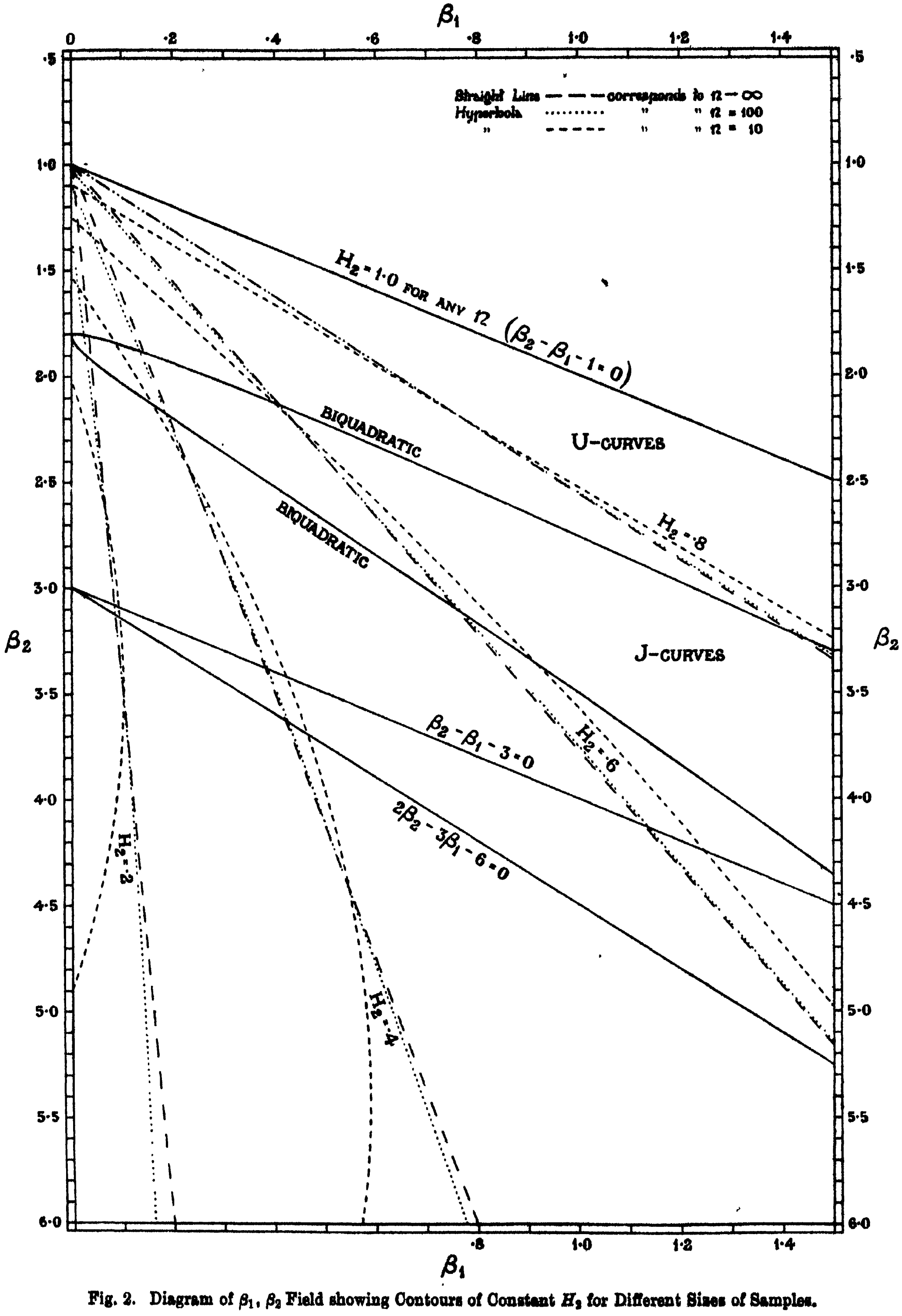

Fig. 2. Diagram of β_1, β_2 Field showing Contours of Constant H_2 for Different Sizes of Samples.

If $n \to \infty$ then this hyperbola degenerates into a straight line

$$\beta_1 = H_2^2(\beta_2 - 1) \qquad \text{.............................(16)},$$

passing through the point $\beta_1 = 0$, $\beta_2 = 1$.

Fig. 2 shows the $\beta_1\beta_2$ field with two sets of hyperbolae (13) corresponding to $n = 10$ and $n = 100$ and to $H_2 = \cdot 2, \cdot 4, \cdot 6, \cdot 8$ and $1 \cdot 0$, and with the set of straight lines (16) corresponding to the same values of H_2. We see that the hyperbolae for $n = 100$ are close to the corresponding straight lines except for very small values of H_2.

Considering this figure we see that although the very high values of H_2 fall on the whole in the area of J- and U-curves, this constant may yet have quite considerable values for populations of types commonly met with, and it is clear that one must be very cautious in drawing conclusions from Student's test, if there be no good reason for supposing that the original frequency distribution is normal. We see further that this is not a case where the position is improved in large samples.

The difficulties involved in the application of Student's test may be illustrated also in the following way. Assume that we are dealing with a sample from a population with $\beta_1 = \cdot 27$ and $\beta_2 = 3 \cdot 5$ *. Suppose further that the variance of the sample shows a deviation from its mean equal to $2\sigma_{\sigma^2}$; for the deviation of the mean in the sample, we can take two cases, first—that it is equal to $+2\sigma_{\bar{x}}$ and second—that it is equal to $-2\sigma_{\bar{x}}$. If we express the deviations of both the variates in common units, for instance in terms of $\sigma_x = \sqrt{\frac{\mu_2}{n}}$, we shall have

$$\sigma = \sqrt{n - 1 + 2\sqrt{\frac{n-1}{n}}\left[(n-1)\beta_2 - n + 3\right]}, \qquad \bar{x} = \pm 2,$$

where σ is the standard deviation of the sample, and $\bar{x}$ the deviation of the mean of the sample from the mean of the population. If now we do not know the moments of the sampled population but only σ and the position of the sample mean, we might use Student's method to test the probability of drawing such a sample from a population with mean at distance $+2$ (or -2) from the sample mean. This we should do by calculating in *both* cases the same ratio, viz.

$z = \frac{|\bar{x}|}{\sigma}$ in Student's notation, where $\sigma = s$, or in that of Fisher†,

$$t = \frac{\sqrt{n-1}\,|\bar{x}|}{s} = \frac{|\bar{x}|}{\left\{1 + 2\sqrt{\frac{1}{n}\left(\beta_2 - \frac{n-3}{n-1}\right)}\right\}^{\frac{1}{2}}} \qquad \text{.............................(17)}.$$

If for instance $n = 11$, we find that $z = \cdot 45$ and $t = 1 \cdot 42$.

* These figures correspond to a set of 200 yields of a serial in a field experiment described by Gorski and Stefaniow in *Rocznik Nauk Rolniczych*, 1917.

† For tables of z, see *Biometrika*, Vol. VI. p. 19 (revised and corrected as Table XXV in Pearson's *Tables for Statisticians and Biometricians*, 2nd Edition), and for t, Fisher's *Statistical Methods for Research Workers*, 1925.

Now let us consider the two cases from the point of view of the correlation between the variance and the mean. We shall see below that the difference between $H_k^2 - H_2^2$, where $k > 2$, is of the order $\frac{1}{n^2}$, therefore we can assume that the actual regression line is well fitted by the second order parabola. If we calculate its ordinates corresponding to the two points $\bar{x} = \pm 2$, we shall have

$$u(-2) = -\cdot 564; \qquad u(+2) = \cdot 688.$$

Taking the ratio of the deviations of the actual value of σ^2 in the sample, from these two means of different arrays, to the corresponding mean array-standard deviation, we have

$$\frac{\sigma^2 - \bar{\sigma}^2 - u(-2)}{\sqrt{1 - H_2^2}} = 2\cdot 7; \qquad \frac{\sigma^2 - \bar{\sigma}^2 - u(+2)}{\sqrt{1 - H_2^2}} = 1\cdot 4,$$

and we see at once that the probability of obtaining the two samples ($\bar{x} = -2$, $\sigma^2 = \bar{\sigma}^2 + 2$) and ($\bar{x} = +2$, $\sigma^2 = \bar{\sigma}^2 + 2$) from the same population is very different, although Student's test gave us the same figure, $t = 1\cdot 42$, for both.

The method described can be considered as a general method of testing whether a sample is likely to have been taken from a population with *a given distribution*, at least, *with given* β_1, β_2 *and* μ_2.

As we have seen, it consists in comparing the deviation $\frac{\sigma^2 - \bar{\sigma}^2 - u(\bar{x})}{\sigma_{\sigma^2}}$ with its standard deviation $\sqrt{1 - H_2^2}$, that is to say, in calculating the ratio

$$L = \frac{\Delta\sigma^2 - u(\bar{x})}{\sqrt{1 - H_2^2}}$$

$$= \frac{(2n+\beta_2-\beta_1-3)[\sqrt{(n-1)\beta_2-n+3}\,\Delta\sigma^2 - \sqrt{(n-1)\beta_1}\,\bar{x}] - \sqrt{n(n-1)}(\beta_2-\beta_1-3)\left(\bar{x}^2 - \sqrt{\frac{\beta_1}{n}}\,\bar{x} - 1\right)}{n\sqrt{2(2n+\beta_2-\beta_1-3)(\beta_2-\beta_1-1)}} \quad \text{.........(18)},$$

where $\bar{x}$ is the deviation of the mean in the sample from the mean of the population measured in terms of $\sqrt{\frac{\mu_2}{n}}$, μ_2 being the variance of the population; $\Delta\sigma^2$ is the deviation of the variance in the sample from its mean value $\frac{n-1}{n}\mu_2$ measured in terms of

$$\sigma_{\sigma^2} = \frac{\mu_2}{n}\sqrt{\frac{n-1}{n}\left((n-1)\beta_2 - n + 3\right)};$$

n is the size of the sample; β_1 and β_2 refer to the sampled population. If L is nearly 3, we may consider it as improbable that the sample has been taken from the population considered; if however $L < 2$ such an assumption will not be in disagreement with the observed mean and standard deviation of the sample. Unfortunately such cases in which we know the constants μ_2, β_1 and β_2 of the

sampled population are very seldom met with. When as usual they are unknown the results of this paper serve to emphasize the caution which it is necessary to exercise in drawing conclusions from the contents of small samples*.

3. We shall now prove that the difference $H_k^2 - H_2^2$, where $k > 2$, is at least of the order $\frac{1}{n^2}$. We shall do it by showing that any term $V_k^2/\Delta_{k-1}\Delta_k$ of the series (35)† is of the order $\frac{1}{n^2}$. As the calculations necessary for this purpose are easier to do than to describe, we shall avoid the details and simply sketch the general method of procedure.

We have

$$\begin{array}{llll} q_0, & q_1, & \ldots & q_k \\ M_0, & M_1, & & M_k \\ M_1, & M_2, & \ldots & M_{k+1} \end{array}$$

$$\begin{array}{llll} M_{k-1}, & M_k, & & M_{2k-1} \end{array}$$

$$\Delta_k = \begin{vmatrix} M_0, & M_1, & \ldots & M_k \\ M_1, & M_2, & \ldots & M_{k+1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ M_k, & M_{k+1}, & \ldots & M_{2k} \end{vmatrix},$$

where the M's are the moments of the mean of the sample about its mean, measured in terms of $\sigma_{\bar{x}}$.

Let us calculate some first terms of M's and q's which will be clearly polynomials in $\frac{1}{n}$. For this purpose let X be the deviation of the mean in the sample from its mean measured in ordinary units and $[A]$ be the mean value of any

* To avoid possible misunderstanding we may here add the following remarks:

Let us consider the variate $\qquad v_k = \sigma^2 - \bar{\sigma}^2 - u_k(\bar{x})$,

where $u_k(\bar{x})$ is the ordinate of the kth order parabola of regression. The variate v_k is a character of the sample in the sense that to every sample there corresponds a perfectly definite value of v_k. Therefore we may calculate the mean $\bar{v}_k$ of v_k and its standard deviation σ_{v_k}. We have, rigorously,

$$\bar{v}_k = 0\,; \quad \sigma_{v_k} = \sqrt{1 - H_k^2},$$

remembering that σ^2 is measured in terms of its own standard deviation.

The proposed method of testing is therefore constructed on the usual principle of comparing the deviation of a variate with the corresponding standard deviation. We notice that if k increases, σ_{v_k} decreases and thus our test becomes more and more accurate.

Of course it would be useful if we knew the value of the standard deviation of v_k for each array corresponding to the different values of $\bar{x}$, because very probably they are not constant. Then it would be possible to consider the variates $v_k(\bar{x})$ corresponding to different values of $\bar{x}$. In any case the proposed L-test seems to be as justified as any other test built on the same principle.

† In my previous paper.

variate A. Then, if $x_1, x_2, \dots x_n$ are the deviations of individuals in the sample from the mean of the population,

$$X^k = \left(\frac{\sum\limits_{r=1}^{n} x_i}{n}\right)^k = \frac{1}{n^k} \sum_{r_1, r_2, \dots r_n} \frac{k!}{r_1! \, r_2! \dots r_n!} x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \quad \text{.........(19)},$$

where the sum is extended over all combinations of non-negative integral values of r's satisfying the condition

$$r_1 + r_2 + \dots + r_n = k.$$

Accordingly

$$[X^k] = \frac{1}{n^k} \sum_{r_1, r_2, \dots r_n} \frac{k!}{r_1! \, r_2! \dots r_n!} \mu_{r_1} \mu_{r_2} \dots \mu_{r_n} \quad \text{..................(20)}.$$

This sum contains several classes of equal terms corresponding each to a single system of values of the r's written in different orders, the number of terms being dependent upon n. We notice that $\mu_1 = 0$ and we assume $\mu_0 = 1$. The most numerous class of non-vanishing terms of (20) corresponds to the following systems of r's:

$$\left.\begin{array}{l} r_i = 2 \quad \left(i = 1, 2, \dots \frac{k}{2}\right) \\ r_j = 0 \quad \left(j = \frac{k}{2} + 1, \dots n\right) \end{array}\right\} \text{if } k \text{ be even,}$$

and

$$\left.\begin{array}{l} r_i = 2 \quad \left(i = 1, 2, \dots \frac{k-3}{2}\right) \\ r_{\frac{k-1}{2}} = 3 \\ r_j = 0 \quad \left(j = \frac{k+1}{2}, \dots n\right) \end{array}\right\} \text{if } k \text{ be odd.}$$

In the first case let $k = 2s$ and in the second $k = 2s + 1$. We have

$$[X^{2s}] = \frac{(2s)!}{2^s \, s!} \frac{n!}{(n-s)!} \frac{\mu_2^s}{n^{2s}} + \dots \left(\text{higher terms in } \frac{1}{n}\right) \quad \text{............(21)},$$

$$[X^{2s+1}] = \frac{(2s+1)!}{3! \, 2^{s-1} (s-1)!} \frac{n!}{(n-s)!} \frac{\mu_3 \mu_2^{s-1}}{n^{2s+1}} + \dots \left(\text{higher terms in } \frac{1}{n}\right) \quad \text{...(22)}.$$

Proceeding in this way and dividing finally the terms of $[X^{2s}]$ by $\left(\frac{\mu_2}{n}\right)^s$ and the terms of $[X^{2s+1}]$ by $\left(\frac{\mu_2}{n}\right)^{\frac{2s+1}{2}}$, we find

$$M_{2s} = \frac{(2s)!}{2^s \, s!} \left[1 + \frac{s(s-1)}{n} \left(\frac{\beta_2}{6} + \frac{(s-2)\beta_1}{9} - \frac{1}{2}\right) + \frac{1}{n^2}(\dots) + \dots\right] \quad \text{...(23)},$$

$$M_{2s+1} = \frac{(2s+1)! \sqrt{\beta_1}}{3! (s-1)! \, 2^{s-1} \sqrt{n}} \left[1 + \frac{s-1}{n} \left(\frac{\beta_3}{10\beta_1} + \frac{(s-2)\beta_2}{3!} + \frac{(s-2)(s-3)\beta_1}{(3!)^2} - \frac{s}{2}\right) + \frac{1}{n^2}(\dots) + \dots\right]$$

$$\text{.........(24)}.$$

In the same way,

$$q_{2s}=\frac{\sqrt{n}}{\sqrt{(n-1)[(n-1)\beta_2-n+3]}}\left[\frac{(2s)!}{2^s(s-1)!}(\beta_2-3)+\frac{(2s)!}{3\times 2^{s-1}(s-2)!}\beta_1+\frac{1}{n}(\ldots)+\ldots\right] \quad \ldots\ldots\ldots(25),$$

$$q_{2s+1}=\frac{n\sqrt{\beta_1}}{\sqrt{(n-1)[(n-1)\beta_2-n+3]}}\frac{(2s+1)!}{2^s s!}\left\{1+\frac{s}{3n}\left[\frac{\beta_2}{\beta_1}+\frac{3(s-1)}{2}\beta_2+\frac{(s-1)(s-2)\beta_1}{3}-\frac{9s^2+11s+6}{2}\right]+\frac{1}{n^2}(\ldots)+\ldots\right\} \quad \ldots(26).$$

Putting these values into Δ_k, we find that in its expansion the terms independent of n do not vanish and so that if $n\to\infty$ we have

$$\Delta_{2s}\to\prod_{t=1}^{2s}(t!);\qquad \Delta_{2s+1}\to\frac{(2s+2)!}{2^{s+1}(s+1)!}\prod_{t=1}^{s}[(2t)!]^2\ldots\ldots\ldots\ldots\ldots(27).$$

As to the expansion of V_k, (where $k\geqslant 3$), not only do the terms independent of n vanish, but the terms with the lowest power of $\frac{1}{n}$, that is with $\frac{1}{\sqrt{n}}$, vanish also. The next power, or $\frac{1}{n}$, in this expansion is the first one. Thus V_k has a common factor $\frac{1}{n}$ and so the kth term of the expansion (35) of my previous paper, for $k\geqslant 3$, has the common factor $\frac{1}{n^2}$. If n be large this can be considered as showing that the second order parabola gives a good fit to the regression line of the variance on the mean of the sample.

In conclusion I wish to express my warm thanks to Dr E. S. Pearson for his help in the English redaction of this paper and for the suggestion to add to it tables and diagrams, the latter of which have been skilfully executed by Miss Ida McLearn of the Galton Laboratory.

TABLE I. *Values of* $H_1=|r|$.

	$n=10$				$n=50$				
β_2 \ β_1	·0	·1	·5	1·0	·0	·1	·5	1·0	β_1 / β_2
1·5	·0000	·3721	·8321	—	·0000	·4300	·9615	—	1·5
2·0	·0000	·2860	·6396	·9045	·0000	·3100	·6931	·9802	2·0
2·5	·0000	·2410	·5388	·7620	·0000	·2548	·5697	·8056	2·5
3·0	·0000	·2121	·4744	·6708	·0000	·2214	·4950	·7000	3·0
3·5	·0000	·1917	·4286	·6061	·0000	·1984	·4436	·6274	3·5
4·0	·0000	·1762	·3939	·5571	·0000	·1813	·4055	·5735	4·0
4·5	·0000	·1639	·3665	·5183	·0000	·1680	·3758	·5314	4·5

N.B. Figures above a continuous line correspond to the U-shaped, and figures between a continuous and a dotted line to the J-shaped curves of Pearson.

TABLE I (*continued*).

	$n=100$				$n=1000$				
$\underbrace{\beta_2}\beta_1\{$	·0	·1	·5	1·0	·0	·1	·5	1·0	$\}\beta_1\underbrace{\beta_2}$
1·5	·0000	·4384	·9804	—	·0000	·4463	·9980	—	1·5
2·0	·0000	·3131	·7001	·9900	·0000	·3159	·7064	·9990	2·0
2·5	·0000	·2565	·5735	·8111	·0000	·2580	·5770	·8160	2·5
3·0	·0000	·2225	·4975	·7036	·0000	·2235	·4997	·7068	3·0
3·5	·0000	·1992	·4454	·6299	·0000	·1999	·4470	·6322	3·5
4·0	·0000	·1820	·4069	·5754	·0000	·1825	·4081	·5772	4·0
4·5	·0000	·1685	·3769	·5330	·0000	·1690	·3779	·5344	4·5

TABLE II. *Values of* H_2.

	$n=10$				$n=50$				
$\underbrace{\beta_2}\beta_1\{$	·0	·1	·5	1·0	0	·1	·5	1·0	$\}\beta_1\underbrace{\beta_2}$
1·5	·4103	·5754	1·0000	—	·2055	·4827	1·0000	—	1·5
2·0	·2075	·3663	·7132	1·0000	·0985	·3284	·7088	1·0000	2·0
2·5	·0863	·2624	·5665	·8070	·0404	·2593	·5754	·8148	2·5
3·0	·0000	·2127	·4804	·6882	·0000	·2215	·4962	·7035	3·0
3·5	·0669	·1990	·4286	·6100	·0313	·2000	·4436	·6281	3·5
4·0	·1216	·2075	·3987	·5571	·0579	·1885	·4065	·5735	4·0
4·5	·1677	·2269	·3836	·5215	·0791	·1836	·3795	·5321	4·5

	$n=100$				$n=1000$				
$\underbrace{\beta_2}\beta_1\{$	·0	·1	·5	1·0	·0	·1	·5	1·0	$\}\beta_1\underbrace{\beta_2}$
1·5	·1476	·4659	1·0000	—	·0474	·4492	1·0000	—	1·5
2·0	·0702	·3235	·7150	1·0000	·0224	·3169	·7072	1·0000	2·0
2·5	·0287	·2588	·5764	·8156	·0091	·2583	·5773	·8164	2·5
3·0	·0000	·2225	·4981	·7053	·0000	·2235	·4998	·7069	3·0
3·5	·0244	·2000	·4454	·6303	·0071	·2000	·4470	·6322	3·5
4·0	·0406	·1856	·4074	·5754	·0129	·1829	·4082	·5772	4·0
4·5	·0563	·1766	·3787	·5333	·0179	·1698	·3780	·5344	4·5

N.B. Figures above a continuous line correspond to the *U*-shaped, and figures between a continuous and a dotted line to the *J*-shaped curves of Pearson.

TABLE III.

Values of $100 \times \dfrac{H_2^2 - H_1^2}{H_2^2}$.

β_2 \ β_1	$n=10$ ·0	$n=10$ ·1	$n=10$ ·5	$n=10$ 1·0	$n=50$ ·0	$n=50$ ·1	$n=50$ ·5	$n=50$ 1·0	β_1 / β_2
1·5	100·0	58·2	30·8	—	100·0	20·6	7·5	—	1·5
2·0	100·0	39·0	19·6	18·2	100·0	10·9	4·4	3·9	2·0
2·5	100·0	15·7	9·5	10·8	100·0	3·5	2·0	2·2	2·5
3·0	?*	·5	2·5	5·0	?*	·1	·5	1·0	3·0
3·5	100·0	7·3	·0	1·3	100·0	1·6	·0	·3	3·5
4·0	100·0	27·9	2·4	·0	100·0	7·4	·5	·0	4·0
4·5	100·0	47·8	8·7	1·2	100·0	16·2	1·9	·2	4·5

β_2 \ β_1	$n=100$ ·0	$n=100$ ·1	$n=100$ ·5	$n=100$ 1·0	$n=1000$ ·0	$n=1000$ ·1	$n=1000$ ·5	$n=1000$ 1·0	β_1 / β_2
1·5	100·0	11·4	3·9	—	100·0	1·3	·4	—	1·5
2·0	100·0	5·7	2·2	2·0	100·0	·6	·2	·2	2·0
2·5	100·0	1·8	1·0	1·1	100·0	·2	·1	·1	2·5
3·0	?*	·0	·2	·5	?*	·0	·0	·1	3·0
3·5	100·0	·8	·0	·1	100·0	·0	·0	·0	3·5
4·0	100·0	3·9	·2	·0	100·0	·4	·0	·0	4·0
4·5	100·0	8·9	1·0	·1	100·0	1·0	·1	·0	4·5

N.B. Figures above a continuous line correspond to the *U*-shaped, and figures between a continuous and a dotted line to the *J*-shaped curves of Pearson.

* The value of $100\,(H_2^2 - H_1^2)/H_2^2$ at $\beta_1=0$, $\beta_2=3$ depends on the manner in which we approach this point. If we take $m=(\beta_2-3)/\beta_1$, then we have

$$100\,(H_2^2 - H_1^2)/H_2^2 = \beta_1 \Big/ \left(\beta_1 + \frac{\beta_1}{m-1} + \frac{2n}{(m-1)^2}\right) = 100 \text{ if } m \to \infty,$$

i.e. if we approach along a curve tangential at $\beta_2=3$ to $\beta_1=0$; but if m be finite, i.e. if we approach along any curve passing through this point ($\beta_1=0$, $\beta_2=3$) but not tangential to $\beta_1=0$, then

$$100\,(H_2^2 - H_1^2)/H_2^2 = 0.$$

MISCELLANEA.

The Use of Biometric Methods applied to Craniology, being a critique of Professor Gordon Harrower's "A Study of the Hokien and the Tamil Skull." *Trans. R. S. Edinburgh,* Vol. LIV. Part 3, No. 13. (Issued July 6, 1926.)

By G. M. MORANT, D.Sc.

The craniometric methods which are used to-day by workers in the Biometric Laboratory are, with some additions and slight modifications, those which were devised in the early days of Biometry, and since Miss Fawcett's paper in the first volume of *Biometrika* several thousand skulls have been measured in precisely the same way. Those methods were drawn up by Professor Karl Pearson with the object of providing data which would be comparable with the measurements of both French and German anthropologists—a consideration almost entirely neglected in earlier British craniometric work—and the fact that many different workers have employed them is a sure guarantee of their practicability. A great advance in descriptive technique was made when Crewdson-Benington published the first type contours of a racial series of skulls and Professor Pearson's more recent coefficient of racial likeness is proving an indispensable aid to the statistical student. But the work involved in providing such complete descriptions and comparisons of adequately long series of crania is very great, and the accumulation of that sufficient bulk of evidence, which is required before statistically reliable conclusions can be drawn, proceeds at a rate which is painfully slow from the individual's point of view. It is with a peculiar pleasure, therefore, that we welcome the publication of a paper discussing a collection of skulls by aid of the methods we are ourselves using. This study of Dr Harrower's is, we believe, the first not issued from the Biometric Laboratory which gives type contours and coefficients of racial likeness and which adopts the measurements used there. The skulls dealt with are those of unclaimed coolies collected in Singapore immediately after death. There are 36 ♂ Chinese from the southern province of Foo-kien and 35 ♂ Tamils from the south of India and these samples, though small, should provide good first approximations to the racial types. In many ways which might seem unimportant to anyone not well acquainted with the technique of measurement Professor Harrower has unfortunately deviated from biometric usage. The usual index letters denoting measurements were adopted but with certain alterations and one or two unusual definitions are particularly likely to lead to confusion. The writer says, "I have followed the greater number of workers in making the measurement (the nasal height) from nasion to the most prominent point on the nasal spine," but we do not know any other modern worker who follows that practice. Such a measurement is certainly not comparable with any taken in the Biometric Laboratory, or with Broca's to the base of the spine or with the Frankfurt height, and it is not to be found in Martin's detailed list. The capacities found by shot are not comparable with those given by Macdonell's seed and water method. But a more serious discordance is found in the definitions of orbital diameters. It has been the custom of all craniometricians to take an orbital width along the major axis of the roughly elliptical cavity, though different investigators have employed different terminal points on the inner margin, and the orbital height perpendicular to this width. But Professor Harrower defines the height he has taken to be the greatest in a vertical plane—presumably perpendicular to the Frankfurt horizontal—and then the breadth is taken from the dacryon at right angles to the height. The latter is certainly not the same as Broca's dacryal breadth and, as far as we are aware, both this height and breadth have not been previously employed in measuring a cranial

series. In protesting against what he calls "non-essential measurements" the writer of this paper remarks that, "the actual difference between the basi-apical (our H) and basi-bregmatic (our H') heights is infinitesimal compared to the range of variation of either measurement in a homogeneous series," and hence he only measures the latter and would apparently consider it comparable with the "basi-apical" height given by other workers. Such a statement displays an entire lack of appreciation of the effort that has been made to make these comparisons rigorously legitimate. The kind of fallacy to which it opens the door may be better illustrated in the case of orbital measurements. The question is, of course, whether the mean differences due to the use of different methods of measurement are infinitesimally small or not compared with the probable errors *of the means*. There are two orbital widths commonly taken in this laboratory, Broca's from the dacryon (our O_1') and the other from the pencil line completing the inner orbital margin (our O_1)*. The second always exceeds the first and the ♂ mean differences for series on which both have been taken are: Tibetan A 2·4 (37), Burmese A 4·6 (41), 1st Dynasty Egyptians 2·8 (33), Farringdon Street English 2·5 (81) and for the last series the standard deviation of O_1 is $1·56 \pm ·08$ and of O_1' $1·67 \pm ·09$. It would obviously be absurd to suppose that the mean orbital measurements found by the two different methods are directly comparable. Considerations such as these have led to the conclusion that the measurements of different workers may never be compared unless they were taken in precisely the same way: the differences due to the use of different methods of measurement are by no means infinitesimally small compared with the probable errors of the means, the inter-racial differences and the inter- and intra-racial standard deviations. Hence we are obliged to duplicate measurements in order to provide comparable data. We can only suppose that conclusions drawn from the direct comparison of measurements that are not identical, as in this paper, are, in all probability, entirely fallacious. In Table II the means of the series of Chinese and Tamil skulls are compared with those of 14 other races and there is no indication of the papers in which the quoted measurements were first published. That omission is all the more to be regretted as the numbers of skulls on which the means were based are not given and the student who wishes to use them in computing coefficients of racial likeness, or for other statistical purposes, must refer to the original studies. The means given for Turner's Scottish skulls certainly represent a heterogeneous population. The Hokien facial index, wrongly given as 76·5, should be 73·8; the Tamil transverse arc, wrongly given as 321·6, should be 313·2 and the Tamil occipital index, wrongly given as 63·95, should be 62·2. In computing coefficients of racial likeness Professor Harrower used the standard deviations of the long series of 26th—30th Dynasty Egyptian skulls measured and preserved in the Biometric Laboratory but no acknowledgement is made of that fact, and it would not be clear to anyone not well acquainted with the craniometric work done there that the greater number of the coefficients given are quoted from the pages of *Biometrika*. The sagittal and transverse type contours—the horizontal types not being provided—were prepared in the customary way with slight modifications. On the former the point of maximum subtense from the nasion-inion line is given instead of the point of maximum subtense from the glabella-inion line and on the latter the auricular points were located further inside the passages than is usual.

The only comparative Asiatic material used in this paper is taken from the papers of Tildesley†—misspelt Tyldesley throughout—and Morant‡. Professor Harrower was apparently not acquainted with a later biometric study of the Nepalese skull in which reduced mean skull measurements for a number of other Oriental races are given§. He says (p. 573): "China covers a vast expanse, much larger than most people realise, and its inhabitants include a large variety of branches which vary widely in craniological features....The dissimilarity between the Cantonese Chinese and the Hylam Chinese skull is, I believe, as great as that between the

* See *Biometrika*, Vol. I. p. 430 and Vol. VIII. pp. 311—312.

† M. L. Tildesley, "A First Study of the Burmese Skull," *Ibid.* Vol. XIII. 1921, pp. 176—262.

‡ G. M. Morant, "A First Study of the Tibetan Skull," *Ibid.* Vol. XIV. 1923, pp. 193—260.

§ G. M. Morant, "A Study of certain Oriental Series of Crania including the Nepalese and Tibetan Series in the British Museum (Natural History)," *Ibid.* Vol. XVI. 1924, pp. 1—104.

Scottish and the German skull." But no comparisons with earlier studied series of Chinese skulls and, indeed, no evidence of any kind are brought forward in support of these statements. In the last-mentioned paper the mean measurements of three fairly long series of Chinese skulls from different parts of the country are given. Though not identical, they proved so similar that the conclusion that all could be considered local varieties of a single type seemed to be quite justified. It was surprising to find that samples drawn from such an enormous and widely scattered population should indicate racial homogeneity and the comparison with the Hokien series is of particular interest as it may confirm or refute Professor Harrower's statement. The coefficients of racial likeness are given in the table below and it may be remarked that the lowest found by Professor Harrower between the Hokien Chinese and an Asiatic race was 5·57 ± ·17 (corrected probable error) with the Tibetans of the A type though that is rather higher than it should be owing to the fallacious comparison of orbital measurements.

Coefficients of Racial Likeness between Series of Chinese Male Skulls.

		Northern Chinese Koganei (69·4)*	Hokien Chinese (36·0)	Northern Chinese (36·4)	Southern Chinese (52·5)
Northern Chinese Koganei (69·4)	All Characters	—	1·54 ± ·21	2·56 ± ·19	6·15 ± ·19
	Indices and Angles	—	0·35 ± ·61	1·58 ± ·32	5·45 ± ·32
Hokien Chinese (36·0)	All Characters	1·54 ± ·21	—	1·62 ± ·17	1·76 ± ·18
	Indices and Angles	0·35 ± ·61	—	2·10 ± ·26	1·24 ± ·29
Northern Chinese (36·4)	All Characters	2·56 ± ·19	1·62 ± ·17	—	0·59 ± ·17
	Indices and Angles	1·58 ± ·32	2·10 ± ·26	—	1·00 ± ·26
Southern Chinese (52·5)	All Characters	6·15 ± ·19	1·76 ± ·18	0·59 ± ·17	—
	Indices and Angles	5·45 ± ·32	1·24 ± ·29	1·00 ± ·26	—

* The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

All the coefficients except those between Koganei's Northern Chinese and the Southern Chinese are of the order of those found between local varieties of the same race or between very closely allied races, so comparison with the Hokien material enables us to re-affirm the belief that the immense population of China belongs predominantly to a single racial type. Making the kind of comparison Professor Harrower suggests between a British (Farringdon Street English (95·6)) and a German (Ranke's *Altbayerisch* (77·0)) ♂ series we find coefficients of 43·92 ± ·19 for 23 characters and 58·97 ± ·33 for 7 indices and angles, results of a widely different order. The longest series of Tamil skulls previously measured appears to have been a short one of 13 ♂ specimens collected by the brothers Sarasin*. The coefficients between the Tamils from Ceylon (12·4) and Harrower's series collected in Singapore (35·0) are 2·25 ± ·19 for 21 characters and 3·54 ± ·32 for 8 indices and angles, but it would be unwise to conclude that there is a real difference between the island and mainland peoples since the samples are so small. The latter series gives coefficients of 5·66 ± ·18 with Veddah means† (24·4) of 27 characters and 4·22 ± ·30

* Paul and Fritz Sarasin, *Ergebnisse naturwissenschaftliche Forschungen auf Ceylon.* Wiesbaden, 1892–4.

† Given in *Biometrika*, Vol. xvi. p. 48 and reduced from the measurements of P. and F. Sarasin (*op. cit.*) with the addition of a few others.

when 9 indices and angles are compared. With the reduced Maravar (Dravidian) means* (25·9) the Tamils (35·0) give coefficients of 5·82 ± ·23 for 16 characters and 2·24 ± ·67 for 3 (!) indices. As these are the lowest coefficients that can be found, little can be said about the affinities of the Tamil skull and no craniological classification of the Dravidian peoples of India can be attempted while the material is so meagre.

Commenting on the use of the coefficient of racial likeness, Professor Harrower says: "a skull may be an exact replica of another, but, say, 10 per cent. smaller in every measurement. The C.R.L. between the two skulls would therefore be much greater than the C.R.L. between two skulls which had half the measurements exactly equal, one-fourth 10 per cent. above and the remaining fourth 10 per cent. below: skulls which would be entirely different in general outline" (p. 586). We may remark, in the first place, that no one has ever suggested finding coefficients between single skulls and that if conditions similar to these hypothetical ones were found for mean types—and from what little we know of inter-racial correlations the probability of finding such would seem to be very small—then it is by no means evident why the first coefficient should be much greater than the second. Indeed, the first might well be smaller than the second. The separate computation of coefficients for indices and angles only was, of course, suggested to provide a measure of differences of shape apart from differences of size. One or two minor points may be noticed. It is contrary to biometric usage to give standard deviations without probable errors as on p. 581. The frontal index defined on the previous page should be preceded by 100 and the legitimacy of its use may be questioned as the median sagittal arc from nasion to bregma has not the form of a catenary if the superciliary ridge is at all well developed. We fail to find any evidence adduced to support the statement—made on p. 597—that "the Tibetan A group is hybrid and the Hokien is a pure type," and it is unworthy of a scientific study to draw conclusions such as the following, involving theories which no one has ever attempted to demonstrate. "Intellectually," we are told, "as judged by his cranial capacity, he (the Hokien) is capable of attaining to the highest standard, but the very narrow frontal region (giving the lowest value of all the groups compared) indicates that his powers of initiative and intelligence are relatively low" (p. 597). We are sorry to see such undisciplined theorising cheek by jowl with valuable biometric measurement. Anthropologists are indebted to Professor Harrower, however, for having provided a craniological record which will be of some permanent value.

* *Biometrika*, Vol. XVI. p. 28.

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